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SOME INTEGRAL FORMULAS FOR COMPACT SURFACES*

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Abstract. In this work a method for deriving new integral formulas for compact surfaces is introduced, in particular, to generalize the famous Gauss-Bonnet, Minkowski, Blaschke and Herglotz formulae.

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1. Introduction

It is known that many famous results of Geometry "in whole" are received by using some integral formulas. Among them the most known and important is, of course, the Gauss-Bonnet formula

$$\iint\limits_S K\ dA = 2\pi\chi,$$

where S is a C^2-smooth compact surface with the area element dA, K is its Gauss curvature and χ is Euler characteristic of S. As a method of proof the integral formulas have been used, for example, by Blaschke for a proof of the infinitesimal rigidity of ovaloids and by Herglotz for a proof of global rigidity of ovaloids too, they compose the essential part of Bochner's technics. In the theory of convex surfaces there are several Minkowski formulas presenting necessary conditions in some theorems of existence. In this article we want to present a method to obtain many new integral relations for compact surfaces of any topological genus g with some concrete formulas and their applications.

2. Generalized Minkowski and Herglotz formulas

The starting point is a method for proving of Herglotz formula given in [1], p. 276, using the integration of the differential $d\omega$ where 1-form ω is equal to the mixed product $(\mathbf{r}, d\mathbf{n}, \mathbf{n})$. The generalization of this method consists in using of a form $f\omega$ with an arbitrary function f determined on the surface S. We suppose that the metric of S is given in isothermic coordinates (u, v) in which

$$ds^2 = \Lambda^2(u, v)(du^2 + dv^2).$$

We'll recall notations from [1]. Let the vectors $\Lambda\mathbf{e}_1 = \mathbf{r}_u, \Lambda\mathbf{e}_2 = \mathbf{r}_v, \mathbf{e}_3 = \mathbf{n}$ compose a positivily oriented moving orthonormal frame on S. We have

$$d\mathbf{r} = \sum_{i=1}^{3} \omega_i \mathbf{e}_i,\ d\mathbf{e}_i = \sum_{j=1}^{3} \omega_{ij}\mathbf{e}_j \tag{1}$$

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with

$$\omega_1 = \Lambda du,\ \omega_2 = \Lambda dv,\ \omega_3 = 0,\ \omega_{ii} = 0,\ \omega_{ij} + \omega_{ji} = 0 \tag{2}$$

and

$$\omega_{12} = -\frac{\Lambda_v}{\Lambda}du + \frac{\Lambda_u}{\Lambda}dv,\ \omega_{13} = \frac{L}{\Lambda}du + \frac{M}{\Lambda}dv,\ \omega_{23} = \frac{M}{\Lambda}du + \frac{N}{\Lambda}dv, \tag{3}$$

where L, M and N are classical notations for coefficients of the second form of the surface,

In addition, between 1-forms ω_i, ω_{ij} and their exterior differentials there are the following structural relations

$$d\omega_i = \sum_{j=1}^{3} \omega_j \wedge \omega_{ji},\ d\omega_{ij} = \sum_{k=1}^{3} \omega_{ik} \wedge \omega_{kj},\ i, j = 1, 2, 3. \tag{4}$$

Let's note $\Omega = f(u, v)\omega$. As in [1], p. 276, we have

$$d\Omega = df \cdot \omega + f(u, v)(d\mathbf{r}, d\mathbf{n}, \mathbf{n}) + f(u, v)(\mathbf{r}, d\mathbf{n}, -d\mathbf{n}).$$

In working with a mixed product we should to take in attention that the presence of two equal vector-valued forms does not mean the equality to zero of this product because the both multiplications, that of vectors as well as of forms, are exterior. Now using formulas (1)-(4) and equalities $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3,\ \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1,\ \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2$ we obtain

$$d\Omega = \left[(\mathbf{r}, \mathbf{e}_1)\frac{Mf_v - Nf_u}{\Lambda^2} + (\mathbf{r}, \mathbf{e}_2)\frac{Mf_u - Lf_v}{\Lambda^2}\right] dA - 2fH\ dA - 2fKp\ dA,$$

where $p = (\mathbf{r}, \mathbf{n})$ denotes the support function of the surfaces, that is the oriented distance from the origin of coordinates to the tangent plane of S at the end point of the vector $\mathbf{r}$. Hence we have the main formula

$$2\iint\limits_S fH\ dA + 2\iint\limits_S fKp\ dA = \\ = \iint\limits_S \left[(\mathbf{r}, \mathbf{e}_1)\frac{Mf_v - Nf_u}{\Lambda^3} + (\mathbf{r}, \mathbf{e}_2)\frac{Mf_u - Lf_v}{\Lambda^3}\right] dA, \tag{5}$$

which in the case $f = 1$ presents the known Minkowski formula prouved by him for convex surfaces.

