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NON-POLYNOMIAL SPLINE APPROXIMATIONS FOR THE SOLUTION OF SINGULARLY-PERTURBED BOUNDARY VALUE PROBLEMS

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Abstract. We consider the self-adjoint singularly perturbed two-point boundary value problems. We know that the numerical methods for solution of such problems based on non-polynomial spline in grid points, can produce the fourth order method only. So that we look for an alternative to obtain higher order methods. We develop the non-polynomial spline in off-step points to rise the order of accuracy. Based on such spline, the purposed new methods are fourth, sixth and eighth-order accurate. These methods are applicable to problems both in singular and non-singular cases. The convergence analysis of the new eight-order method is proved. We applied the presented methods to test problems which have been solved by other existing methods in our references, for comparison of our methods with the existing methods. Numerical results are given to illustrate the efficiency of our methods.

Keywords: Self-adjoint singularly-perturbed boundary value problem, Non-polynomial spline, Convergence analysis, Numerical conclusion.

AMS Subject Classification: 34B15; 33F05; 65D20; 65L.

1. Introduction

We consider the following self-adjoint singularly perturbed boundary value problem:

$$-\epsilon u'' + p(x)u = q(x), \quad p(x) > 0,$$
$$u(a) = \eta_1, \quad u(b) = \eta_2, \tag{1}$$

where η_1, η_2 are given constants and ϵ is a small positive parameter such that $0 < \epsilon \leq 1$ and $p(x)$, $q(x)$ are small bounded real functions. This problems occur naturally in various fields of science and engineering, for example, combustion, nuclear engineering, control theory, elasticity, fluid mechanics, fluid dynamics, quantum mechanics, optimal control, chemical-reactor theory, hydrodynamics, convection-diffusion process, geophysics, etc. A few notable examples are boundary-layer problems, WKB Theory, the modeling of steady and unsteady viscous flow problems with large Reynolds number and convective heat transport problems with large Peclet number. Secondly, the occurrence of sharp boundary-layers as ϵ, the coefficient of highest derivative, approaches zero creates difficulty for most standard numerical schemes.

The application of splines for the numerical solution of singularly-perturbed boundary- value problems has been described in many papers [3]-[5], [7],[8],[11]-[16]. Recently Aziz and khan [1, 2] and Khan et. al. [6] solve this problem by cubic spline in comparison, quintic spline and sextic spline with assuming $p(x) = constant$. A generalized scheme based on quartic non-polynomial spline functions was proposed by Tirmizi et. al. [6].

In this paper non-polynomial sextic spline relations have been derived using off-step points. In Section 2, we derive the formulation of our spline function approximation. We present the

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spline relations to be used for discretization of the given problem (1). In section 3, we present the formulation of our method. We drive boundary formulas in section 4. In section 5, convergence of the new eight-order method is stablished. Finally, in section 6, numerical evidence is included to demonstrate the efficiency of the method.

2. Non-polynomial spline functions

We introduce the set of grid points in the interval $[a, b]$

$$x_0 = a, \;\; x_{i-\frac{1}{2}} = a + (i - \frac{1}{2})h, \;\; h = \frac{b-a}{N}, \;\; i = 1, 2, ..., N, \;\; x_N = b.$$

Definition. *A non-polynomial spline function $S_l(x)$, interpolating to a function $u(x)$ on $[a, b]$ defined as:*

(1) *In each subinterval $[x_l, x_{l+1}]$, $S_l(x)$ is a polynomial of degree at most six.*

(2) *The first-fifth derivatives of $S_l(x)$ are continuous on $[a, b]$.*

(3) $S_l(x_{l-\frac{1}{2}}) = u(x_{l-\frac{1}{2}})$, $l = 1(1)N$.

For each segment $[x_{l-\frac{1}{2}}, x_{l+\frac{1}{2}}]$, $l = 1, 2, ..., N-1$ the non-polynomial spline $S_l(x)$, is define as

$$S_l(x) = a_l + b_l(x - x_l) + c_l(x - x_l)^2 + d_l(x - x_l)^3 + e_l(x - x_l)^4 + \\ + f_l \sin \tau(x - x_l) + g_l \cos \tau(x - x_l), \quad l = 0, 1, 2, ..., N, \qquad (2)$$

where a_l, b_l, c_l, d_l, e_l, f_l and g_l are constants and τ is free parameter.

We further require that the values of the first-, second-, third-, fourth- and fifth-order derivatives be the same for the pair of segments that join at each point (x_l, u_l).

To derive expression for the coefficients of (2) in terms of $u_{l-\frac{1}{2}}$, $u_{l+\frac{1}{2}}$, $m_{l-\frac{1}{2}}$, $M_{l-\frac{1}{2}}$, $M_{l+\frac{1}{2}}$, $F_{l-\frac{1}{2}}$ and $F_{l+\frac{1}{2}}$ we first define:

$$\begin{array}{ll} (i) & S_l(x_{l-\frac{1}{2}}) = u_{l-\frac{1}{2}}, \\ (ii) & S_l(x_{l+\frac{1}{2}}) = u_{l+\frac{1}{2}}, \\ (iii) & S_l'(x_{l-\frac{1}{2}}) = m_{l-\frac{1}{2}}, \\ (iv) & S_l''(x_{l-\frac{1}{2}}) = M_{l-\frac{1}{2}}, \\ (v) & S_l''(x_{l+\frac{1}{2}}) = M_{l+\frac{1}{2}}, \\ (vi) & S_l^{(4)}(x_{l-\frac{1}{2}}) = F_{l-\frac{1}{2}}, \\ (vii) & S_l^{(4)}(x_{l+\frac{1}{2}}) = F_{l+\frac{1}{2}}. \end{array} \qquad (3)$$

From algebraic manipulation we get the following expression:

$$a_l = \frac{1}{96\tau^4\theta[-24 + 24\cos\theta + \theta(12 + \theta^2)\sin\theta]}\{2[72\theta(8 + \theta^2) + 3\theta(192 + 24\theta^2 + 5\theta^4)\cos\theta + \\ + (-1152 - 48\theta^2 - 27\theta^4 + 2\theta^6)\sin\theta]F_{l-\frac{1}{2}} - [6\theta(192 + 24\theta^2 + 5\theta^4) + 144\theta(8 + \theta^2)\cos\theta + \\ + (-2304 - 96\theta^2 + 42\theta^4 + \theta^6)\sin\theta]F_{l+\frac{1}{2}} + \tau^4\{6h\sin\theta(384 + 48\theta^2 + 5\theta^4)m_{l-\frac{1}{2}} + 4h^2[36\theta(1 - \\ - \cos\theta) + (192 + 6\theta^2 + \theta^4)\sin\theta]M_{l-\frac{1}{2}} + h^2[144\theta(1 - \cos\theta) + (384 - 24\theta^2 - \theta^4)\sin\theta]M_{l+\frac{1}{2}} + \\ + [-1152(1 - \cos\theta) + (2304 + 864\theta^2 + 78\theta^4)\sin\theta]u_{l-\frac{1}{2}} + [-1152(1 - \cos\theta) + (-2304 + 288\theta^2 + \\ + 18\theta^4)\sin\theta]u_{l+\frac{1}{2}}\}\},$$

$$b_l = \frac{1}{24\tau^3\theta}\{(24+\theta^2)(F_{l-\frac{1}{2}} - F_{l+\frac{1}{2}}) + \tau^4[h^2(M_{l-\frac{1}{2}} - M_{l+\frac{1}{2}}) + 24(u_{l+\frac{1}{2}} - u_{l-\frac{1}{2}})]\},$$

