TWMS J. Pure Appl. Math. V.1, N.2, 2010, pp. 224-235

ON A NONNEGATIVE SOLUTIONS OF THE HEAT EQUATION WITH SINGULAR POTENTIAL IN THE CONICAL DOMAIN

B.A. HUDAYKULIYEV [1]

ABSTRACT. In this paper we study the behavior of nonnegative solution of the Cauchy-Dirichlet problem for the heat equation with a singular potential in the domain $\Omega_\nu = G \cap B_\nu = G \cap B_\nu(0, r) \subset R^n, n \geq 3$, where G be a cone in R^n and $r < e_\nu^{-1}$. Existence and nonexistence of nonnegative solutions are analyzed.

Keywords: heat equation, singular potential, nonnegative solution, existence and nonexistence.

AMS Subject Classification: 58J35

1. Introduction

In this paper we consider the problem

$$\frac{\partial u}{\partial t} - \Delta u = V(x)u + f(x,t), \tag{1}$$

$$u|_{\partial\Omega_\nu} = 0, \quad t > 0, \tag{2}$$

$$u(x,t)|_{t=0} = u_0(x), \quad x \in \Omega_\nu \tag{3}$$

in the domain $\Omega_\nu \times (0,T)$, where $\Omega_\nu = G \cap B_\nu \subset R^n (n \geq 3); e_0 = 1, e_1 = e, ..., e_\nu = \exp e_{\nu-1}, \nu \geq 1, x = (x_1, ..., x_n) \in \Omega_\nu, B_\nu = B_\nu(0, e_\nu^{-1}) = \{x \in R^n : |x| < e_\nu^{-1}\} \subset R^n$ and $\partial\Omega_\nu -$ the boundary of $\Omega_\nu, 0 < T \leq \infty, G$ be a cone with vertex at the origin. We suppose that the boundary of Ω_ν, except the origin, is smooth enough.

Under solution to the equation (1) we mean the generalized function $u(x,t) \in D'(\Omega_\nu \times (0,T))$, such that $u(x,t) \geq 0, Vu \in L_{1,loc}(\Omega_\nu \times (0,T))$. Assumed that $0 \leq V(x) \in L_1(\Omega_\nu), 0 \leq u_0(x) \in L_1(\Omega_\nu)$ and $f(x,t) \in L_1(\Omega_\nu \times (0,T))$, where $L_{1,loc}(\Omega_\nu \times (0,T))-$ is the space of locally integrable functions, $L_1(\Omega)-$ is the space of integrable functions. We denote by $D'-$ the space of generalized functions.

The condition (3) means that

$$ess \lim_{t \to 0} \int_{\Omega_\nu} u(x,t)\phi(x)dx = \int_{\Omega_\nu} u_0(x)\phi(x)dx$$

for any $\phi(x) \in D(\Omega_\nu) = C_0^\infty(\Omega_\nu)$.

In the polar coordinates (r, ω), where $r = |x|, \omega = (\omega_1, \omega_2, ..., \omega_{n-1})$, the Laplace operator is given by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_\omega,$$

where Δ_ω the Beltrami operator. Let λ_1 be a first eigenvalue of the operator $-\Delta_\omega$ on $G \cap \partial B_\nu$ with zero Dirichlet condition on $\partial G \cap \partial B_\nu$, $Y_1(\omega)$ be a eigenfunction, corresponding to λ_1.

Let $F_0(x) = |x|, F_\nu(x) = \ln|F_{\nu-1}(x)|, \nu \geq 1, x \neq 0$. If we set

$$\varphi(x) = |x|^{-(n-2)/2}|F_1(x)|^{1/2}...|F_{\nu-1}(x)|^{1/2}|F_\nu(x)|^{\alpha/2}Y_1(\omega), \tag{4}$$

[1] Turkmen State University, Ashgabat, Turkmenistan,
e-mail: bazargeldyh@yandex.ru
Manuscript received 29 June 2010.

then it is easy to show that

$$-\Delta\varphi = (\frac{(n-2)^2}{4F_0^2(x)} + \frac{1}{4F_0^2(x)F_1^2(x)} + ... + \frac{1}{4F_0^2(x)...F_{\nu-1}^2(x)} + $$
$$+\frac{\alpha(2-\alpha)}{4F_0^2(x)...F_{\nu-1}^2(x)F_\nu^2(x)} + \frac{\lambda_1}{F_0^2(x)})\varphi(x),$$

so that

$$-\frac{\Delta\varphi}{\varphi} = \frac{(n-2)^2}{4F_0^2(x)} + \frac{1}{4F_0^2(x)F_1^2(x)} + ... + \frac{1}{4F_0^2(x)...F_{\nu-1}^2(x)} + \frac{c}{4F_0^2(x)...F_{\nu-1}^2(x)F_\nu^2(x)} + \frac{\lambda_1}{F_0^2(x)},$$

where $c = \alpha(2-\alpha)$. Note that the smaller root α of $\alpha(2-\alpha) = c$ is given by $\alpha = 1 - \sqrt{1-c}$ and $\Delta\varphi \in L_1(\Omega_\nu)$, when $0 < \alpha \leq 1$.

Put

$$V_0(x) = \frac{(n-2)^2}{4F_0^2(x)} + \frac{1}{4F_0^2(x)F_1^2(x)} + ... + \frac{c}{4F_0^2(x)...F_{\nu-1}^2(x)F_\nu^2(x)} + \frac{\lambda_1}{F_0^2(x)}, x \in \Omega_\nu. \qquad (5)$$

In this paper is studied the behavior of nonnegative solutions to the problem (1)-(3), when $V_0(x)$ is given by (5), and is proved that if $0 \leq c \leq 1$ and $V(x) \leq V_0(x)$ in Ω_ν, then the problem has a nonnegative solution; if $c > 1$ and $V(x) \geq V_0(x)$ in Ω_ν, then the problem does not have nonnegative solution if either $u_0(x) > 0$ or $f(x,t) > 0$.