We can take an another form $\tilde{\Omega} = f \cdot (\mathbf{r}, \omega_{13}^* \mathbf{e}_1 + \omega_{23}^* \mathbf{e}_2, \mathbf{n})$ where the sign $*$ means that the corresponding value is taken for a surface S^* isometric to S. This time we have

$$d\tilde{\Omega} = \left[(\mathbf{r}, \mathbf{e}_1)\frac{N^* f_u - M^* f_v}{\Lambda^3} + (\mathbf{r}, \mathbf{e}_2)\frac{L^* f_v - M^* f_u}{\Lambda^3}\right] dA + \\ + 2fH^*\ dA + 2fKp\ dA - f\begin{vmatrix} l^* - l & m^* - m \\ m^* - m & n^* - n \end{vmatrix} p\ dA,$$

where

$$l^* - l = \frac{L^* - L}{\Lambda^2}, m^* - m = \frac{M^* - M}{\Lambda^2}, n^* - n = \frac{N^* - N}{\Lambda^2}.$$

We arrive to the second formula

$$2\iint\limits_S fH^*\ dA + 2\iint\limits_S fKp\ dA - \iint\limits_S f\begin{vmatrix} l^* - l & m^* - m \\ m^* - m & n^* - n \end{vmatrix} p\ dA = \\ = \iint\limits_S \left[(\mathbf{r}, \mathbf{e}_1)\frac{M^* f_v - N^* f_u}{\Lambda^3} + (\mathbf{r}, \mathbf{e}_2)\frac{M^* f_u - L^* f_v}{\Lambda^3}\right] dA. \tag{6}$$

From (5) and (6) we have the followig generalization of Herglotz formula

$$2\iint\limits_S fH\;dA - 2\iint\limits_S fH^*\;dA + \iint\limits_S pf \begin{vmatrix} l^*-l & m^*-m \\ m^*-m & n^*-n \end{vmatrix} dA =$$

$$= \iint\limits_S \left[(\mathbf{r},\mathbf{r}_u)\frac{(M-M^*)f_v-(N-N^*)f_u}{\Lambda^4} + (\mathbf{r},\mathbf{r}_v)\frac{(M-M^*)f_u-(L-L^*)f_v}{\Lambda^4}\right] dA. \tag{7}$$

There are also many other sources to obtain some new integral equalities for compact surfaces. For example let's take the other Minkowski equality[1]

$$\iint\limits_S Hp\;dA + A = 0\ (A \text{ is the area of the surface}), \tag{8}$$

which is proved in [1] starting from the relation

$$d(\mathbf{n},\mathbf{r},d\mathbf{r}) = (2+2pH)dA.$$

If we start from the form $f\cdot(\mathbf{n},\mathbf{r},d\mathbf{r})$ we arrive to the equality

$$2\iint\limits_S fHp\;dA + 2\iint\limits_S f\;dA =$$

$$= -\iint\limits_S \left[\frac{(\mathbf{r},\mathbf{r}_u)f_u}{\Lambda^2} + \frac{(\mathbf{r},\mathbf{r}_v)f_v}{\Lambda^2}\right]\;dA \tag{9}$$

generalizing the Minkowski formula (8). Evidently there are many others combinations to use for new relations, f.e. instead of a scalar function f one can take some vector-functions $\mathbf{f}$ related with the surface and its different deformations.

3. A GENERALIZATION OF BLASCHKE FORMULA

The formula (7) for the case $f = 1$ gives immediately a new very simple proof (see [3]) of the known fact of invariance of integral mean curvature during bending of a bendable surface S. From this fact we have the following theorem

Theorem 1. *If two isometric surfaces have the same integral mean curvature then for them we have*

$$\iint\limits_S p\Delta\;dA = 0, \tag{10}$$

where

$$\Delta = \begin{vmatrix} l^*-l & m^*-m \\ m^*-m & n^*-n \end{vmatrix}.$$

The formula (10) is an analog of Blaschke integral formula for the field of rotation of an infinitesimal deformation established by him for a surface of genus $g = 0$, [2]. It is valid also for Bonnet mate surfaces which are isometric and have the same mean curvature. In this last case $\Delta \le 0$. One should remark that this theorem is purely hypothetic for the moment because still it is not known the existence of a compact bendable surface neither compact Bonnet mate.

