

IDENTITIES OF DUAL LEIBNIZ ALGEBRAS*

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ABSTRACT. We prove that in characteristic 0 any proper subvariety of the variety of dual Leibniz algebras is nilpotent. Consequently, the variety of dual Leibniz algebras is Shpekhtian and has base rank 1.

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1. INTRODUCTION

Recall that an algebra \mathfrak{g} over an arbitrary field k equipped with a bilinear operation $[-, -]$ is called *Leibniz* if it satisfies the (right) *Leibniz identity*

$$[x, [y, z]] = [[x, y], z] - [[x, z], y].$$

An algebra A over k is called *dual Leibniz* if it satisfies the identity

$$(xy)z = x(zy + yz). \tag{1}$$

The Leibniz algebras form a Koszul operad in the sense of V. Ginzburg and M. Kapranov [4]. Under the Koszul duality the operad of Lie algebras is dual to the operad of associative and commutative algebras. The notion dual Leibniz algebra defined by J.-L. Loday [7] is precisely the dual operad of Leibniz algebras in this sense. Moreover, any dual algebra A with respect to symmetrization

$$a * b = ab + ba \tag{2}$$

is an associative and commutative algebra [7]. This defines a functor

$$(\text{dual Leibniz}) \longrightarrow (\text{Com})$$

between the categories of algebras, which is dual to the inclusion functor

$$(\text{Lie}) \longrightarrow (\text{Leibniz}).$$

Dual Leibniz algebras are called also Zinbiel (Leibniz is written in reverse order) algebras.

In this paper we study identities of dual Leibniz algebras. Let \mathfrak{M} be a variety of linear algebras over a field k and $k_{\mathfrak{M}} \langle x_1, x_2, \dots, x_n \rangle$ be the free algebra of \mathfrak{M} in the variables x_1, x_2, \dots, x_n . The least natural number n such that the variety $\text{Var}(k_{\mathfrak{M}} \langle x_1, x_2, \dots, x_n \rangle)$ of algebras generated by $k_{\mathfrak{M}} \langle x_1, x_2, \dots, x_n \rangle$ is equal to \mathfrak{M} is called the *base rank* $rb(\mathfrak{M})$ of the variety \mathfrak{M} . If such a number does not exist then we say that $rb(\mathfrak{M}) = \infty$.

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It is obvious that the base rank of the variety of associative and commutative algebras is equal to 1. In 1952 A.I. Malcev [9] proved that any associative algebra of countable dimension can be embedded in an associative algebra with two generators. The same result for Lie algebras was proved by A.I. Shirshov [13] in 1958. Consequently, the base ranks of the variety of associative algebras and Lie algebras are equal to 2. In 1977 I.P. Shestakov [10] proved that the base ranks of the variety of alternative and Malcev algebras are infinite (see, also [3]).

A variety of algebras \mathfrak{M} is said to be *Spekhtian* if each of its subvarieties is defined by a finite system of identities. It is equivalent that the variety satisfies the descending chain condition for subvarieties. The famous result by A.R. Kemer [8] says that the variety of associative algebras is Shpekhtian in characteristic 0. In the case of nonassociative algebras there are many partial results (see, for example [1, 6, 11, 12]).

A.S. Dzhumadildaev and K.M. Tulenbaev [2] proved an analog of the Nagata-Higman theorem [5] for dual Leibniz algebras. In particular, they proved that any dual Leibniz algebra with bounded nil-index is nilpotent and every finite-dimensional dual Leibniz algebra over a field of characteristic 0 is nilpotent.

J.-L. Loday proved [7] that the set of all nonassociative words with left arranged parenthesis compose a basis of free dual Leibniz algebras. So, there is a one-to-one correspondence between this basis of free dual Leibniz algebras and the set of all nonempty associative words. In this sense the properties of the variety of dual Leibniz algebras must be close to properties of the variety of associative algebras. This opinion contradicts to the results of this paper.

In this paper we prove that in characteristic 0 the free dual Leibniz algebras in a single variable do not satisfy any nontrivial identity. We also prove that any proper subvariety of dual Leibniz algebras is nilpotent. Consequently, the variety of dual Leibniz algebras is Shpekhtian and has base rank 1. It looks like there is some parallel between free dual Leibniz algebras and polynomial algebras which approves Loday's functor (*dual Leibniz*) \longrightarrow (*Com*).

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2. HOMOMORPHISMS INTO ONE GENERATED FREE ALGEBRAS

Throughout this paper denote by k an arbitrary fixed field of characteristic 0. Denote by $DL \langle x_1, x_2, \dots, x_n, \dots \rangle$ the free dual Leibniz algebra over k in the variables $x_1, x_2, \dots, x_n, \dots$.