In several reaction-diffusion problems involving the heat equation with supercritical reaction term, it appears a stationary singular solution. For instance, this is the case for $u_t - \Delta u = \eta \cdot e^u$, and $u_t - \Delta u = \eta \cdot u + u^{\beta-1}$, where $2n/(n-2) < \beta$. The linearization on this singular solution gives a linearized equation of the type $u_t - \Delta u = \frac{c}{|x|^2} \cdot u$. This linear equation is a borderline case with respect to the classical theory of parabolic equations, namely, the potential $c \cdot |x|^{-2}$ belongs to L^p_{loc} if and only if $1 \leq p < n/2$; therefore the standard uniqueness and regularity theories do not apply to this case. For this reason the study of this kind of equation is interesting. The linear equation $u_t - \Delta u = \frac{c}{|x|^2} \cdot u$ was studied by Baras-Goldstein in [2], where it was obtained the behavior of the solutions depending on the values of the parameter c. More precisely Baras-Goldstein prove that the critical value $C_*(n) = (n-2)^2/4$, determines the behavior of the solutions to the equation $u_t - \Delta u = \frac{c}{|x|^2} \cdot u$. They found that if $c > C_*(n)$, then the above problem has no nonnegative solutions except $u(x,t) \equiv 0$ and if $c \leq C_*(n)$, positive weak solutions do exist. The result in [2] stimulated several interesting results in the study of heat equation with singular potentials; see [4], [3], [1], [6].

2. Main results

The following theorem is our main result:

Theorem 2.1. *1. If $0 \leq c \leq 1$ and $V(x) \leq V_0(x)$ in Ω_ν, then the problem (1)-(3) has a nonnegative solution $u(x,t)$ if*

$$\int_{\Omega_\nu} u_0(x)\varphi(x)dx < \infty, \int_0^T \int_{\Omega_\nu} f(x,t)\varphi(x)dxdt < \infty,$$

where $\varphi(x)$ is given by (4)
2. If either $u_0(x) > 0$ or $f(x,t) > 0$ in $\Omega_\nu \times (0,\varepsilon)$ for each $\varepsilon \in (0,T)$ and $V(x) \geq V_0(x)$, then given $\Omega' \subset \Omega_\nu$ such that $\partial\Omega' \cap \partial\Omega_\nu = \{0\}$, there is a constant $C = C(\epsilon, \Omega') > 0$ such that

$$u(x,t) \geq C\varphi(x)$$

if $(x,t) \in \Omega' \times [\varepsilon, T)$.
3. If $c > 1$ and $V(x) \geq V_0(x)$ in Ω_ν, then the problem does not have nonnegative solution if either $u_0(x) > 0$ or $f(x,t) > 0$.

Proof of theorem. 1). We first prove the existence part. We shall attack (1)-(3) by studying the approximate problem

$$\frac{\partial u_m}{\partial t} - \Delta u_m = V_m(x)u_m + f_m, \qquad (1_m)$$

$$u_m|_{\partial\Omega_\nu} = 0, \quad t > 0, \qquad (2_m)$$

$$u_m|_{t=0} = u_0(x), \quad x \in \Omega_\nu, \qquad (3_m)$$

where $V_m(x) \in L_\infty(\Omega_\nu), 0 \le V_m(x) \le V(x)$, and $V_m(x) \uparrow V(x)$ a.e in Ω_ν, $f_m = \min\{f, m\}$. The problem $(1_m) - (3_m)$ has a unique bounded nonnegative solution (see [5]) which satisfies the integral equation

$$u_m(x,t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}V_m u_m(s)ds + \int_0^t e^{(t-s)\Delta}f_m(s)ds, \qquad (6)$$

where $\{e^{t\Delta}; t > 0\}$ denotes the semigroup generated by Δ with Dirichlet boundary conditions; note that the perturbation V_m defines a bounded multiplication operator on $L_p(\Omega_\nu)$ for all $p \ge 1$. Also,

$$(e^{t\Delta}u)(x) = \int_{\Omega_\nu} e^{t\Delta}\delta_x(y)u(y)dy, \qquad (7)$$

where $\delta_x(y)-$ the Dirak's function.
The sequence of nonnegative functions $\{u_m(x,t)\}$ is clearly increasing.

We first show that assumptions on the data implies the existence of a solution. Let $p \in C^2(R)$ be a convex function satisfying $p(0) = p'(0) = 0$. Multiply the equation (1_m) by $p'(u_m)\varphi$, where $\varphi = \varphi(x)$ is given by (4), and integrate over $\Omega_\nu \times [\delta, t)$ for $0 < \delta < t < T$. One gets, using integration by parts,

$$\int_{\Omega_\nu} p(u_m(t))\varphi dx + \int_\delta^t \int_{\Omega_\nu} \nabla u_m \nabla(p'(u_m)\varphi)dxdt = \int_\delta^t \int_{\Omega_\nu} (V_m u_m + f_m)p'(u_m)\varphi dxdt + \int_{\Omega_\nu} p(u_m(\delta))\varphi dx,$$

whence, since p is convex,

$$\int_{\Omega_\nu} p(u_m(t))\varphi dx + \int_\delta^t \int_{\Omega_\nu} p(u_m)(-\Delta\varphi)dxdt \le \int_\delta^t \int_{\Omega_\nu} (V_m u_m + f_m)p'(u_m)\varphi dxdt + \int_{\Omega_\nu} p(u_m(\delta))\varphi dx.$$

Replace $p(r)$ by a sequence $p_l(r)$ satisfying the hypotheses for p and converging to $|r|$ as $l \to \infty$. We obtain the limiting inequality

$$\int_{\Omega_\nu} u_m(t)\varphi dx + \int_\delta^t \int_{\Omega_\nu} u_m(-\Delta\varphi)dxdt \le \int_\delta^t \int_{\Omega_\nu} (V_m u_m + f_m)\varphi dxdt + \int_{\Omega_\nu} u_m(\delta)\varphi dx. \qquad (8)$$

We want to let $\delta \to 0$. First we claim that

$$\int_{\Omega_\nu} u_m(\delta)\varphi dx \to \int_{\Omega_\nu} u_0(x)\varphi dx.$$