$$c_l = -\frac{1}{4\tau^2\theta[-24+24\cos\theta+\theta(12+\theta^2)\sin\theta]}\boldsymbol{\{}2[24\theta + 6\theta(4+\theta^2)\cos\theta + (-48-$$
$$-2\theta^2+\theta^4)\sin\theta]F_{l-\frac{1}{2}} - [6\theta(4+\theta^2) + 24\theta\cos\theta - (24+\theta^2)\sin\theta]F_{l+\frac{1}{2}} + \tau^3\{6\theta\sin\theta\times$$
$$\times(8+\theta^2)m_{l-\frac{1}{2}} + h[24(1-\cos\theta) + \theta(4+\theta^2)\sin\theta]M_{l-\frac{1}{2}} + h[24(1-\cos\theta)-$$
$$-4\theta\sin\theta]M_{l+\frac{1}{2}} + \tau[(48+6\theta^2)\sin\theta](u_{l-\frac{1}{2}} - u_{l+\frac{1}{2}})\}\boldsymbol{\}},$$

$$d_l = \frac{1}{6\tau h}[F_{l+\frac{1}{2}} - F_{l-\frac{1}{2}} + \tau^2(M_{l+\frac{1}{2}} - M_{l-\frac{1}{2}})],$$

$$e_l = \frac{1}{6\theta[-24(1-\cos\theta)+\theta(12+\theta^2)\sin\theta]}\boldsymbol{\{}2[3\theta\cos\theta + (-3+\theta^2)\sin\theta]F_{l-\frac{1}{2}} + [-6\theta+$$
$$+(6+\theta^2)\sin\theta]F_{l+\frac{1}{2}} + \tau^4\sin\theta[6hm_{l-\frac{1}{2}} + h^2(2M_{l-\frac{1}{2}} + M_{l+\frac{1}{2}}) + 6(u_{l-\frac{1}{2}} - u_{l+\frac{1}{2}})]\boldsymbol{\}},$$

$$f_l = \frac{1}{2\tau^4\sin\frac{\theta}{2}}(F_{i+\frac{1}{2}} - F_{l-\frac{1}{2}}),$$

and

$$g_l = \frac{1}{\tau^4\theta[-24(1-\cos\theta)+\theta(12+\theta^2)\sin\theta]}\boldsymbol{\{}[-24\theta\cos\frac{\theta}{2} + (48-4\theta^2+\theta^4)\sin\frac{\theta}{2}](F_{l-\frac{1}{2}}-$$
$$-F_{l+\frac{1}{2}}) - 8\tau^4\sin\frac{\theta}{2}[6hm_{l-\frac{1}{2}} + h^2(2M_{l-\frac{1}{2}} + M_{l+\frac{1}{2}}) + 6(u_{l-\frac{1}{2}} - u_{l+\frac{1}{2}})]\boldsymbol{\}}, \quad (4)$$

where $\theta = \tau h$ and $l = 1, 2, ..., N-1$.

From continuity of the first derivative at the point $x = x_l$, we can obtain the following spline relation:

$$m_{l-\frac{3}{2}} = -\frac{1}{24\tau^3\theta\sin\theta(-24\theta+\theta^3+48\sin\frac{\theta}{2})}\boldsymbol{\{} -3[192+8\theta^2+8(24+4\theta^2+\theta^4)\cos\frac{\theta}{2} - 8(24-$$
$$-23\theta^2+\theta^4)\cos\theta - 64(3-\theta^2)\cos\frac{3\theta}{2} - 3\theta(96-20\theta^2+\theta^4)\sin\theta - 192\theta\sin\frac{3\theta}{2}]F_{l-\frac{3}{2}} + 2[576+$$
$$+24\theta^2 - 12\theta^4 + 12(24+16\theta^2+\theta^4)\cos\frac{\theta}{2} + 24(-24+11\theta^2)\cos\theta - 12(24-8\theta^2-\theta^4)\cos\frac{3\theta}{2}-$$
$$-288\theta\sin\frac{\theta}{2} - \theta(576-72\theta^2-13\theta^4)\sin\theta - 288\theta\sin\frac{3\theta}{2}]F_{l-\frac{1}{2}} + [-576-24\theta^2-24\theta^2(12+\theta^2)\times$$
$$\times\cos\frac{\theta}{2} + 24(24+\theta^2)\cos\theta + 576\theta\sin\frac{\theta}{2} + \theta(288+36\theta^2+\theta^4)\sin\theta]F_{l+\frac{1}{2}}) + 6\tau^2\theta^2\sin\frac{\theta}{2}[3\theta\times$$
$$\times(-20+\theta^2)\cos\frac{\theta}{2} - 8(\sin\frac{\theta}{2} - 8\sin\theta)]M_{l-\frac{3}{2}} + 4\tau^2\theta^2\sin\frac{\theta}{2}[\theta(84+13\theta^2)\cos\frac{\theta}{2} - 264\sin\frac{\theta}{2}+$$
$$+48\sin\theta]M_{l-\frac{1}{2}} + \tau^2\theta^2[-24(1-\cos\theta)+\theta(12+\theta^2)\sin\theta]M_{l+\frac{1}{2}} + 576\tau^4\sin\frac{\theta}{2}[-3\theta\cos\frac{\theta}{2}+$$

$$+2(\sin\frac{\theta}{2}+\sin\theta)]u_{l-\frac{3}{2}}+48\tau^4\sin\frac{\theta}{2}[\theta(48+\theta^2)\cos\frac{\theta}{2}-24(2\sin\frac{\theta}{2}+\sin\theta)]u_{l-\frac{1}{2}}-$$
$$-24\tau^4[-24(1-\cos\theta)+\theta(12+\theta^2)\sin\theta]u_{l+\frac{1}{2}}\}. \quad (5)$$

Also continuity of the third derivative at the point $x=x_l$, implies that:

$$m_{l-\frac{3}{2}}=-\frac{1}{24\tau^3\theta\sin\theta(-\theta+2\sin\frac{\theta}{2})}\{-[24+(24+4\theta^2+\theta^4)\cos\frac{\theta}{2}+24(-1+\theta^2)\cos\theta-$$
$$-8(3-\theta^2)\cos\frac{3\theta}{2}-\theta(36-7\theta^2)\sin\theta-24\theta\sin\frac{3\theta}{2}]F_{l-\frac{3}{2}}+[48+24\theta^2+(24+$$
$$+16\theta^2+\theta^4)\cos\frac{\theta}{2}-48\cos\theta-(24-8\theta^2-\theta^4)\cos\frac{3\theta}{2}-24\theta\sin\frac{\theta}{2}-6\theta(8+\theta^2)\times$$
$$\times\sin\theta-24\theta\sin\frac{3\theta}{2}]F_{l-\frac{1}{2}}+[-24-\theta^2(12+\theta^2)\cos\frac{\theta}{2}+24\cos\theta+24\theta\sin\frac{\theta}{2}+\theta\times$$
$$\times(12+\theta^2)\sin\theta]F_{l+\frac{1}{2}}+2\tau^2\sin\frac{\theta}{2}[\theta(12-7\theta^2)\cos\frac{\theta}{2}-24\sin\frac{\theta}{2}+8\theta^2\sin\theta]M_{l-\frac{3}{2}}+$$
$$+4\tau^2\sin\frac{\theta}{2}[-3\theta(4+\theta^2)\cos\frac{\theta}{2}+24\sin\frac{\theta}{2}+2\theta^2\sin\theta]M_{l-\frac{1}{2}}+\tau^2[-24(1-\cos\theta)+\theta(12+\theta^2)\times$$
$$\times\sin\theta]M_{l+\frac{1}{2}}+24\tau^4(-\theta+2\sin\frac{\theta}{2})\sin\theta(u_{l-\frac{3}{2}}-u_{l-\frac{1}{2}})\}. \quad (6)$$