A linear basis of free dual Leibniz algebras is given in [7]. The set of all nonempty nonassociative words with left arranged parenthesis

$$x_{i_1}(x_{i_2} \dots (x_{i_{n-1}}x_{i_n}) \dots), \quad n \geq 1,$$

compose a basis of $DL \langle x_1, x_2, \dots, x_n, \dots \rangle$.

In characteristic 0 any identity is equivalent to the set of multilinear homogeneous identities [15]. Then any nontrivial dual Leibniz identity can be written as

$$\sum_{\sigma \in S_n} \alpha_{\sigma}(x_{\sigma 1}(x_{\sigma 2} \dots (x_{\sigma n-1}x_{\sigma n}) \dots)) = 0, \quad (3)$$

where S_n is the n th symmetric group, $\alpha_{\sigma} \in k$.

Let A be an arbitrary dual Leibniz algebra and $a \in A$. We define a^i by induction on i as follows: $a^1 = a, a^{i+1} = aa^i$ for all $i \geq 1$. It is proved in [2] that

$$a^i a^j = \binom{i+j-1}{j} a^{i+j} \quad (4)$$

for all $i, j \geq 1$.

Theorem 1. *The free dual Leibniz algebras in a single variable over a field of characteristic zero do not satisfy any nontrivial dual Leibniz identity.*

Proof. Denote by $A = DL \langle x \rangle$ the free dual Leibniz algebra in the variable x . Put $X_i = i!x^i$. By (4),

$$X_i X_j = \frac{i}{i+j} X_{i+j}. \quad (5)$$

Consider a homomorphism

$$\psi : DL \langle x_1, x_2, \dots, x_n \rangle \longrightarrow D \langle x \rangle$$

such that $\psi(x_i) = X_{\lambda_i}$ for all $1 \leq i \leq n$. Put also

$$P_n(\lambda_1, \lambda_2, \dots, \lambda_n) = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \dots + \lambda_n} \frac{\lambda_2}{\lambda_2 + \dots + \lambda_n} \cdots \frac{\lambda_{n-2}}{\lambda_{n-2} + \lambda_{n-1} + \lambda_n} \frac{\lambda_{n-1}}{\lambda_{n-1} + \lambda_n}.$$

Note that

$$P_n(\lambda_1, \lambda_2, \dots, \lambda_n) = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \dots + \lambda_n} P_{n-1}(\lambda_2, \dots, \lambda_n).$$

Using this property and (5), it can be easily proved that

$$\psi(x_1(x_2 \dots (x_{n-1}x_n) \dots)) = P_n(\lambda_1, \lambda_2, \dots, \lambda_n) X_{\lambda_1 + \lambda_2 + \dots + \lambda_n}.$$

Put also

$$Q_n(\lambda_1, \lambda_2, \dots, \lambda_n) = \frac{1}{(\lambda_2 + \dots + \lambda_n)(\lambda_3 + \dots + \lambda_n) \cdots (\lambda_{n-2} + \lambda_{n-1} + \lambda_n)(\lambda_{n-1} + \lambda_n)\lambda_n}.$$

Then,

$$P_n(\lambda_1, \lambda_2, \dots, \lambda_n) = \frac{\lambda_1 \lambda_2 \cdots \lambda_n}{\lambda_1 + \lambda_2 + \dots + \lambda_n} Q_n(\lambda_1, \lambda_2, \dots, \lambda_n).$$

Now assume, in contrary, that A satisfies a nontrivial multilinear identity of the form (3) with the least possible natural n . We may assume $\alpha_1 = 1$. Applying homomorphism ψ , from (3) we get

$$\sum_{\sigma \in S_n} \alpha_\sigma P_n(\lambda_{\sigma_1}, \lambda_{\sigma_2}, \dots, \lambda_{\sigma_n}) X_{\lambda_1 + \lambda_2 + \dots + \lambda_n} = 0.$$

Consequently,

$$\sum_{\sigma \in S_n} \alpha_\sigma P_n(\lambda_{\sigma_1}, \lambda_{\sigma_2}, \dots, \lambda_{\sigma_n}) = 0$$

and

$$\sum_{\sigma \in S_n} \alpha_\sigma Q_n(\lambda_{\sigma_1}, \lambda_{\sigma_2}, \dots, \lambda_{\sigma_n}) = 0.$$

The last equation can be written as

$$\sum_{\sigma \in S_n, \sigma_n = n} \alpha_\sigma Q_n(\lambda_{\sigma_1}, \lambda_{\sigma_2}, \dots, \lambda_{\sigma_{(n-1)}}, \lambda_n) + \sum_{\sigma \in S_n, \sigma_n \neq n} \alpha_\sigma Q_n(\lambda_{\sigma_1}, \lambda_{\sigma_2}, \dots, \lambda_{\sigma_n}) = 0. \quad (6)$$

Thus, every system of positive integers $\lambda_1, \lambda_2, \dots, \lambda_n$ satisfies equation (6). Let's fix arbitrary positive integers $\lambda_1, \lambda_2, \dots, \lambda_{n-1}$ and consider this equation with respect to the variable λ_n . Multiplying the right hand side of the equation by common denominator, we get a polynomial with respect to λ_n . This polynomial is identically zero since it has infinite number of positive integer roots. Consequently, equation (6) holds for every $\lambda_n \in k$ if the right hand side just defined.