To see why this is so, note that

$$e^{\delta\Delta}u_0 \le u_m(\delta) = e^{\delta(\Delta+V_m)}u_0 + \int_0^\delta e^{(\delta-s)(\Delta+V_m)}f_m(s)ds \le e^{\delta\lambda}e^{\delta\Delta}u_0 + e^{\delta\lambda}\int_0^\delta e^{(\delta-s)\Delta}f_m(s)ds,$$

if $\|V_m\|_\infty \le \lambda$, since $e^{\delta(\Delta+V_m)}u_0 = \lim\limits_{i\to\infty}(e^{\delta\Delta/i}e^{\delta V_m/i})^i u_0 \le e^{\delta\lambda}e^{\delta\Delta}u_0$ by the positivity preserving property of $\{e^{\delta\Delta}\}$. Thus

$$\int\limits_{\Omega_\nu}(e^{\delta\Delta}u_0)\varphi dx \le \int\limits_{\Omega_\nu} u_m(\delta)\varphi dx \le e^{\delta\lambda}\int\limits_{\Omega_\nu}(e^{\delta\Delta}u_0)\varphi dx + e^{\delta\lambda}\delta\|f_m\|_\infty \int\limits_{\Omega_\nu}\varphi dx,$$

whence

$$\int\limits_{\Omega_\nu}(e^{\delta\Delta}u_0)\varphi dx = \int\limits_{\Omega_\nu}(e^{\delta\Delta}\varphi)u_0 dx \to \int\limits_{\Omega_\nu}\varphi u_0 dx$$

as $\delta \to 0$, as asserted. Letting $\delta \to 0$ in (8), we deduce

$$\int\limits_{\Omega_\nu} u_m(t)\varphi dx + \int\limits_0^t\int\limits_{\Omega_\nu} u_m(-\Delta\varphi)dxdt \le \int\limits_0^t\int\limits_{\Omega_\nu} V_m u_m\varphi dxdt + \int\limits_0^t\int\limits_{\Omega_\nu} f_m\varphi dxdt + \int\limits_{\Omega_\nu} u_0(x)\varphi dx.$$

But $-\Delta\varphi \ge V_m(x)\varphi$. Consequently

$$\int\limits_{\Omega_\nu} u_m(t)\varphi dx \le \int\limits_0^t\int\limits_{\Omega_\nu} f_m\varphi dxdt + \int\limits_{\Omega_\nu} u_0(x)\varphi dx,$$

and therefore if

$$\int\limits_0^t\int\limits_{\Omega_\nu} f_m\varphi dxdt + \int\limits_{\Omega_\nu} u_0(x)\varphi dx < \infty$$

we conclude that $u_m(x,t)$ increases to a finite limit $u(x,t)$ as $m\to\infty$, for all $t\in(0,T)$ and for a.e. $x\in\Omega_\nu$.

Pick a point (x_0,t_0) such that $u(x_0,t_0)$ is finite. Let $\upsilon_m = e^t u_m$. Then

$$\frac{\partial \upsilon_m}{\partial t} - \Delta\upsilon_m = (V_m+1)\upsilon_m + e^t f_m.$$

Applying (6) and (7) to υ_m gives

$$e^{t_0}u_m(x_0,t_0) \ge \int\limits_0^{t_0}\int\limits_{\Omega_\nu}(e^{(t_0-s)\Delta}\delta_{x_0})(y)(V_m(y)+1)u_m(y,s)e^s dyds. \tag{9}$$

If $\Omega' \subset \Omega_\nu$ such that $\partial\Omega' \cap \partial\Omega_\nu = \{0\}$ and $0<\varepsilon<T$,

$$\inf\{(e^{s\Delta}\delta_{x_0})(y) : (y,s)\in\Omega'\times[\varepsilon,T]\} = c_0 > 0.$$

Therefore

$$c_0\int\limits_0^{t_0-\varepsilon}\int\limits_{\Omega'} V_m(y)u_m(y,s)dyds + c_0\int\limits_0^{t_0-\varepsilon}\int\limits_{\Omega'} u_m(y,s)dyds \le e^{t_0}u_m(x_0,t_0). \tag{10}$$

By hypothesis, u_m increases to u and $V_m u_m$ increases to Vu in $L_1(\Omega'\times(0,t_0-\varepsilon))$, and $u(x,t)$ is a solution (1)-(3) in the sense of generalized functions. This solution $u(x,t)$ satisfies the integral equation

$$u(x,t) = \int\limits_{\Omega_\nu} e^{t\Delta}\delta_x(y)u_0(y)dy + \int\limits_0^t\int\limits_{\Omega_\nu} e^{(t-s)\Delta}\delta_x(y)V(y)u(y,s)dyds +$$

$$+\int\limits_0^t\int\limits_{\Omega_\nu} e^{t\Delta}\delta_x(y)f(y,s)dyds$$

a.e. in $\Omega_\nu \times (0, t_0)$. By (9),

$$(y, s) \mapsto e^{(t_0-s)\Delta}\delta_x(y)V(y)u(y, s) \in L_1(\Omega_\nu \times (0, t_0))$$

since $\lim_{m\to\infty} u_m(x, t) = u(x, t) < \infty$ a.e. in $\Omega_\nu \times (0, t_0)$. The first part of theorem is proven.