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[1] By the way from this equality we have for a compact surface with constant mean curvature H_0 its volume $V = -\frac{3A}{H_0}$ and using the isoperimetric inequalty we have that the area A of the surface and its constant mean curvature H_0 are related by the inequality $AH_0^2 \ge 36\pi$ (take in attention that the formula (7) and the isoperimetric inequality are valid for immersed surfaces too).

4. A GENERALIZATION OF GAUSS-BONNET FORMULA

Now choosing different functions $f(u,v)$ we can obtain a lot of new integral formulas for compact surfaces. Let's consider some cases.

1) Choose $f = 1$. Then the formula (5) gives the known equality, see [1], p. 279

$$\iint\limits_S H\ dA + \iint\limits_S Kp\ dA = 0. \tag{11}$$

Suppose we translate the surface to a constant vector $\mathbf{C}$. Then the previous formula takes the form

$$\iint\limits_S H\ dA + \iint\limits_S K(p + (\mathbf{C}, \mathbf{n}))\ dA = 0,$$

from which we have

$$\iint\limits_S K\mathbf{n}\ dA = 0. \tag{12}$$

Analogically, from the Minkowski equality we have

$$\iint\limits_S H\mathbf{n}\ dA = 0.$$

Let's suppose that the surface S is flexible (or bendable, in other therminology). Because the integral mean curvature remains constant during a flexion, from (5) we have for flexing surfaces

$$\iint\limits_S Kp\ dA = \iint\limits_S Kp^*\ dA.$$

From (7) we have analogical equality with the mean curvatures of flexing surfaces

$$\iint\limits_S Hp\ dA = \iint\limits_S H^* p^*\ dA.$$

2) Choose $f = p^k, k \in \mathbf{N}$. Then

$$f_u = -kp^{k-1}\frac{L(\mathbf{r}, \mathbf{e}_1) + M(\mathbf{r}, \mathbf{e}_2)}{\Lambda},\ \ f_v = -kp^{k-1}\frac{M(\mathbf{r}, \mathbf{e}_1) + N(\mathbf{r}, \mathbf{e}_2)}{\Lambda},$$

and the formula (5) gives the equality

$$\begin{aligned} 2\iint\limits_S p^k H\ dA + 2\iint\limits_S Kp^{k+1}\ dA = \\ = k\iint\limits_S [(\mathbf{r}, \mathbf{e}_1)^2 + (\mathbf{r}, \mathbf{e}_2)^2] p^{k-1} K\ dA \end{aligned} \tag{13}$$

Because $\mathbf{r} = \sum_{i=1}^{3} (\mathbf{r}, \mathbf{e}_i)\mathbf{e}_i$ we have

$$(\mathbf{r}, \mathbf{e}_1)^2 + (\mathbf{r}, \mathbf{e}_2)^2 = \mathbf{r}^2 - p^2,$$

so we can present the equation (13) as follows

$$\begin{aligned} 2\iint\limits_S p^k H\ dA + (2+k)\iint\limits_S Kp^{k+1}\ dA = \\ = k\iint\limits_S (\mathbf{r}^2) p^{k-1} K\ dA. \end{aligned} \tag{14}$$

For the case $k = 1$ using the equation (7) we have the following equality

$$3 \iint\limits_S Kp^2 dA - \iint\limits_S K\mathbf{r}^2 dA = 2A.$$

Multiplying the equation (14) by ε^k and summarising for all $k = 0, 1, 2, ...$ we obtain the equality

$$2 \iint\limits_S \frac{H}{1 - \varepsilon p} dA + \iint\limits_S \frac{2p - \varepsilon p^2}{(1 - \varepsilon p)^2} K \ dA = \iint\limits_S \frac{\varepsilon \mathbf{r}^2}{(1 - \varepsilon p)^2} K \ dA. \tag{15}$$

All three integrals in (15) can be considered as some analytical functions $F_1(\varepsilon), F_2(\varepsilon)$ and $F_3(\varepsilon)$ of complex variable ε with the possible singularities only on the real axe and with the relation $F_1(\varepsilon) + F_2(\varepsilon) = F_3(\varepsilon)$. The nature of these functions is studied not at all, for example, what is passing when $Re(\varepsilon) = 0$ and $\varepsilon \to \infty$?