Finally, from continuity of the fifth derivative at the point $x=x_l$, we have the following spline relation:

$$m_{l-\frac{3}{2}}=-\frac{1}{96\tau^3\theta\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}}\{\{(24+4\theta^2+\theta^4)\cos\frac{\theta}{2}+8[(-3+\theta^2)\cos\frac{3\theta}{2}-3\theta\sin\frac{3\theta}{2}]\}\times$$
$$\times F_{l-\frac{3}{2}}-[(24+16\theta^2+\theta^4)\cos\frac{\theta}{2}+(-24+8\theta^2+\theta^4)\cos\frac{3\theta}{2}-96\theta\cos^2\frac{\theta}{2}\sin\frac{\theta}{2}]F_{l-\frac{1}{2}}+$$
$$+\theta[\theta(12+\theta^2)\cos\frac{\theta}{2}-24\sin\frac{\theta}{2}]F_{l+\frac{1}{2}}-16\tau^4\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}[h^2(2M_{l-\frac{3}{2}}+M_{l-\frac{1}{2}})+$$
$$+6(u_{l-\frac{3}{2}}-u_{l-\frac{1}{2}})]\}, \quad (7)$$

where $l=2(1)N-1$.

From Eqs. (5) and (6) we have

$$u_{l-\frac{3}{2}}-2u_{l-\frac{1}{2}}+u_{l+\frac{1}{2}}=\frac{1}{48\tau^4(1-\cos\theta)}\{[48-22\theta^2+\theta^4-2(24+\theta^2)\cos\theta]F_{l-\frac{3}{2}}+[-96+$$
$$+44\theta^2+(96+4\theta^2-2\theta^4)\cos\theta]F_{l-\frac{1}{2}}+[48-22\theta^2+\theta^4-2(24+\theta^2)\cos\theta]F_{l+\frac{1}{2}}+$$
$$+2\tau^2\theta^2(1-\cos\theta)(M_{l-\frac{3}{2}}-2M_{l-\frac{1}{2}}+M_{l+\frac{1}{2}})\}. \quad (8)$$

Also from Eqs. (5) and (7) we have

$$\theta^2\{[-2+\theta^2+2\cos\theta]F_{l-\frac{3}{2}}-2[-2+(2+\theta^2)\cos\theta]F_{l-\frac{1}{2}}+[-2+\theta^2+2\cos\theta]F_{l+\frac{1}{2}}-$$
$$-2\tau^2(1-\cos\theta)(M_{l-\frac{3}{2}}-2M_{l-\frac{1}{2}}+M_{l+\frac{1}{2}})\}=0. \quad (9)$$

From Eqs. (8) and (9) we obtain

$$h^4F_{l-\frac{1}{2}} = -\frac{1}{2[-12+5\theta^2+(12+\theta^2)\cos\theta]}\{h^2\{[24-12\theta^2+\theta^4-24\cos\theta](M_{l-\frac{3}{2}}+M_{l+\frac{1}{2}})+$$
$$+2[-24+5\theta^4+\theta^4+12(2+\theta^2)\cos\theta]M_{l-\frac{1}{2}}+12\theta^2[2-\theta^2-2\cos\theta](u_{l-\frac{3}{2}}-2u_{l-\frac{1}{2}}+u_{l+\frac{1}{2}})\}. \quad (10)$$

The elimination of F_l's from (9) and (10) gives

$$\frac{1}{h^2}[u_{i-\frac{5}{2}}+\beta_1(u_{i-\frac{3}{2}}+u_{l+\frac{1}{2}})+\gamma_1 u_{i-\frac{1}{2}}+u_{i+\frac{3}{2}}] = \alpha(M_{i-\frac{5}{2}}+M_{i+\frac{3}{2}})+\beta(M_{i-\frac{3}{2}}+$$
$$+M_{i+\frac{1}{2}})+\gamma M_{i-\frac{1}{2}}, \qquad i=3(1)N-2, \quad (11)$$

where

$$\alpha = \frac{24(1-\cos\theta)+\theta^2(-12+\theta^2)}{12\theta^2(-2+\theta^2+2\cos\theta)},$$
$$\beta = \frac{-48+\theta^2(12+5\theta^2)+(48+12\theta^2+\theta^4)\cos\theta}{6\theta^2(-2+\theta^2+2\cos\theta)},$$
$$\gamma = \frac{72+\theta^2(-12+\theta^2)-2(36+12\theta^2+5\theta^4)\cos\theta}{6\theta^2(-2+\theta^2+2\cos\theta)},$$
$$\beta_1 = \frac{-2[-4+\theta^2+(4+\theta^2)\cos\theta]}{-2+\theta^2+2\cos\theta},$$
$$\gamma_1 = \frac{2[-6+\theta^2+2(3+\theta^2)\cos\theta]}{-2+\theta^2+2\cos\theta}. \quad (12)$$

When $\tau \to 0$, that $\theta \to 0$, then $(\alpha,\beta,\gamma) \to \frac{1}{30}(1,56,246)$, $(\beta_1,\gamma_1) \to (8,-18)$ and the relations defined by (11) and (12) reduce into sextic polynomial spline functions.

3. Description of methods

Now we consider the equation (1). At the grid points x_i, the proposed differential equation (1) may be discretized by

$$-\epsilon u_i'' + p_i u_i = q_i, \quad (13)$$

where $p_i = p(x_i)$ and $q_i = q(x_i)$.

By using moment of spline in (13) we obtain

$$-\epsilon M_i + p_i u_i = q_i. \quad (14)$$

We discretize the given system (14) at the grid points $x_{i-\frac{1}{2}}$, $i = 1,2,...,N$ and using the spline relation (11). We obtain $(N-4)$ linear algebraic equation in the (N) unknowns $u_{i-\frac{1}{2}}$, $i = 1,2,...,N$ as

$$(-\epsilon+\alpha h^2 p_{i-\frac{5}{2}})u_{i-\frac{5}{2}}+(-\epsilon\beta_1+\beta h^2 p_{i-\frac{3}{2}})u_{i-\frac{3}{2}}+(-\epsilon\gamma_1+\gamma p_{i-\frac{1}{2}})h^4 u_{i-\frac{1}{2}}+$$
$$+(-\epsilon\beta_1+\beta h^2 p_{i+\frac{1}{2}})u_{i+\frac{1}{2}}+(-\epsilon+\alpha h^2 p_{i+\frac{3}{2}})u_{i+\frac{3}{2}} = h^2(\alpha q_{i-\frac{5}{2}}+\beta q_{i-\frac{3}{2}}+\gamma q_{i-\frac{1}{2}}+$$
$$+\beta q_{i+\frac{1}{2}}+\alpha q_{i+\frac{3}{2}}), \qquad i=3(1)N-2, \quad (15)$$