2). Our next assertion is that If $V(x) \geq V_0(x)$ and $u_0(x)$ is not identically zero, for $\varepsilon > 0$ and $\Omega' \subset \Omega_\nu$ with $\partial\Omega' \cap \partial\Omega_\nu = \{0\}$, there is a constant $C = C(\epsilon, \Omega') > 0$ such that

$$u(x, t) \geq C\varphi(x) \tag{11}$$

for all $x \in \Omega'$ and $t \in [\varepsilon, T)$.
For the proof we first recall that if $u_0 > 0$, there is a positive constant C_0 such that $e^{t\Delta}u_0(y) \geq C_0$ if $x \in \Omega'$ and $t \in [\varepsilon/2, T)$. Next u is bounded below by the solution w of

$$\frac{\partial w}{\partial t} - \Delta w = V_0 w \quad in \quad D'(\Omega_\nu \times [\varepsilon/2, T)),$$

$$w = 0 \quad on \quad \partial\Omega_\nu, \quad w(y, \varepsilon/2) = C_0\chi_{\Omega'}(y) \quad in \quad \Omega_\nu,$$

and w is the (increasing) limit of the unique nonnegative solution w_m of

$$\frac{\partial w_m}{\partial t} - \Delta w_m = V_m w_m \quad in \quad D'(\Omega_\nu \times [\varepsilon/2, T)),$$

$$w_m = 0 \quad on \quad \partial\Omega_\nu, \quad w_m(y, \varepsilon/2) = C_0\chi_{\Omega'}(y) \quad in \quad \Omega_\nu.$$

Choose a ball $B = B_0 = B(0, r_0), r_0 < e_\nu^{-1}$. Let $\Omega_0 = \Omega' \cap B_0, \Omega_0 \subset \Omega'$. Then $w_m \geq \upsilon_m$ where

$$\frac{\partial \upsilon_m}{\partial t} - \Delta\upsilon_m = V_m\upsilon_m \quad in \quad D'(\Omega_0 \times [\varepsilon/2, T)), \tag{12}$$

$$\upsilon_m = 0 \quad on \quad \partial\Omega_0, \quad \upsilon_m(y, \varepsilon/2) = C_0 \quad in \quad \Omega_0,$$

where here and in the sequel $V_m = \inf\{V_0, m\}$. Multiply (12) by $\upsilon_m^{p-1}\varphi^{2-p}$ for $p > 1$ and integrate to obtain

$$\frac{\partial}{\partial t}\left(p^{-1}\int_{\Omega_0}(\frac{\upsilon_m}{\varphi})^p\varphi^2 dy\right) + \int_{\Omega_0}\nabla\upsilon_m \cdot \nabla(\upsilon_m^{p-1}\varphi^{2-p})dy = \int_{\Omega_0} V_m(\frac{\upsilon_m}{\varphi})^p\varphi^2 dy.$$

Setting $k_m = \upsilon_m/\varphi$ we get

$$\frac{\partial}{\partial t}\left(p^{-1}\int_{\Omega_0} k_m^p\varphi^2 dy\right) + \frac{4(p-1)}{p^2}\int_{\Omega_0}|\nabla k_m^{p/2}|^2\varphi^2 dy + \int_{\Omega_0} k_m^p(-\Delta\varphi)\varphi dy = \int_{\Omega_0} V_m k_m^p\varphi^2 dy.$$

Recall that $V_m \leq V_0(x) = -\Delta\varphi/\varphi$. Thus $V_m\varphi^2 \leq (-\Delta\varphi)\varphi$ and consequently

$$\frac{\partial}{\partial t}\left(p^{-1}\int_{\Omega_0} k_m^p\varphi^2 dy\right) \leq 0,$$

whence for $\varepsilon/2 \leq t < T$,

$$\left(\int_{\Omega_0}\upsilon_m^p\varphi^{2-p}dy\right)^{1/p} \leq C_0\left(\int_{\Omega_0}\varphi^{2-p}dy\right)^{1/p},$$

the right side being the value of the left side for $t = \varepsilon/2$. Letting $p \to \infty$ it follows that $k_m \leq C_0$ a.e. in Ω_0, which is equivalent to $\upsilon_m \leq C_0\varphi$ a.e. in Ω_0 We are now justified in setting

$$\upsilon = \lim_{m\to\infty}\upsilon_m, \quad k = \lim_{m\to\infty} k_m.$$

We will show that

$$C_0 \geq k(x, t) \geq C_1 \quad for \quad \varepsilon < t < T \quad and \quad a.e. \quad x \in \frac{1}{2}\Omega_0 = \Omega_0 \cap B(0, \frac{r_0}{2}) \tag{13}$$

(Here $k(x,t) \le C_0$ is already proven.) Since $u \ge w \ge w_m \ge \upsilon_m \ge k_m\varphi$, (13) implies (12) with $y \in \Omega' = \frac{1}{2}\Omega_0$. And for $y \in \Omega' \setminus \frac{1}{2}\Omega_0$ we have (since $u \ge e^{t\Delta}u_0$)

$$k(y,t) \ge \varphi^{-1}(y)(e^{t\Delta}u_0)(y) \ge C_2 > 0$$

for all $y \in \Omega'$, $\varphi^{-1}(y) \ge C_3 > 0$ in $\Omega' \setminus \frac{1}{2}\Omega_0$, where C_2 and C_3 are suitable constants.

Let $g : [0,\infty[\to [0,\infty[$ be convex and of class C^2. Multiply (12) by $g'(k_m)g(k_m)\varphi\psi^2$, where $k_m = \frac{\upsilon_m}{\varphi}, \psi = \psi(x,t) \in C_0^\infty(\Omega_0 \times (\varepsilon/2,T))$, and integrate over $Q = \Omega_0 \times (\varepsilon/2,T))$:

$$\int\limits_Q \frac{\partial \upsilon_m}{\partial t} g'(k_m)g(k_m)\varphi\psi^2 dxdt - \int\limits_Q \Delta\upsilon_m g'(k_m)g(k_m)\varphi\psi^2 dxdt = \int\limits_Q V_m\upsilon_m g'(k_m)g(k_m)\varphi\psi^2 dxdt. \tag{14}$$

Straightforward computations give

$$\int\limits_Q \frac{\partial \upsilon_m}{\partial t} g'(k_m)g(k_m)\varphi\psi^2 dxdt = \frac{1}{2}\left(\int\limits_{\Omega_0} g^2(k_m)\varphi^2\psi^2 dx\right)(t) - \int\limits_Q g^2(k_m)\varphi^2\psi\frac{\partial\psi}{\partial t}dxdt;$$

$$-\int\limits_Q \Delta\upsilon_m g'(k_m)g(k_m)\varphi\psi^2 dxdt = \int\limits_Q \nabla(k_m\varphi)\nabla(g'(k_m)g(k_m)\varphi\psi^2)dxdt =$$

$$= \int\limits_Q |\nabla g(k_m)|^2\varphi^2\psi^2 dxdt + \int\limits_Q g''(k_m)|\nabla k_m|^2 g(k_m)\varphi^2\psi^2 dxdt +$$

$$+ \int\limits_Q \nabla g(k_m)g(k_m)\varphi^2\nabla\psi^2 dxdt + \int\limits_Q g'(k_m)g(k_m)k_m\varphi\psi^2(-\Delta\varphi)dxdt.$$