The equality (14) doesn't depend on the position of the surface S in the space. This means that we can translate it on any constant vector $\mathbf{C}$ and the equality still will be valid. Let's mark the values in an initial position by the subscript 0: $\mathbf{r}_0, p_0$ etc. After translation to a constant vector $\mathbf{C}$ we have

$$2 \iint\limits_S [p_0 + (\mathbf{C}, \mathbf{n})]^k H \ dA + (2 + k) \iint\limits_S K[p_0 + (\mathbf{C}, \mathbf{n})]^{k+1} \ dA =$$

$$== k \iint\limits_S \left[(\mathbf{r}_0)^2 + 2(\mathbf{r}_0, \mathbf{C}) + \mathbf{C}^2\right] [p_0 + (\mathbf{C}, \mathbf{n})]^{k-1} K \ dA. \tag{16}$$

Let the unit normal be $\mathbf{n} = \{n_1, n_2, n_3\}$ in the standart orthonormal basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$. Take k equal to 1. Let the vector of translation be $C_1\mathbf{i}$. Then we have

$$2 \iint\limits_S (p_0 + C_1 n_1) H \ dA + 3 \iint\limits_S (p_0^2 + 2p_0 C_1 n_1 + C_1^1 n_1^2) K \ dA =$$

$$= \iint\limits_S (\mathbf{r}_0^2 + 2x_0 C_1 + C_1^2) K \ dA. \tag{17}$$

The members in (17) at the C_1^2 give the equality

$$3 \iint\limits_S K n_i^2 = \iint\limits_S K \ dA, \ i = 1, 2, 3,$$

that is

$$\iint\limits_S K n_i^2 \ dA = \frac{2\pi\chi}{3}, \tag{18}$$

where χ is Euler characteristic of the surface.

Now consider the members at the first degree of C_1. We have

$$2 \iint\limits_S H n_1 \ dA + \iint\limits_S 3p_0 n_1 K \ dA = \iint\limits_S x_0 K \ dA. \tag{19}$$

After the translation to the vector $C_2\mathbf{j}$ the coefficients at C_2 give the equality

$$\iint\limits_S K n_1 n_2 \ dA = 0. \tag{20}$$

By the analogical considerations one can obtain the vectorial equality

$$\iint\limits_S K\mathbf{r}\ dA = 3\iint\limits_S Kp\mathbf{n}\ dA.$$

Let's consider now the case of arbitrary value of k and take in (16) the members with the greatest degree of $\mathbf{C}$. Then we obtain

$$(2+k)\iint\limits_S Kn_i^{k+1}\ dA = k\iint\limits_S Kn_i^{k-1}\ dA.$$

If k is an odd number, $k = 2m-1$ then using the received recurrent relation and the formula (18) we find

$$\iint\limits_S Kn_i^{2m}\ dA = \frac{2\pi\chi}{2m+1}. \tag{21}$$

If k is an even number then the recurrent relation leads us finally to the formula (12) and we have for all odd degrees equality

$$\iint\limits_S Kn_i^{2m-1}\ dA = 0.$$

Using the formula (21) we can find the integral $\iint\limits_S Kn_i^{2m}n_j^2\ dA, i \neq j$. Indeed let's take $i = 1$ then

$$\iint\limits_S Kn_1^{2m}n_2^2\ dA = \iint\limits_S Kn_1^{2m}n_3^2\ dA = J.$$

From (21) we know

$$\iint\limits_S Kn_1^{2m+2}\ dA = \iint\limits_S Kn_1^{2m}n_1^2\ dA = \frac{2\pi\chi}{2m+3}.$$

Then

$$\iint\limits_S Kn_1^{2m}\ dA = \iint\limits_S Kn_1^{2m+2}\ dA + 2J$$

and

$$J = \iint\limits_S Kn_1^{2m}n_2^2\ dA = \iint\limits_S Kn_1^{2m}n_3^2\ dA = \frac{2\pi\chi}{(2m+1)(2m+3)}.$$

For example, we have $\iint\limits_S Kn_1^2n_2^2\ dA = \dfrac{2\pi\chi}{15}.$

Now we consider in (16) the members with the least positive degree of $\mathbf{C}$ equal to 1. We have the equality

$$2k\iint\limits_S p_0(\mathbf{C},\mathbf{n})H\ dA + (2+k)(k+1)\iint\limits_S p_0^k(\mathbf{C},\mathbf{n})K\ dA =$$

$$= k\iint\limits_S [(k-1)(\mathbf{r}_0)^2p_0^{k-2}(\mathbf{C},\mathbf{n}_0) + 2(\mathbf{C},\mathbf{r}_0)p_0^{k-1}]K\ dA. \tag{22}$$