where $p_i = p(x_i)$ and $q_i = q(x_i)$, with local truncation error

$$T_i(h) = -\epsilon(2+2\beta_1+\gamma_1)u_i+\frac{\epsilon}{2}(2+2\beta_1+\gamma_1)hu_i'+\frac{\epsilon}{8}(16\alpha+16\beta+8\gamma-34-10\beta_1-$$
$$-\gamma_1)h^2u_i''-\frac{\epsilon}{48}(48\alpha+48\beta+24\gamma-98-26\beta_1-\gamma_1)h^3u_i'''+\frac{\epsilon}{384}(1632\alpha+480\beta+$$
$$+48\gamma-706-82\beta_1-\gamma_1)h^4u_i^{(4)}-\frac{\epsilon}{3840}(7840\alpha+2080\beta+80\gamma-2882-242\beta_1-$$

$$-\gamma_1)h^5u_i^{(5)} + \frac{\epsilon}{46080}(84720\alpha + 9840\beta + 120\gamma - 16354 - 730\beta_1 - \gamma_1)h^6u_i^{(6)} +$$
$$+\frac{\epsilon}{645120}(484176\alpha + 40656\beta + 168\gamma - 75938 - 2186\beta_1 - \gamma_1)h^7u_i^{(7)} - \frac{\epsilon}{10321920}\times$$
$$\times(3663296\alpha + 163520\beta + 224\gamma - 397186 - 6562\beta_1 - \gamma_1)h^8u_i^{(8)} - \frac{\epsilon}{185794560}\times$$
$$\times(21870144\alpha + 629568\beta + 288\gamma - 1933442 - 19682\beta_1 - \gamma_1)h^9u_i^{(9)} + \frac{\epsilon}{3715891200}\times$$
$$\times(142986960\alpha + 2362320\beta + 360\gamma - 9824674 - 59050\beta_1 - \gamma_1)h^{10}u_i^{(10)} + O(h^{11}),$$
$$\text{where} \quad u_i^{(k)} = u^{(k)}(\xi_i) \quad x_{i-1} < \xi_i < x_{i+1}. \tag{16}$$

By choosing suitable values of parameters β_1, γ_1, α, β and γ we obtain the following methods:

(1) If we choose $2+2\beta_1+\gamma_1 = 0$ *and* $2\alpha+2\beta+\gamma-4-\beta_1 = 0$ *in (15) we obtain second-order schemes for the solution of (1).*

(2) If we choose $(\beta_1, \gamma_1, \alpha, \beta, \gamma) = (4, -10, \frac{1}{18}, \frac{13}{9}, 5)$ *in (15) we obtain a fourth-order scheme with truncation error* $T_i = \frac{\epsilon}{180}h^6u^{(6)}(\xi_i)$.

(3) If we choose $(\beta_1, \gamma_1, \alpha, \beta, \gamma) = (4, -10, \frac{1}{20}, \frac{22}{15}, \frac{149}{30})$ *in (15) we obtain a sixth-order scheme with truncation error* $T_i = \frac{\epsilon}{15120}h^8u^{(8)}(\xi_i)$.

(4) If we choose $(\beta_1, \gamma_1, \alpha, \beta, \gamma) = (\frac{128}{31}, -\frac{318}{31}, \frac{23}{465}, \frac{688}{465}, \frac{786}{155})$ *in (15) we obtain a eight-order scheme with truncation error* $T_i = \frac{79\epsilon}{585900}h^{10}u^{(10)}(\xi_i)$, *which is the the highest order among the methods for solution of (1).*

4. Development of boundary formula

The relation (15) gives $N-4$ linear algebraic equations in the n unknowns $u_{i-\frac{1}{2}}$, for $i = 1, 2, ..., n$. We need four more equations, two at each end of interval, for the direct computation of $u_{i-\frac{1}{2}}$.

By using Taylor series and method of undetermined coefficients the boundary formulas associated with boundary conditions can be determined as follows. We drive the following equations to be associated with fourth-order method as:

$$-10u_0 + 14u_{\frac{1}{2}} - 3u_{\frac{3}{2}} - u_{\frac{5}{2}} = \frac{h^2}{48}[2M_0 - 154M_{\frac{1}{2}} - 75M_{\frac{3}{2}} - M_{\frac{5}{2}}] + \frac{221}{15360}h^6u^{(6)}(x_0) +$$
$$+O(h^7), \quad i = 1, \tag{17}$$

$$-2u_0 - 3u_{\frac{1}{2}} + 10u_{\frac{3}{2}} - 4u_{\frac{5}{2}} - u_{\frac{7}{2}} = \frac{h^2}{48}[10M_0 - 75M_{\frac{1}{2}} - 230M_{\frac{3}{2}} - 76M_{\frac{5}{2}} - M_{\frac{7}{2}}] +$$
$$+\frac{449}{15360}h^6u^{(6)}(x_0) + O(h^7), \quad i = 2, \tag{18}$$

and for right boundary we obtain:

$$-2u_N - 3u_{N-\frac{1}{2}} + 10u_{N-\frac{3}{2}} - 4u_{N-\frac{5}{2}} - u_{N-\frac{7}{2}} = \frac{h^2}{48}[10M_N - 75M_{N-\frac{1}{2}} - 230M_{N-\frac{3}{2}} -$$
$$-76M_{N-\frac{5}{2}} - M_{N-\frac{7}{2}}] + \frac{449}{15360}h^6u^{(6)}(x_N) + O(h^7), \quad i = N-1, \tag{19}$$

$$-10u_N+14u_{N-\frac{1}{2}}-3u_{N-\frac{3}{2}}-u_{N-\frac{5}{2}}=\frac{h^2}{48}[2M_N-154M_{N-\frac{1}{2}}-75M_{N-\frac{3}{2}}-M_{N-\frac{5}{2}}]+$$
$$+\frac{221}{15360}h^6u^{(6)}(x_N)+O(h^7),\quad i=N. \tag{20}$$

In same manner for sixth-order method we derive the following boundary formula:

$$-10u_0+14u_{\frac{1}{2}}-3u_{\frac{3}{2}}-u_{\frac{5}{2}}=\frac{h^2}{13440}[-608M_0-40439M_{\frac{1}{2}}-24059M_{\frac{3}{2}}+2009M_{\frac{5}{2}}-869M_{\frac{7}{2}}+$$
$$+126M_{\frac{9}{2}}]-\frac{15767}{3096576}h^8u^{(8)}(x_0)+O(h^9),\quad i=1, \tag{21}$$

$$-2u_0-3u_{\frac{1}{2}}+10u_{\frac{3}{2}}-4u_{\frac{5}{2}}-u_{\frac{7}{2}}=\frac{h^2}{120960}[10784M_0-157059M_{\frac{1}{2}}-612759M_{\frac{3}{2}}-170163M_{\frac{5}{2}}-$$
$$-8649M_{\frac{7}{2}}+406M_{\frac{9}{2}}]-\frac{38551}{15482880}h^8u^{(8)}(x_0)+O(h^9),\quad i=2, \tag{22}$$

$$-2u_N-3u_{N-\frac{1}{2}}+10u_{N-\frac{3}{2}}-4u_{N-\frac{5}{2}}-u_{N-\frac{7}{2}}=\frac{h^2}{120960}[10784M_N-157059M_{N-\frac{1}{2}}-$$
$$-612759M_{N-\frac{3}{2}}-170163M_{N-\frac{5}{2}}-8649M_{N-\frac{7}{2}}+406M_{N-\frac{9}{2}}]-\frac{38551}{15482880}h^8u^{(8)}(x_N)+$$
$$+O(h^9),\quad i=N-1, \tag{23}$$

$$-10u_N+14u_{N-\frac{1}{2}}-3u_{N-\frac{3}{2}}-u_{N-\frac{5}{2}}=\frac{h^2}{13440}[-608M_N-40439M_{N-\frac{1}{2}}-24059M_{N-\frac{3}{2}}+$$
$$+2009M_{N-\frac{5}{2}}-869M_{N-\frac{7}{2}}+126M_{N-\frac{9}{2}}]-\frac{15767}{3096576}h^8u^{(8)}(x_N)+$$
$$+O(h^9),\quad i=N. \tag{24}$$