Whence

$$\frac{1}{2}\left(\int\limits_{\Omega_0} g^2(k_m)\varphi^2\psi^2 dx\right)(t) - \int\limits_Q g^2(k_m)\varphi^2\psi\frac{\partial\psi}{\partial t}dxdt +$$

$$+ \int\limits_Q \nabla g(k_m)g(k_m)\varphi^2\nabla\psi^2 dxdt + \int\limits_Q g''(k_m)|\nabla k_m|^2 g(k_m)\varphi^2\psi^2 dxdt +$$

$$+ \int\limits_Q |\nabla g(k_m)|^2\varphi^2\psi^2 dxdt = \int\limits_Q (\Delta\varphi + V_m\varphi)g'(k_m)g(k_m)k_m\varphi\psi^2 dxdt.$$

The fourth term on the left is nonnegative since g is convex and nonnegative; for the third term we will use the Cauchy's inequality:

$$2\left|\int\limits_Q \nabla g(k_m)g(k_m)\varphi^2\psi\nabla\psi dxdt\right| \le \frac{1}{2}\int\limits_Q |\nabla g(k_m)|^2\varphi^2\psi^2 dxdt + 2\int\limits_Q g^2(k_m)\varphi^2|\nabla\psi|^2 dxdt.$$

Therefore

$$\frac{1}{2}\left(\int\limits_{\Omega_0} g^2(k_m)\varphi^2\psi^2 dx\right)(t) + \frac{1}{2}\int\limits_Q |\nabla g(k_m)|^2\varphi^2\psi^2 dxdt \le$$

$$\le \int\limits_Q \left(2|\nabla\psi|^2 + \psi\frac{\partial\psi}{\partial t}\right) g^2(k_m)\varphi^2 dxdt + \int\limits_Q (\Delta\varphi + V_m\varphi)g'(k_m)g(k_m)k_m\varphi\psi^2 dxdt.$$

Take $B_r = B(0,r)$ to have sufficient by small radius, i.e. $r < r_0 < e_\nu^{-1}, \Omega_r = \Omega' \cap B_r$. Since $V_m(x) \le V_0(x) = -\Delta\varphi/\varphi$ the second term on the right side of the above inequality tends to zero as $m \to \infty$ by Lebesgue's dominated convergence theorem. (Here we are using $\|k_m\|_\infty \le Const$ in Ω_0 and the hypotheses on g). Thus when $m \to \infty$ we obtain

$$\left(\int\limits_{\Omega_r} g^2(k)\varphi^2\psi^2 dx\right)(t) + \int\limits_Q |\nabla g(k)|^2\varphi^2\psi^2 dxdt \le 2\int\limits_Q \left(2|\nabla\psi|^2 + \psi\frac{\partial\psi}{\partial t}\right) g^2(k)\varphi^2 dxdt. \quad (15)$$

Now choose $\psi(x,t)$ so that: $0 \le \psi(x) \le 1; \psi(x,t) = 1$ in $\Omega_{r-\delta} \times [s+\delta, T], \psi(x,t) = 0$ in $((\Omega_0 \setminus \Omega_r) \times [0,T]) \cup (\Omega_0 \times [0,s])$, where $s > 0, \delta > 0$. We further suppose that $|\nabla\psi|^2 \le C_4\delta^{-2}, |\frac{\partial\psi}{\partial t}| \le C_4\delta^{-1}$, where the constant $C_4 > 0$ is independent of the pair (s,δ). Inequality (15) then yields

$$\int\limits_{\Omega_{r-\delta}} g^2(k(t))\varphi^2\psi^2 dx + \int\limits_{s+\delta}^{T}\int\limits_{\Omega_{r-\delta}} |\nabla g(k)|^2\varphi^2 dxdt \le 6C_4\delta^{-2}\int\limits_s^T\int\limits_{\Omega_r} g^2(k)\varphi^2 dxdt. \quad (16)$$

for all $t \in [s+\delta, T)$. Now we will prove the following inequality

Lemma. *Let $0 < r \le e_\nu^{-1}, h(s) \in C^1[0,r]$. Then for $2 \le q \le 4$, $0 < \alpha \le 1$ the inequality is true*

$$\left(\int\limits_0^r |h(s)|^q s|F_1(s)...F_{\nu-1}(s)||F_\nu(s)|^\alpha ds\right)^{2/q} \le K\int\limits_0^r [|h'(s)|^2 + h^2(s)]s|F_1(s)...F_{\nu-1}(s)||F_\nu(s)|^\alpha ds, \quad (17)$$

where the constant $K = K(n,\alpha,\nu) > 0$, and α is defined by $\alpha(2-\alpha) = c$.

Proof. We first prove the inequality: Let $0 < r \le e_\nu^{-1}, 0 < h(s) \in C^1[0,r]$ and $h(r) = 0$. Then for $2 \le q \le 4$ and $0 < \alpha \le 1$ the inequality is true

$$\left(\int\limits_0^r |h(s)|^q s|F_1(s)...F_{\nu-1}(s)||F_\nu(s)|^\alpha ds\right)^{2/q} \le K\int\limits_0^r |h'(s)|^2 s|F_1(s)...F_{\nu-1}(s)||F_\nu(s)|^\alpha ds, \quad (18)$$