Choose here $\mathbf{C} = C_1\mathbf{i}$. Then we have

$$2k\iint\limits_S p_0^{k-1}Hn_1\ dA + (2+k))(1+k)\iint\limits_S p_0^kKn_1\ dA =$$

$$= k\iint\limits_S [2x_0p_0^{k-1} + (k-1)\mathbf{r}_0^2p_0^{k-2}n_1]K\ dA. \tag{23}$$

The formula (23) is valid for any "initial" values $p_0, x_0, \mathbf{r}_0$. Add to the initial position vector $C_2\mathbf{j}$. Then the formula takes the following form

$$2k\iint\limits_S (p_0+C_2n_2)^{k-1}Hn_1\ dA+(2+k)(1+k)\iint\limits_S (p_0+C_2n_2)^kKn_1\ dA=$$

$$=k\iint\limits_S [2x_0(p_0+C_2n_2)^{k-1}+(k-1)(\mathbf{r}_0+C_2bfj)^2(p_0+C_2n_2)^{k-2}n_1]K\ dA. \tag{24}$$

Consider the coefficients at the greatest degree of C_2 equal to k. Then we obtain the following equality

$$(k+1)(k+2)\iint\limits_S Kn_1n_2^k\ dA=k(k-1)\iint\limits_S Kn_1n_2^{k-2}\ dA.$$

Using this reccurence we succeed to express $\iint\limits_S Kn_1n_2^k\ dA$ as the product of $\iint\limits_S Kn_1n_2\ dA$ or $\iint\limits_S Kn_1\ dA$ by a coefficient in dependence of parity of k. But in both cases because of the formulas (14) and (20) we have that

$$\iint\limits_S Kn_1n_2^k\ dA=0.$$

We remark that the method of reducing of integrals $\iint\limits_S Kn_1^{s_1}n_2^{s_2}n_3^{s_3}dA$ to some seeming integrals with the smaller degrees of n_1, n_2 and n_3 doesn't depend on the view of concrete surfaces and it gives always the product of $2\pi\chi$ by a coefficient which is the same for any surface. So we can calculate this coefficient for the unit sphere and we obtain

Theorem 2. *For any compact C^2-smooth surface we have*

$$\iint\limits_S Kn_1^ln_2^mn_3^n\ dA=0,$$

if one of degrees l, m, n is odd. For even degrees we have

$$\iint\limits_S Kn_1^{2l}n_2^{2m}n_3^{2n}\ dA=\frac{(2l)!(2m)!(2n)!(l+m+n)!}{l!m!n!(2l+2m+2n+1)!}2\pi\chi.$$

For example, we have $\displaystyle\iint\limits_S Kn_1^2n_2^2n_3^2\ dA=\frac{2\pi\chi}{105}$.

Corollary 3. *If in the integral $\iint\limits_S (Kn_1^ln_2^mn_3^n)\mathbf{n}\ dA$ all degrees are even numbers or there are two or three odd degrees then it is equal to 0.*

5. Another formula

Now we choose $f=(\mathbf{r}^2)^k\equiv r^{2k}$ in (5) and obtain the equality

$$\iint\limits_S r^{2k}H\ dA+\iint\limits_S r^{2k}Kp\ dA=-k\iint\limits_S Pr^{2k-2}dA, \tag{25}$$

where

$$P=\frac{L(\mathbf{r},\mathbf{r}_v)^2-2M(\mathbf{r},\mathbf{r}_u)(\mathbf{r},\mathbf{r}_v)+N(\mathbf{r},\mathbf{r}_u)^2}{\Lambda^4}.$$

Again multiplying the equation (25) by ε^k and taking the sum over all $k = 0, 1, 2, ...$ we obtain a new relation

$$\iint\limits_S \frac{H}{1-\varepsilon F} dA + \iint\limits_S \frac{Kp}{1-\varepsilon F} dA = \iint\limits_S \frac{\varepsilon P}{(1-\varepsilon F)^2} dA, \ F \equiv \mathbf{r}^2, \tag{26}$$

which can be extended to complex values of ε too. This time we can suppose $\mathbf{r}^2 \neq 0$ on S and then the relations (25) will be valid also for $k = -1, -2, ...$ By the way the same equations can be obtained considering the behavior of integrals in (26) for values $\varepsilon \to \infty$. More, using (26) we can obtain other integral relations. Indeed replace ε in (26) by $-\varepsilon$ and take the sum of two formulas. We have

$$\iint\limits_S \frac{H}{1-(\varepsilon F)^2} dA + \iint\limits_S \frac{Kp}{1-(\varepsilon F)^2} dA = \iint\limits_S \frac{4\varepsilon^2 F}{(1-(\varepsilon F)^2)^2} P \ dA. \tag{27}$$