Note that the denominator of $Q_n(\lambda_{\sigma_1}, \lambda_{\sigma_2}, \dots, \lambda_{\sigma_n})$ is divisible by λ_n if and only if $\sigma_n = n$. Note also that by multiplying $Q_n(\lambda_1, \lambda_2, \dots, \lambda_n)$ by λ and substituting $\lambda_n = 0$ we obtain

$Q_{n-1}(\lambda_1, \lambda_2, \dots, \lambda_{n-1})$. Then, by multiplying equation (6) by λ_n and substituting $\lambda_n = 0$, we get

$$\begin{aligned} & \sum_{\sigma \in S_n, \sigma_n = n} \alpha_\sigma Q_{n-1}(\lambda_{\sigma_1}, \lambda_{\sigma_2}, \dots, \lambda_{\sigma_{(n-1)}}) = \\ & = \sum_{\delta \in S_{n-1}} \alpha_\delta Q_{n-1}(\lambda_{\delta_1}, \lambda_{\delta_2}, \dots, \lambda_{\delta_{(n-1)}}) = 0 \end{aligned}$$

for all positive integers $\lambda_1, \lambda_2, \dots, \lambda_{n-1}$. Consequently,

$$\sum_{\delta \in S_{n-1}} \alpha_\delta P_{n-1}(\lambda_{\delta_1}, \lambda_{\delta_2}, \dots, \lambda_{\delta_{(n-1)}}) = 0$$

and

$$\begin{aligned} & \psi\left(\sum_{\delta \in S_{n-1}} \alpha_\delta (x_{\delta_1}(x_{\delta_2} \dots (x_{\delta_{(n-2)}}x_{\delta_{(n-1)}}) \dots))\right) = \\ & = \sum_{\delta \in S_{n-1}} \alpha_\delta P_{n-1}(\lambda_{\delta_1}, \lambda_{\delta_2}, \dots, \lambda_{\delta_{(n-1)}}) X_{\lambda_1 + \lambda_2 + \dots + \lambda_{n-1}} = 0. \end{aligned}$$

this means that algebra $A = DL \langle x \rangle$ satisfies the identity

$$\sum_{\delta \in S_{n-1}} \alpha_\delta (x_{\delta_1}(x_{\delta_2} \dots (x_{\delta_{(n-2)}}x_{\delta_{(n-1)}}) \dots)) = 0.$$

This identity is nontrivial since $\alpha_1 = 1$. It contradicts the minimality of n .

Corollary 1. *The base rank of the variety of dual Leibniz algebras is equal to one.*

Proof. By Theorem 1, the variety of algebras $Var(DL \langle x \rangle)$ generated by the free dual Leibniz algebra $DL \langle x \rangle$ in the variable x is defined only by identity (1). This means that the base rank of the variety of dual Leibniz algebras equals one.

It is easy to see that free dual Leibniz algebras in more than one variable cannot be embedded into one generated free dual Leibniz algebras.

Theorem 2. *Any proper subvariety of dual Leibniz algebras in characteristic zero is nilpotent.*

Proof. Let \mathfrak{M} be an arbitrary proper subvariety of the variety of dual Leibniz algebras and $B = \mathfrak{M} \langle y \rangle$ be the free algebra of this variety in the variable y . Consider the homomorphism

$$\varphi : DL \langle x \rangle \longrightarrow B$$

such that $\varphi(x) = y$. If φ is isomorphism then $DL \langle x \rangle \in \mathfrak{M}$, which is impossible by Theorem 1. Consequently, $Ker(\varphi) \neq 0$. Then there exists a natural n such that $x^n \in Ker(\varphi)$ since φ is a homogeneous homomorphism of homogeneous algebras. This means that \mathfrak{M} satisfies the identity

$$y^n = 0,$$

i.e., the variety of algebras \mathfrak{M} has nil index n . It was proved in [2] that nil algebras of bounded nil-index are nilpotent. Consequently, the variety \mathfrak{M} is nilpotent.

Note that every nilpotent variety of algebras is Shpekhtian, i.e., its every subvariety has finite basis of identities.

Corollary 2. *The variety of dual Leibniz algebra in characteristic zero is Shpekhtian.*

It is well known that any finite dimensional algebra satisfies an analogue of the standard identity (see definition, for example in [8]). Then next corollary immediately follows from Theorem 2.

Corollary 3. *Every finite dimensional dual Leibniz algebra in characteristic zero is nilpotent.*

This result was proved also in [2].

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