Integrating by parts and using the Hölder's inequality, it is easy to show that

$$\int\limits_0^r h^q(s)s|F_1(s)...F_{\nu-1}(s)||F_\nu(s)|^\alpha ds \le K\int\limits_0^r h^{q-1}(s)|h'(s)|s|F_1(s)...F_{\nu-1}(s)||F_\nu(s)|^\alpha ds \le$$

$$\le K_1\left(\int\limits_0^r h^{2(q-1)}(s)s^3|F_1(s)...F_{\nu-1}(s)||F_\nu(s)|^\alpha ds\right)^{\frac{1}{2}}\left(\int\limits_0^r |h'(s)|^2 s|F_1(s)...F_{\nu-1}(s)||F_\nu(s)|^\alpha ds\right)^{\frac{1}{2}} \le$$

$$\le K_1\left(\int\limits_0^r |h'(s)|^2 s|F_1(s)...F_{\nu-1}(s)||F_\nu(s)|^\alpha ds\right)^{\frac{1}{2}}\left(\int\limits_0^r h^q(s)s|F_1(s)...F_{\nu-1}(s)||F_\nu(s)|^\alpha ds\right)^{\frac{1}{2}} \times$$

$$\times\left(\sup_{s\in[0,r]}\{h^{q-2}(s)s^2\}\right)^{\frac{1}{2}},$$

where $K_1 = K_1(n,\alpha,\nu) > 0$. Whence

$$\int\limits_0^r h^q(s)s|F_1(s)...F_{\nu-1}(s)||F_\nu(s)|^\alpha ds \le K_1^2\int\limits_0^r |h'(s)|^2 s|F_1(s)...F_{\nu-1}(s)||F_\nu(s)|^\alpha ds \times$$

$$\times \sup_{s\in[0,r]}\{h^{q-2}(s)s^2\}.$$

Now we will show that

$$\sup_{s\in[0,r]}\{h^{q-2}(s)s^2\}\le K_2\left(\int_0^r |h'(s)|^2 s|F_1(s)...F_{\nu-1}(s)||F_\nu(s)|^\alpha ds\right)^{\frac{q-2}{2}}.$$

We have (note that $h^{q-2}(s)s^2=[h(s)s^{\frac{2}{q-2}}]^{\frac{q-2}{2}}$.)

$$\sup_{s\in[0,r]} s^{\frac{2}{q-2}}h(s)=\sup_{s\in[0,r]} s^{\frac{4-q}{q-2}}\{h(s)s-h(r)r\}=\sup_{s\in[0,r]} s^{\frac{4-q}{q-2}}\left\{-\int_s^r (h(\tau)\tau)'d\tau\right\}\le$$

$$\le\sup_{s\in[0,r]} s^{\frac{4-q}{q-2}}\left\{\left(\int_s^r |h'(\tau)|^2\tau|F_1(\tau)...F_{\nu-1}(\tau)||F_\nu(\tau)|^\alpha d\tau\right)^{\frac{1}{2}}\left(\int_s^r \frac{\tau d\tau}{|F_1(\tau)...F_{\nu-1}(\tau)||F_\nu(\tau)|^\alpha}\right)^{\frac{1}{2}}\right\}\le$$

$$\le\sup_{s\in[0,r]} M(s)\left(\int_0^r |h'(\tau)|^2\tau|F_1(\tau)...F_{\nu-1}(\tau)||F_\nu(\tau)|^\alpha d\tau\right)^{\frac{1}{2}},$$

where

$$M(s)=s^{\frac{4-q}{q-2}}\left(\int_s^r \tau d\tau\right)^{1/2}=s^{\frac{4-q}{q-2}}\left(\frac{r^2-s^2}{2}\right)^{1/2},$$

since $|F_1(\tau)...F_{\nu-1}(\tau)||F_\nu(\tau)|^\alpha\ge 1$, when $0<s<r<e_\nu^{-1}$. It is clear that there is a constant $K_3>0$, such that $\sup_{s\in[0,r]} M(s)\le K_3$. This proves (18). Next we deduce (17). Fix $\rho>0$ and let $r\ge\rho$. Let $h\in C^1(0,r)$. Let $\xi\in C^1[r,2r]$ satisfy $0\le\xi\le 1, \xi\equiv 0$ in $[r+\rho/2,2r], \xi\equiv 1$ in $[r,r+\rho/4]$, and $0\ge\xi'\ge -5\rho^{-1}$ in $[r,2r]$. Let $\psi(s)$ be $h(s)$ or $h(2r-s)\xi(s)$ according as $s\in[0,r)$ or $s\in[r,2r]$. Then by (18)

$$\left(\int_0^r h^q(s)s|F_1(s)\cdot...\cdot F_{\nu-1}(s)||F_\nu(s)|^\alpha ds\right)^{2/q}\le\left(\int_0^{2r}\psi^q(s)s|F_1(s)\cdot...\cdot F_{\nu-1}(s)||F_\nu(s)|^\alpha ds\right)^{2/q}\le$$

$$\le K_0\int_0^{2r}(\psi'(s))^2 s|F_1(s)\cdot...\cdot F_{\nu-1}(s)||F_\nu(s)|^\alpha ds\le K_0[\int_0^r (h'(s))^2 s|F_1(s)\cdot...\cdot F_{\nu-1}(s)||F_\nu(s)|^\alpha ds+$$

$$+2\int_r^{2r}(h'(2r-s))^2\xi^2(s)s|F_1(s)\cdot...\cdot F_{\nu-1}(s)||F_\nu(s)|^\alpha ds+$$

$$+2\int_r^{2r}h^2(2r-s)(\xi'(s))^2 s|F_1(s)\cdot...\cdot F_{\nu-1}(s)||F_\nu(s)|^\alpha ds]\le$$

$$\le K_0[\int_0^r (h'(s))^2 s|F_1(s)\cdot...\cdot F_{\nu-1}(s)||F_\nu(s)|^\alpha ds+$$

$$+2\int_{r-\rho/2}^{r}(h'(\sigma))^2(2r-\sigma)|F_1(2r-\sigma)\cdot...\cdot F_{\nu-1}(2r-\sigma)||F_\nu(2r-\sigma)|^\alpha d\sigma+$$

$$+2\int\limits_{r-\rho/2}^{r-\rho/4} h^2(\sigma)(\xi'(2r-\sigma))^2(2r-\sigma)|F_1(2r-\sigma)\cdot ...\cdot F_{\nu-1}(2r-\sigma)||F_\nu(2r-\sigma)|^\alpha d\sigma] \leq$$