Finally for eight order method we develop the following boundary equation:

$$-10u_0+14u_{\frac{1}{2}}-3u_{\frac{3}{2}}-u_{\frac{5}{2}}=h^2[-\frac{108205}{1297296}M_0-\frac{561643}{193536}M_{\frac{1}{2}}-$$
$$-\frac{7677301}{3870720}M_{\frac{3}{2}}+\frac{175867}{430080}M_{\frac{5}{2}}-\frac{547021}{1935360}M_{\frac{7}{2}}+\frac{694187}{5806080}M_{\frac{9}{2}}-\frac{144323}{4730880}M_{\frac{11}{2}}+$$
$$+\frac{177067}{50319360}M_{\frac{13}{2}}]+\frac{16404163}{7431782400}h^{10}u^{(10)}(x_0)+O(h^{11}),\quad i=1, \tag{25}$$

$$-2u_0-3u_{\frac{1}{2}}+10u_{\frac{3}{2}}-4u_{\frac{5}{2}}-u_{\frac{7}{2}}=h^2[\frac{440147}{6486480}M_0-\frac{399481}{322560}M_{\frac{1}{2}}-$$
$$-\frac{20033969}{3870720}M_{\frac{3}{2}}-\frac{1621507}{129240}M_{\frac{5}{2}}-\frac{14347}{71680}M_{\frac{7}{2}}+\frac{407023}{5806080}M_{\frac{9}{2}}-\frac{271477}{14192640}M_{\frac{11}{2}}+$$
$$+\frac{38821}{16773120}M_{\frac{13}{2}}]+\frac{13124647}{7431782400}h^{10}u^{(10)}(x_0)+O(h^{11}),\quad i=2, \tag{26}$$

$$-2u_N-3u_{N-\frac{1}{2}}+10u_{N-\frac{3}{2}}-4u_{N-\frac{5}{2}}-u_{N-\frac{7}{2}}=h^2[\frac{440147}{6486480}M_N-$$
$$-\frac{399481}{322560}M_{N-\frac{1}{2}}-\frac{20033969}{3870720}M_{N-\frac{3}{2}}-\frac{1621507}{1290240}M_{N-\frac{5}{2}}-\frac{14347}{71680}M_{N-\frac{7}{2}}+$$
$$+\frac{407023}{5806080}M_{N-\frac{9}{2}}-\frac{271477}{14192640}M_{N-\frac{11}{2}}+\frac{38821}{16773120}M_{N-\frac{13}{2}}]+$$

$$+\frac{13124647}{7431782400}h^{10}u^{(10)}(x_N)+O(h^{11}), \quad i=N-1, \tag{27}$$

$$-10u_N+14u_{N-\frac{1}{2}}-3u_{N-\frac{3}{2}}-u_{N-\frac{5}{2}}=h^2[-\frac{108205}{1297296}M_N-$$

$$-\frac{561643}{193536}M_{N-\frac{1}{2}}-\frac{7677301}{3870720}M_{N-\frac{3}{2}}+\frac{175867}{430080}M_{N-\frac{5}{2}}-\frac{547021}{1935360}M_{N-\frac{7}{2}}+$$

$$+\frac{694187}{5806080}M_{N-\frac{9}{2}}-\frac{144323}{4730880}M_{N-\frac{11}{2}}+\frac{177067}{50319360}M_{N-\frac{13}{2}}]+$$

$$+\frac{16404163}{7431782400}h^{10}u^{(10)}(x_N)+O(h^{11}), \quad i=N. \tag{28}$$

5. Convergence analysis

In this section, we investigate the convergence analysis of our eight order method. The eight order quintic spline solution of (1) is based on the linear equations given by (15) and boundary formulas (25)-(28). Let $Z=(z_i)$ be N dimensional column vector and $A=(a_{ij})$ be an $(N\times N)$ matrix such that

$$z_1=(10\epsilon-\tfrac{108205}{1297296}h^2p_0)\eta_1+h^2[\tfrac{108205}{1297296}q_0+\tfrac{561643}{193536}q_{\frac{1}{2}}+\tfrac{7677301}{3870720}q_{\frac{3}{2}}-\tfrac{175867}{430080}q_{\frac{5}{2}}+$$
$$+\tfrac{547021}{1935360}q_{\frac{7}{2}}-\tfrac{694187}{5806080}q_{\frac{9}{2}}+\tfrac{144323}{4730880}q_{\frac{11}{2}}-\tfrac{177067}{50319360}q_{\frac{13}{2}}],$$
$$z_2=(2\epsilon+\tfrac{440147}{6486480}h^2p_0)\eta_1+h^2[-\tfrac{440147}{6486480}q_0+\tfrac{399481}{322560}q_{\frac{1}{2}}+\tfrac{20033969}{3870720}q_{\frac{3}{2}}+\tfrac{1621507}{129240}q_{\frac{5}{2}}+$$
$$+\tfrac{14347}{71680}q_{\frac{7}{2}}-\tfrac{407023}{5806080}q_{\frac{9}{2}}+\tfrac{271477}{14192640}q_{\frac{11}{2}}-\tfrac{38821}{16773120}q_{\frac{13}{2}}],$$
$$z_i=\tfrac{h^2}{465}(23q_{i-\frac{5}{2}}+688q_{i-\frac{3}{2}}+2358q_{i-\frac{1}{2}}+688q_{i+\frac{1}{2}}+23q_{i+\frac{3}{2}}), \quad i=3(1)N-2,$$
$$z_{N-1}=(2\epsilon+\tfrac{440147}{6486480}h^2p_N)\eta_2+h^2[-\tfrac{440147}{6486480}q_N+\tfrac{399481}{322560}q_{N-\frac{1}{2}}+\tfrac{20033969}{3870720}q_{N-\frac{3}{2}}+$$
$$+\tfrac{1621507}{129240}q_{N-\frac{5}{2}}+\tfrac{14347}{71680}q_{N-\frac{7}{2}}-\tfrac{407023}{5806080}q_{N-\frac{9}{2}}+\tfrac{271477}{14192640}q_{N-\frac{11}{2}}-\tfrac{38821}{16773120}q_{N-\frac{13}{2}}],$$
$$z_N=(10\epsilon-\tfrac{108205}{1297296}h^2p_N)\eta_2+h^2[\tfrac{108205}{1297296}q_N+\tfrac{561643}{193536}q_{N-\frac{1}{2}}+\tfrac{7677301}{3870720}q_{N-\frac{3}{2}}-$$
$$-\tfrac{175867}{430080}q_{N-\frac{5}{2}}+\tfrac{547021}{1935360}q_{N-\frac{7}{2}}-\tfrac{694187}{5806080}q_{N-\frac{9}{2}}+\tfrac{144323}{4730880}q_{N-\frac{11}{2}}-\tfrac{177067}{50319360}q_{N-\frac{13}{2}}],$$

and

$$a_{i,i\pm k}=\text{coefficient of } \ u_{i-k},\ k=\pm\tfrac{3}{2},\pm\tfrac{1}{2},\tfrac{5}{2}.$$