Now let's take the equation in the point $i\varepsilon$ (instead of ε) and add this new equation to (27) (or simply replace ε^2 by t and $-t$ and add two equations). We obtain

$$\iint\limits_S \frac{H}{1-(\varepsilon F)^4} dA + \iint\limits_S \frac{Kp}{1-(\varepsilon F)^4} dA = \iint\limits_S \frac{16\varepsilon^4 F^3}{(1-(\varepsilon F)^4)^2} P \ dA.$$

We can continue this process and for any n arrive to the equation

$$\iint\limits_S \frac{H+Kp}{1-(\varepsilon F)^{m(n)}} dA = \iint\limits_S \frac{m^2(n)\varepsilon^{m(n)} F^{m(n)-1}}{(1-(\varepsilon F)^{m(n)})^2} P \ dA, \ n = 1, 2, \dots, \tag{28}$$

where $m(n) = 2^{n-1}$. The independence of these equations on a translation of the surface in space gives many new integral formulas valid which define in their turn many functions on complexe variable ε. So any compact surface generates many holomorphe functions on complexe variable which are not studied yet.

6. A VOLUME FORMULA

We could find many other integral formulas using the equality (9) with different choices of function f. But we restrict to show only that the volume bounded by an immersed surface can be calculate if we know the metric, the second fundamental form and the distances from a point to points of the surface[2]. Let's recall that the algebraic (or oriented) volume V of a body B with the boundary $\partial B = S$ is integral $\frac{1}{3}\iint\limits_S p \ dA$. In the case when a surface is an immersion only so it doesn't bound any body the above formula gives *by definition* the generalized oriented volume restricted by this surface. Then the formula (9) with the choice $f = p$ gives us the equality

$$2\iint\limits_S Hp^2 dA + 6V = \iint\limits_S \frac{L(\mathbf{r},\mathbf{r}_u)^2 + 2M(\mathbf{r},\mathbf{r}_u)(\mathbf{r},\mathbf{r}_v) + N(\mathbf{r},\mathbf{r}_v)^2}{\Lambda^4} dA. \tag{29}$$

If in this formula we replace p^2 by the expression

$$p^2 = \mathbf{r}^2 - (\mathbf{r},\mathbf{e}_1)^2 - (\mathbf{r},\mathbf{e}_2)^2 = \mathbf{r}^2 - \frac{(\mathbf{r},\mathbf{r}_u)^2 + (\mathbf{r},\mathbf{r}_v)^2}{\Lambda^2}$$

we arrive to the theorem

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[2] Of course if we know two fundamental forms of a surface we can find the surface itself but it is possible only theoretically meanwhile for our formula we need to know $|\mathbf{r}|^2 = \mathbf{r}^2$ and not $\mathbf{r}$.

Theorem 4. *The algebraic volume V bounded by an immersed surface S with the position vector $\mathbf{r}(u,v)$ is given by the formula*

$$V=-\frac{1}{3}\iint\limits_S HF dA+\frac{1}{3}\iint\limits_S H|grad\ F|^2 dA+\frac{1}{12}\iint\limits_S \frac{<Qgrad\ F, grad\ F>}{\Lambda^2}dA,$$

where $F=\mathbf{r}^2, Q$ is the bilinear form corresponding to the second fundamental form of the surface and $grad\ F$ is the gradient of F on the surface.

We can present the formula for the volume by an other one. For this we have to take the difference between the equation (29) and the equation (25) for the case $k=1$ Then we have

$$6V=\iint\limits_S[(Kp-H)\mathbf{r}^2+H|grad\ \mathbf{r}^2|^2]dA.$$

The last formula is looking better but it includes the support function p however if we know p we can find the volume immediately as integral $\iint\limits_S p\ dA$. The theorem 4 gives an answer without using the function p.

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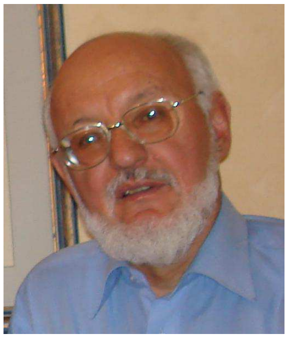

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