$$\leq K_0\left[1+2\cdot\frac{r+\rho/2}{r-\rho/2}\right]\int\limits_0^r (h'(s))^2 s|F_1(s)\cdot ...\cdot F_{\nu-1}(s)||F_\nu(s)|^\alpha ds+$$

$$+50\rho^{-2}K_0\frac{r+\rho/2}{r-\rho/2}\int\limits_0^r h^2(s)s|F_1(s)\cdot ...\cdot F_{\nu-1}(s)||F_\nu(s)|^\alpha ds \leq$$

$$\leq K_4\int\limits_0^r (h'(s))^2 s|F_1(s)\cdot ...\cdot F_{\nu-1}(s)||F_\nu(s)|^\alpha ds + K_5\int\limits_0^r h^2(s)s|F_1(s)\cdot ...\cdot F_{\nu-1}(s)||F_\nu(s)|^\alpha ds,$$

where $\sigma = 2r - s$. The inequality (17) is proven. The lemma is proven.

Let λ_r be the first eigenvalue of the operator $-\Delta_\omega$ on $G\cap\partial B_r$ with zero Dirichlet condition on $\partial G\cap\partial B_r$, $Y_r(\omega)$ be a eigenfunction, corresponding to λ_r. From (17) for any nonnegative function $h(x)\in C^1(\Omega_r)$, we get

$$\int\limits_{G\cap\partial B_r}\int\limits_0^r |h(s)|^q s|F_1(s)...F_{\nu-1}(s)||F_\nu(s)|^\alpha Y_r^2(\omega)dsd\omega \leq$$

$$\leq \left(K\int\limits_{G\cap\partial B_r}\int\limits_0^r\left[\left|\frac{\partial h}{\partial s}\right|^2 + h^2(s)\right]s|F_1(s)...F_{\nu-1}(s)||F_\nu(s)|^\alpha Y_r^2(\omega)dsd\omega\right)^{q/2} \leq$$

$$\leq \left(K\int\limits_{G\cap\partial B_r}\int\limits_0^r[|\nabla h|^2 + h^2(s)]s|F_1(s)...F_{\nu-1}(s)||F_\nu(s)|^\alpha Y_r^2(\omega)dsd\omega\right)^{q/2},$$

whence (by (4))

$$\left(\int\limits_{\Omega_r}|h(x)|^q\varphi^2(x)dx\right)^{2/q} \leq C_5\int\limits_{\Omega_r}[|\nabla h(x)|^2 + h^2(x)]\varphi^2(x)dx.$$

Define β by $\beta+\frac{2}{q}=1$, where $2<q\leq 4$. By Hölder's inequality and last inequality we obtain, for a nonnegative function h,

$$\int\limits_{\Omega_r}h^{2+2\beta}\varphi^2 dx \leq \left(\int\limits_{\Omega_r}h^q\varphi^2 dx\right)^{2/q}\left(\int\limits_{\Omega_r}h^2\varphi^2 dx\right)^\beta \leq$$

$$\leq C_5\left(\int\limits_{\Omega_r}|\nabla h|^2\varphi^2 dx + \int\limits_{\Omega_r}h^2\varphi^2 dx\right)\left(\int\limits_{\Omega_r}h^2\varphi^2 dx\right)^\beta,$$

whence

$$\int\limits_a^b\int\limits_{\Omega_r}h^{2+2\beta}\varphi^2 dxdt \leq C_5\left(\int\limits_a^b\int\limits_{\Omega_r}|\nabla h|^2\varphi^2 dxdt + \int\limits_a^b\int\limits_{\Omega_r}h^2\varphi^2 dxdt\right)\sup_{a\leq t\leq b}\left(\int\limits_{\Omega_r}h^2\varphi^2 dx\right)^\beta, \quad (19)$$

From (16) we deduce

$$\sup_{t\in[s+\delta,T]}\int\limits_{\Omega_r} g^2(k(t))\varphi^2dx \le 6C_4\delta^{-2}\int\limits_s^T\int\limits_{\Omega_r} g^2(k)\varphi^2dxdt.$$

Whence replacing h by $g(k)$ and applying (19) with $[a,b]=[s+\delta,T]$ and with $\Omega_{r-\delta}$ in place Ω_r, we get

$$\int\limits_{s+\delta}^T\int\limits_{\Omega_{r-\delta}} g^{2+2\beta}(k)\varphi^2dxdt \le C_5(6C_4\delta^{-2}+1)\left(\int\limits_s^T\int\limits_{\Omega_{r-\delta}}|\nabla g(k)|^2\varphi^2dxdt\right)\times$$

$$\times\left(6C_4\delta^{-2}\int\limits_s^T\int\limits_{\Omega_r} g^2(k)\varphi^2dxdt\right)^{\beta},$$

whence

$$\left(\int\limits_{s+\delta}^T\int\limits_{\Omega_{r-\delta}} g^{2+2\beta}(k)\varphi^2dxdt\right)^{1/(2+2\beta)} \le [C_5^{1/2}(6C_4+1)]^{1/(1+\beta)}\delta^{-\gamma}\left(\int\limits_s^T\int\limits_{\Omega_r} g^2(k)\varphi^2dxdt\right)^{1/2} =$$

$$= C_6\delta^{-1}\left(\int\limits_s^T\int\limits_{\Omega_r} g^2(k)\varphi^2dxdt\right)^{1/2}. \tag{20}$$

Let $a>0$ be a small number and let

$$\delta=\frac{a}{2^j}, r_1=r, r_{j+1}=r_j-\frac{a}{2^j}, g_{j+1}=g_j^{1+\beta}, s_{j+1}=s_j+\frac{a}{2^j}$$

$$H_j=\left(\int\limits_{s_j}^T\int\limits_{\Omega_{r_j}} g_j^2(k)\varphi^2dxdt\right)^{1/2}, j=1,2,3,...,$$

where $g_1=g$, and r_1 and s_1 are given positive numbers. With this notation the estimate (20) yields

$$H_{j+1}^{1/(1+\beta)} \le C_7 2^j a^{-1} H_j,$$

whence, by induction

$$H_j^{1/(1+\beta)} \le (C_7a^{-1})^{\alpha_j}2^{\gamma_j}H_1^{(1+\beta)^{j-2}},$$

where $\alpha_j=(1+\beta)^{j-2}\sum\limits_{\mu=0}^{j-2}(1+\beta)^{-\mu}$; $\gamma_j=\sum\limits_{\mu=0}^{j-1}(1+\mu)(1+\beta)^{j-2-\mu}$.