It is easily seen that the system of N linear algebraic equations given by (15) and boundary formulas (25)-(28) may be written in the form

$$\begin{aligned}&(i)\quad \mathbf{A}\overline{\mathbf{U}}=Z+\mathbf{T},\\ &(ii)\quad \mathbf{A}\mathbf{U}=Z.\end{aligned} \tag{29}$$

The vector T is the local truncation error vector and define as

$$t_i = \begin{cases} \frac{16404163}{7431782400}\epsilon h^{10}u^{(10)}(x_0) + O(h^{11}), & & i = 1, \\ \frac{13124647}{7431782400}\epsilon h^{10}u^{(10)}(x_0) + O(h^{11}), & & i = 2, \\ \frac{79}{585900}\epsilon h^{10}u^{(10)}(x_i) + O(h^{11}), & x_{i-\frac{5}{2}} < x_i < x_{i+\frac{3}{2}} & i = 3, ..., N-2, \\ \frac{13124647}{7431782400}\epsilon h^{10}u^{(10)}(x_N) + O(h^{11}), & & i = N-1, \\ \frac{16404163}{7431782400}\epsilon h^{10}u^{(10)}(x_N) + O(h^{11}), & & i = N. \end{cases} \tag{30}$$

The error equation is obtained in the usal manner in the form

$$\mathbf{AE} = \mathbf{T}, \tag{31}$$

where $E = (e_i)$, with $e_i = (\overline{u_i} - u_i)$ and $T = (t_i)$ are the N dimentional vectors.

Now the matrix A may be written as

$$\mathbf{A} = \epsilon(\mathbf{A}_0 + \mathbf{A}_1) + h^2\mathbf{BP}, \tag{32}$$

where

$$\mathbf{A}_0 = \begin{bmatrix} 10 & -5 & 1 & 0 & . & . & . & . & 0 \\ -5 & 6 & -4 & 1 & 0 & . & . & . & . \\ 1 & -4 & 6 & -4 & 1 & 0 & . & . & . \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & . & . \\ . & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & . \\ . & . & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ . & . & . & 0 & 1 & -4 & 6 & -4 & 1 \\ . & . & . & . & 0 & 1 & -4 & 6 & -5 \\ 0 & .. & . & . & . & 0 & 1 & -5 & 10 \end{bmatrix}, \tag{33}$$

$$\mathbf{A}_1 = \begin{bmatrix} 4 & 2 & -2 & 0 & . & . & . & . & 0 \\ 2 & 4 & 0 & -2 & 0 & . & . & . & . \\ -2 & -\frac{4}{31} & \frac{132}{31} & -\frac{4}{31} & -2 & 0 & . & . & . \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & . & . \\ . & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & . \\ . & & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ . & . & . & 0 & -2 & -\frac{4}{31} & \frac{132}{31} & -\frac{4}{31} & -2 \\ . & . & . & . & 0 & -2 & 0 & 4 & 2 \\ 0 & . & . & . & . & 0 & -2 & 2 & 4 \end{bmatrix}, \tag{34}$$

$$\mathbf{B}=\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} & b_{17} & 0 & 0 \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} & b_{26} & 0 & 0 \\ \frac{23}{465} & \frac{688}{465} & \frac{786}{155} & \frac{688}{465} & \frac{23}{465} & 0 & . & . & . \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & . & . \\ . & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & . \\ . & . & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ . & . & . & 0 & \frac{23}{465} & \frac{688}{465} & \frac{786}{155} & \frac{688}{465} & \frac{23}{465} \\ 0 & 0 & b_{N-11} & b_{N-12} & b_{N-13} & b_{N-14} & b_{N-15} & b_{N-16} & b_{N-17} \\ 0 & 0 & b_{N1} & b_{N2} & b_{N3} & b_{N4} & b_{N5} & b_{N6} & b_{N7} \end{bmatrix} \tag{35}$$

with

$(b_{11}, b_{12}, b_{13}, b_{14}, b_{15}, b_{16}, b_{16}) = (b_{N7}, b_{N6}, b_{N5}, b_{N4}, b_{N3}, b_{N2}, b_{N1}) = (\frac{561643}{193536},$

$\frac{7677301}{3870720}, -\frac{175867}{430080}, \frac{547021}{1935360}, -\frac{694187}{5806080}, \frac{144323}{4730880}, -\frac{177067}{50319360}),$

$(b_{21}, b_{22}, b_{23}, b_{24}, b_{25}, b_{26}, b_{27}) = (b_{N-17}, b_{N-16}, b_{N-15}, b_{N-14}, b_{N-13}, b_{N-12},$

$b_{N-11}) = (\frac{399481}{322560}, \frac{20033969}{3870720}, \frac{1621507}{1290240}, \frac{14347}{71680}, -\frac{407023}{5806080}, \frac{271477}{14192640}, -\frac{38821}{16773120})$

and $P = diag(p_i)$.

Theorem 1. *The symmetric matrix A_0 os irreducible and monotone and*

$$\|A_0^{-1}\|_\infty \le \frac{1}{384h^4}[5(b-a)^4 + 10(b-a)^2h^2 + 9h^4]. \tag{36}$$

This theorem proved by Usmani [17].

Theorem 2. *Let u be the solution of singularly perturbed boundary value problem (1), and u_i be the numerical solution obtained from the difference scheme (29)(ii). Then, we have*

$$\|E\|_\infty = O(h^8)$$

provided $\|p\| < \frac{\epsilon}{252\omega h^2}(31 - 264\omega)$ where $\omega = \frac{1}{384h^4}[5(b-a)^4 + 10(b-a)^2h^2 + 9h^4]$ and $\|p\| = Max|p(x_i)|$, $a \le p(x_i) \le b$.

Proof. From (31) we have

$$\|E\| = \|A^{-1}T\| \le \|A^{-1}\|_\infty \|T\| \le$$

$$\le \|[\epsilon(A_0 + A_1) + h^2BP]^{-1}\|\|T\| \le$$

$$\le \epsilon^{-1}\|I + A_0^{-1}A_1 + \epsilon^{-1}h^2A_0^{-1}BP]^{-1}\|\|A_0^{-1}\|\|T\|.$$

From (32), (34) and (35) we have

$$\|A_1 + \epsilon^{-1}h^2A_0^{-1}BP\| \le \frac{12}{31}(22 + 21\epsilon^{-1}h^2\|p\|). \tag{37}$$

Also from (30) we have

$$\|T\| \le \frac{16404163}{7431782400}\epsilon h^{10}L_{10},$$

where $L_{10} = Max|u^{(10)}(x_i)|$, $a \le x_i \le b$.

By using $\|(I+W)^{-1}\| \leq \frac{1}{1-\|W\|}$ provided $\|W\| < 1$, we get

$$\|E\| \leq \frac{\epsilon^{-1}\|A_0^{-1}\|\|T\|}{1-\|A_0^{-1}A_1+\epsilon^{-1}h^2A_0^{-1}BP]^{-1}\|} \leq$$

$$\leq \frac{\epsilon^{-1}\|A_0^{-1}\|\|T\|}{1-\|A_0^{-1}A_1+\epsilon^{-1}h^2A_0^{-1}BP\|},$$

provided that $\|A_0^{-1}A_1 + \epsilon^{-1}h^2A_0^{-1}BP\| < 1$.

Therefore we obtain

$$\|E\| \leq \frac{508529053\omega h^{10}L_{10}}{7431782400[31-12\omega(22+21\epsilon^{-1}h^2\|p\|)]} \equiv O(h^8), \quad (38)$$

provided

$$\|p\| < \frac{\epsilon}{252\omega h^2}(31-264\omega). \quad (39)$$

So that the given method for solving the boundary value problem (1) for sufficiently small h is a eight-order convergent method.

6. Numerical illustrations

In order to test the viability of the proposed methods and to demonstrate their convergence computationally we have solved the following singularly perturbed boundary value problems which their exact solutions are known to us.

The maximum absolute errors at the nodal points, $\max|u(x_i) - u_i|$ are tabulated in Tables.

Example 1. Consider the following singularly perturbed boundary value problem [2],[5],[6], [13]-[17]:

$$-\epsilon u'' + u = -\cos^2 \pi x - 2\epsilon \cos 2\pi x,$$
$$u(0) = u(1) = 0$$

with the exact solution

$$u(x) = [exp((x-1)/\sqrt{\epsilon}) + exp(-x/\sqrt{\epsilon})]/[1 + exp(-1/\sqrt{\epsilon})] - \cos^2 \pi x.$$

This problem has been solved using our methods with different values of $N = 16, 32, 64, 128, 256$ and $\epsilon = \frac{1}{16}, ..., \frac{1}{128}$, computed solutions are compared with the exact solutions in nodal points and the maximum absolute errors in solutions are tabulated in Table 1, and compared with the methods in Ref. [2],[5],[6], [13]-[17] in Tables 1-2. Our results shows the efficiency and accuracy of our methods.

ϵ	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$
Our eight order method					
$\frac{1}{16}$	6.91(−9)	2.04(−11)	6.86(−14)	2.51(−15)	3.07(−16)
$\frac{1}{32}$	8.48(−9)	1.57(−10)	3.71(−14)	1.44(−15)	3.08(−16)
$\frac{1}{64}$	5.86(−8)	1.36(−9)	1.92(−13)	3.44(−15)	1.27(−15)
$\frac{1}{128}$	8.35(−7)	2.84(−9)	4.97(−12)	1.36(−14)	1.06(−14)
Our sixth order method					
$\frac{1}{16}$	1.18(−7)	6.74(−10)	4.25(−12)	6.15(−14)	1.00(−14)
$\frac{1}{32}$	4.08(−8)	2.18(−10)	3.00(−12)	4.64(−14)	1.04(−14)
$\frac{1}{64}$	5.84(−7)	4.92(−9)	2.81(−11)	1.27(−13)	1.68(−14)
$\frac{1}{128}$	6.47(−6)	6.39(−8)	4.24(−10)	2.09(−12)	3.01(−14)
Our fourth order method					
$\frac{1}{16}$	1.83(−4)	4.44(−5)	7.13(−6)	9.97(−7)	1.31(−7)
$\frac{1}{32}$	5.84(−4)	1.30(−4)	2.07(−5)	2.89(−6)	3.82(−7)
$\frac{1}{64}$	1.50(−3)	3.47(−4)	5.69(−5)	8.09(−6)	1.07(−6)
$\frac{1}{128}$	3.37(−3)	8.67(−4)	1.50(−4)	2.21(−5)	3.00(−6)
Sixth order method in [16]					
$\frac{1}{16}$	1.19(−7)	6.74(−10)	4.26(−12)	6.18(−14)	2.09(−14)
$\frac{1}{32}$	4.08(−8)	2.19(−10)	3.00(−12)	4.61(−14)	9.60(−15)
$\frac{1}{64}$	5.84(−7)	4.92(−9)	2.81(−11)	1.27(−13)	1.62(−14)
$\frac{1}{128}$	6.48(−6)	6.41(−8)	4.24(−10)	2.10(−12)	2.84(−14)
Fifth order method in [14]					
$\frac{1}{16}$	1.22(−6)	6.45(−9)	3.40(−11)	1.03(−12)	8.88(−15)
$\frac{1}{32}$	2.00(−6)	1.68(−8)	1.36(−10)	1.09(−12)	1.90(−14)
$\frac{1}{64}$	8.89(−6)	1.16(−7)	1.20(−9)	1.08(−11)	9.07(−14)
$\frac{1}{128}$	5.74(−5)	9.98(−7)	1.18(−8)	1.14(−10)	9.91(−13)

TABLE 1. The maximum absolute errors for example 1.

Example 2. Consider following boundary value problem [13]:

$$-\epsilon u'' + u = x,$$
$$u(0) = 1, \qquad u(1) = 1 + \exp(-1/\sqrt{\epsilon})$$

with the exact solution, $u(x) = x + \exp(-x/\sqrt{\epsilon})$.

This problem has been solved using our method with different values of $N = 8, 16, 32, 64, 128$ and $\epsilon = \frac{1}{16}, ..., \frac{1}{512}$. The maximum absolute errors in solutions are tabulated in Table 3 and compared with the methods in Ref. [13].

ϵ	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$
Fourth order method in [12]					
$\frac{1}{16}$	1.57(−5)	8.79(−7)	5.32(−8)	3.30(−9)	2.05(−10)
$\frac{1}{32}$	8.27(−6)	4.41(−7)	2.62(−8)	1.62(−9)	1.00(−10)
$\frac{1}{64}$	1.84(−5)	8.67(−7)	6.65(−8)	4.39(−9)	2.78(−10)
$\frac{1}{128}$	1.03(−4)	2.61(−6)	2.23(−7)	1.54(−8)	9.44(−10)
Fourth order method in [13]					
$\frac{1}{16}$	4.07(−5)	2.53(−6)	1.58(−7)	9.87(−9)	6.17(−10)
$\frac{1}{32}$	2.00(−5)	1.24(−6)	7.74(−8)	4.83(−9)	3.02(−10)
$\frac{1}{64}$	5.45(−5)	3.42(−6)	2.14(−7)	1.34(−8)	8.39(−10)
$\frac{1}{128}$	1.83(−4)	1.22(−5)	7.68(−7)	4.81(−8)	3.01(−9)
The method in [4]					
$\frac{1}{16}$	8.06(−3)	2.02(−3)	5.08(−4)	1.27(−4)	3.17(−5)
$\frac{1}{32}$	7.11(−3)	1.79(−3)	4.48(−4)	1.12(−4)	2.80(−5)
$\frac{1}{64}$	6.58(−3)	1.66(−3)	4.15(−4)	1.04(−4)	2.60(−5)
$\frac{1}{128}$	6.36(−3)	1.61(−3)	4.03(−4)	1.01(−4)	2.52(−5)
The method in [5]					
$\frac{1}{16}$	4.14(−3)	1.02(−3)	2.54(−4)	6.35(−5)	1.58(−5)
$\frac{1}{32}$	3.68(−3)	9.03(−4)	5.61(−5)	1.40(−5)	3.50(−6)
$\frac{1}{64}$	3.45(−3)	8.40(−4)	2.08(−4)	5.20(−5)	1.30(−5)
$\frac{1}{128}$	3.43(−3)	8.21(−4)	2.03(−4)	5.06(−5)	1.26(−5)
The method in [6]					
$\frac{1}{16}$	1.20(−4)	7.47(−6)	4.67(−7)	2.90(−8)	4.39(−9)
$\frac{1}{32}$	1.28(−4)	8.00(−6)	5.00(−7)	3.14(−8)	1.99(−9)
$\frac{1}{64}$	1.60(−4)	1.00(−5)	6.26(−7)	3.92(−8)	2.31(−9)
$\frac{1}{128}$	2.34(−4)	1.47(−5)	9.23(−7)	5.77(−8)	3.72(−9)
The method in [7]					
$\frac{1}{16}$	7.09(−3)	1.77(−3)	4.45(−4)	1.11(−4)	2.78(−5)
$\frac{1}{32}$	5.68(−3)	1.42(−3)	3.55(−4)	8.89(−5)	2.22(−5)
$\frac{1}{64}$	4.07(−3)	1.01(−3)	2.54(−4)	6.35(−5)	1.58(−5)
$\frac{1}{128}$	6.97(−3)	1.75(−3)	4.33(−4)	1.08(−4)	2.71(−5)

TABLE 2. The maximum absolute errors for example 1.