Now let $j\to\infty$. Since $g_j=g^{(1+\beta)^{j-1}}$ we get

$$\sup_{\Omega_{r_1-a}\times[s_1+a,T]} g(k(x,t)) \le (C_7a^{-1}2^{(1+\beta)/\beta})^{(1+\beta)/\beta}\left(\int\limits_{s_1}^T\int\limits_{\Omega_r} g^2(k)\varphi^2dxdt\right)^{1/2}.$$

Replace g by a sequence $\{g_l\}$ satisfying the hypotheses and tending to $k^{-\gamma}$ as $l\to\infty$. We then obtain

$$\sup_{\Omega_{r_1-a}\times[s_1+a,T]} k^{-\gamma}(x,t) \le (C_7a^{-1}2^{(1+\beta)/\beta})^{(1+\beta)/\beta}\left(\int\limits_{s_1}^T\int\limits_{\Omega_r} k^{-2\gamma}\varphi^2dxdt\right)^{1/2}.$$

Now set $s_1 = 3\varepsilon/4, a = \varepsilon/4, r_1 < r_0$, where $\varepsilon > o$ is given. Note that

$$k(x,t) = \frac{\upsilon}{\varphi} \geq \varphi^{-1}(x)(e^{t\Delta}\upsilon_0)(x) \geq C_0 C_8 \varphi^{-1}(x)$$

for $(x,t) \in \Omega_{r_1} \times (3\varepsilon/4, T)$, where the constant C_8 is independent of r_1 and ε (but C_0 depends on ε, as before). Thus we obtain

$$\sup_{\Omega_{r_1-\varepsilon/4}\times[\varepsilon,T]} k^{-\gamma}(x,t) \leq C_9 C_0^{-\gamma} \varepsilon^{-1-1/\beta} \left(\int_{3\varepsilon/4}^{T} \int_{\Omega_{r_1}} \varphi^{2+2\gamma} dx dt \right)^{1/2}.$$

which implies the estimate

$$k(x,t) \geq C_{10} C_0 \varepsilon^{(1+1/\beta)/\gamma} \left(\int_{\Omega_{r_1}} \varphi^{2+2\gamma} dx \right)^{-1/2\gamma} \tag{21}$$

for a.e. $x \in \Omega_{r_1-\varepsilon/4}$ and for all $t \in [\varepsilon, T]$, where the constant $C_{10} > 0$ is independent of the pair (r_1, ε). The inequality (13), consequently and the inequality (11) is proven.

3). Now we prove the last part of theorem.

Let $c > 1, V(x) \geq V_0(x)$. If (1)-(3) has a nonzero solution, then one has

$$\frac{\partial u}{\partial t} - \Delta u = \left(\frac{(n-2)^2}{4F_0^2(x)} + \frac{1}{4F_0^2(x)F_1^2(x)} + ... + \frac{1}{4F_0^2(x)...F_\nu^2(x)} + \frac{\lambda_1}{F_0^2(x)} \right) u + \frac{c-1}{4F_0^2(x)...F_\nu^2(x)} u$$

in $D'(\Omega_\nu \times (0,T))$. From first part we know that the solution exists only if

$$\frac{c-1}{4F_0^2(x)...F_\nu^2(x)} u\varphi \in L_1(\Omega' \times (0, T-\varepsilon))$$

for $\Omega' \subset \Omega_\nu$ and $\varepsilon > 0$ (where we assume $\partial\Omega' \cap \partial\Omega_\nu = \{0\}$). From (11) follows that for any $\Omega' \subset \Omega_\nu$:

$$u(x,t) \geq Const \cdot \varphi(x) = Const \cdot |x|^{-(n-2)/2} |F_1(x)|^{1/2} ... |F_{\nu-1}(x)|^{1/2} |F_\nu(x)|^{1/2} Y_1(\omega),$$

therefore

$$\int_0^{T-\varepsilon} \int_{\Omega'} \frac{c-1}{4F_0^2(x)...F_\nu^2(x)} u |x|^{-(n-2)/2} |F_1(x)|^{1/2} ... |F_{\nu-1}(x)|^{1/2} |F_\nu(x)|^{1/2} Y_1(\omega) dx dt \geq$$

$$\geq Const \int_{\Omega'} |x|^{-n} |F_1(x)...F_\nu(x)|^{-1} Y_1^2(\omega) dx = \infty.$$

This proves the last part of our theorem. The Theorem is proven.

References

[1] Azorero Garsia, J., Peral Alonso, I., (1998), Hardy inequalities and some critical elliptic and parabolic problems, Differential Equations, 144, pp. 441-476.

[2] Baras, P., Goldstein, J.A., (1984), The heat equation with a singular potential, Trans. of the American Mathematical Society, 284:1, pp. 121-139.

[3] Cabre, X., Martel, Y., (1999), Existence versus explosion instantanee pour des equations de la chaleur lineaires avec potentiel singulier, C.R. Acad. Sci. Paris Ser I. Math., 329, pp. 973-978.

[4] Crespo Aguilar, J.A., Peral Alonso, (2000), Global behavior of the Cauchy problem for some critical nonlinear parabolic equations, SIAM J. Math. Analysis, 31, pp. 1270-1294.

[5] Lieberman, G.M., (1996), Second Order Parabolic Differential Equations, World Scientific Publishing Co. Inc. River. Edge, NJ.

[6] Vazquez, J.L., Zuazua, E., (2000), The Hardy inequality and the asymptotic behavior the heat equation with an inverse-square potential, J. Funct. Analysis, 173, pp. 103-153.

Hudaykuliev Bazargeldy- was born in 1961. He graduated from Moscow State University in 1988. Presently he works in Turkmen State University. His research interests include the elliptic and parabolic equations with singular potential.