Example 3. Finally consider the following boundary value problem [17]:

$$-\epsilon u'' + [1+x(1-x)]u = 1+x(1-x)+[2\sqrt{\epsilon}-x^2(1-x)]\exp[-\frac{(1-x)}{\sqrt{\epsilon}}]+$$

$$+[2\sqrt{\epsilon}-x(1-x)^2]\exp[-x/\sqrt{\epsilon}],$$

$$u(0)=u(1)=0,$$

ϵ	$N=8$	$N=16$	$N=32$	$N=64$	$N=128$
Our eight order method					
$\frac{1}{8}$	2.97(−9)	5.02(−12)	1.59(−14)	5.45(−15)	1.76(−15)
$\frac{1}{16}$	5.48(−8)	1.27(−10)	1.47(−13)	8.32(−15)	2.12(−15)
$\frac{1}{32}$	8.35(−7)	2.84(−9)	4.96(−12)	7.27(−15)	5.99(−15)
$\frac{1}{64}$	9.84(−6)	5.39(−8)	1.26(−10)	1.60(−13)	1.38(−14)
$\frac{1}{128}$	8.43(−5)	8.31(−7)	2.84(−9)	4.96(−12)	8.60(−15)
$\frac{1}{256}$	5.01(−4)	9.83(−6)	5.39(−8)	1.26(−10)	1.61(−13)
$\frac{1}{512}$	2.03(−3)	8.43(−5)	8.31(−7)	2.84(−9)	4.96(−12)
Our sixth order method					
$\frac{1}{8}$	6.56(−8)	4.28(−10)	2.11(−12)	1.70(−14)	1.33(−14)
$\frac{1}{16}$	6.71(−7)	5.50(−9)	3.08(−11)	1.36(−13)	2.93(−14)
$\frac{1}{32}$	6.55(−6)	6.45(−8)	4.26(−10)	2.10(−12)	2.37(−14)
$\frac{1}{64}$	5.30(−5)	6.70(−7)	5.49(−9)	3.08(−11)	1.36(−13)
$\frac{1}{128}$	3.36(−4)	6.55(−6)	6.44(−8)	4.26(−10)	2.10(−12)
$\frac{1}{256}$	1.55(−3)	5.30(−5)	6.70(−7)	5.49(−9)	3.08(−11)
$\frac{1}{512}$	4.88(−3)	3.36(−4)	6.55(−6)	6.44(−8)	4.26(−10)
Our fourth order method					
$\frac{1}{8}$	7.83(−3)	2.27(−3)	6.20(−4)	1.62(−4)	4.15(−5)
$\frac{1}{16}$	1.25(−2)	3.95(−3)	1.13(−3)	3.04(−4)	7.88(−5)
$\frac{1}{32}$	1.95(−2)	7.01(−3)	2.12(−3)	5.84(−4)	1.53(−4)
$\frac{1}{64}$	2.95(−2)	1.19(−2)	3.92(−3)	1.12(−3)	3.00(−4)
$\frac{1}{128}$	4.27(−2)	1.94(−2)	7.02(−3)	2.12(−3)	5.35(−4)
$\frac{1}{256}$	5.88(−2)	2.93(−2)	1.20(−2)	3.93(−3)	1.12(−3)
$\frac{1}{512}$	7.62(−2)	4.27(−2)	1.95(−2)	7.05(−3)	2.13(−3)
The method in [11]					
$\frac{1}{8}$	1.10(−5)	7.01(−7)	4.38(−8)	2.74(−9)	1.71(−10)
$\frac{1}{16}$	4.70(−5)	2.96(−6)	1.85(−7)	1.15(−8)	7.24(−10)
$\frac{1}{32}$	1.78(−4)	1.18(−5)	7.54(−7)	4.67(−8)	2.96(−9)
$\frac{1}{64}$	7.41(−4)	4.74(−5)	2.96(−6)	1.86(−7)	1.16(−8)
$\frac{1}{128}$	2.70(−3)	1.78(−4)	1.18(−5)	7.46(−7)	4.67(−8)
$\frac{1}{256}$	8.31(−3)	7.41(−4)	4.74(−5)	2.98(−6)	1.86(−7)
$\frac{1}{512}$	2.05(−2)	2.70(−3)	1.78(−4)	1.18(−5)	7.46(−7)

TABLE 3. The maximum absolute errors for example 2.

with the exact solution, $u(x) = 1 + (x-1)\exp[-x/\sqrt{\epsilon}] - x\exp[-(1-x)/\sqrt{\epsilon}]$.

This problem has been solved using our method with different values of $N = 8, 16, 32, 64, 128$ and $\epsilon = \frac{1}{16}, ..., \frac{1}{256}$ the maximum absolute errors in solutions are tabulated in Table 4.

ϵ	$N = 16$	$N = 32$	$N = 64$	$N = 128$
Our eight order method				
$\frac{1}{16}$	4.40(−10)	6.00(−13)	4.77(−15)	1.22(−16)
$\frac{1}{32}$	7.39(−9)	1.33(−11)	2.62(−14)	2.94(−15)
$\frac{1}{64}$	1.10(−7)	2.74(−10)	3.97(−13)	9.10(−15)
$\frac{1}{128}$	1.39(−6)	5.08(−9)	9.18(−12)	1.35(−14)
Our sixth order method				
$\frac{1}{16}$	1.62(−8)	9.21(−11)	4.14(−13)	7.43(−15)
$\frac{1}{32}$	1.47(−7)	1.00(−9)	5.04(−12)	2.78(−14)
$\frac{1}{64}$	1.25(−6)	1.05(−8)	6.07(−11)	2.75(−13)
$\frac{1}{128}$	1.03(−5)	1.05(−7)	7.13(−10)	3.57(−12)
Our fourth order method				
$\frac{1}{16}$	5.25(−4)	7.79(−5)	1.05(−5)	1.37(−6)
$\frac{1}{32}$	1.20(−3)	1.91(−4)	2.70(−5)	3.59(−6)
$\frac{1}{64}$	2.62(−3)	4.64(−4)	6.91(−5)	9.45(−6)
$\frac{1}{128}$	5.37(−3)	1.08(−3)	1.74(−4)	2.48(−5)
The method in [17]				
$\frac{1}{16}$	1.63(−8)	9.22(−11)	4.14(−13)	2.43(−15)
$\frac{1}{32}$	1.47(−7)	1.00(−9)	5.04(−12)	2.78(−14)
$\frac{1}{64}$	1.25(−6)	1.06(−8)	6.07(−11)	2.75(−13)
$\frac{1}{128}$	1.03(−5)	1.05(−7)	7.14(−10)	3.57(−12)

Table 4. The maximum absolute errors for example 3.

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