TWMS J. Pure Appl. Math. V.1, N.2, 2010, pp. 252-256

# A NOTE ON SELF SIMILAR VECTOR FIELDS IN PLANE SYMMETRIC STATIC SPACE-TIMES

GHULAM SHABBIR[1], SUHAIL KHAN[1]

Abstract. A different approach is developed to study self similar vector fields in plane symmetric static space-times by using algebraic and direct integration techniques. Here we discuss self similar vector fields of first, second, zeroth and infinite kinds for the above space-times in both tilted and non tilted cases. We obtained self similar vector fields of first, second and infinite kinds in tilted case. In case of non tilted we obtained only self similar vector fields in zeroth kind.

Keywords: direct integration techniques, tilted and non tilted self similar vector fields.

AMS Subject Classification: 83C05, 83C15, 83C20.

## 1. Introduction

Over the past few years there has been much interest in studying symmetries in general relativity. These symmetries arise in the exact solutions of Einstein field equations (EFES) given by

$$G_{ab} = R_{ab} - \frac{1}{2} R\, g_{ab} = 8\pi\, T_{ab}, \tag{1}$$

where $G_{ab}$ represents the components of Einstein tensor, $R_{ab}$ are the components of Ricci tensor, $T_{ab}$ are the components of energy momentum tensor, $R$ is the Ricci scalar and $g_{ab}$ is the metric tensor. Here we are considering cosmological constant is $\wedge = 0$.

In general relativity self similarity is defined by the existence of a homothetic vector field. Such similarity is called of the first kind. Much work has been done in studying self similar solutions in some well known space-times [1-3,5-7,9]. Self similar solutions of the EFES are widely studied for two very important reasons; first, the governing differential equations have some mathematical complexity which is often reduced by the assumption of self similarity and the system of partial differential equations is reduced to ordinary differential equations. Second, self similarity solutions are extensively used for cosmological perturbations, star formation, gravitational collapse, primordial black holes, cosmological voids and cosmic censorship [8]. Self similar solutions describe asymptotic behaviour of more general solutions this feature is discussed in detail for homogenous cosmological models in [4].

A vector field $X$ is said to be self similar if it satisfies the following two conditions [4]

$$\underset{X}{L}\, u_a = \alpha\, u_a, \tag{2}$$

$$\underset{X}{L}\, h_{ab} = 2\,\delta\, h_{ab}. \tag{3}$$

[1] Faculty of Engineering Sciences, GIK Institute of Engineering Sciences and Technology, Topi, Swabi, NWFP, Pakistan,
e-mail: shabbir@giki.edu.pk
*Manuscript received 3 April 2010.*

where $u^a$ is the four-velocity of the fluid satisfying $u^a u_a = \in$ and $h_{ab} = g_{ab} + \in u_a u_b$ is the projection tensor, $\in = \pm 1$ and $\alpha, \delta \in R$. If $\in = 1$ the vector field $u^a$ is said to be space-like otherwise timelike. If $\delta \neq 0$, the similarity transformation is characterized by the scale-independent ratio $\alpha/\delta$, which is called the similarity index. If the ratio is unity, $X$ turns out to be a homothetic vector field. In the context of self similarity, homothety is referred to as self similarity of the First kind. If $\alpha = 0$ and $\delta \neq 0$, it is referred to as self similarity of the Zeroth kind. If the ratio is not equal to zero or one, it is referred to as self similarity of the Second kind. If $\alpha \neq 0$ and $\delta = 0$, it is referred to as self similarity of the Infinite kind. If $\delta = \alpha = 0$, $X$ turns out to be a Killing vector fields. If a self similar vector field $X$ is in the direction of the four velocity of the fluid then it is said to be non-tilted parallel self similar vector field. If a self similar vector field $X$ is along the hyper surface, then it will be perpendicular to the fluid flow and is called non-tilted orthogonal self similar vector field. If a self similar vector field $X$ is neither orthogonal nor parallel to the fluid flow it is said to be tilted.

Throughout in this paper $M$ represents a four dimensional, connected, hausdorff space-time manifold with Lorentz metric $g$ of signature (-, +, +, +). The curvature tensor associated with $g_{ab}$, through the Levi-Civita connection, is denoted in component form by $R^a_{bcd}$. The usual covariant, partial and Lie derivatives are denoted by a semicolon, a comma and the symbol $L$, respectively. Here, $M$ is assumed non flat in the sense that the curvature tensor does not vanish over any non empty open subset of $M$.

## 2. Main results

Consider the plane symmetric static space-times in the usual coordinate system $(t, x, y, z)$ (labeled by $(x^0, x^1, x^2, x^3)$,respectively) with the line element

$$ds^2 = -e^{A(x)}dt^2 + dx^2 + e^{B(x)}(dy^2 + dz^2). \tag{4}$$

The Ricci tensor Segre type of the above space-times is {1,1(11)} or one of its degeneracies. Without losing the generality we choose $u_a \equiv e^{\frac{A}{2}}\, \delta^0_a$, where $u^a u_a = -1$. The line element (4) becomes

$$ds^2 = -du^2 + dx^2 + e^{B(x)}(dy^2 + dz^2). \tag{5}$$

The above space-times (5) become 1+3 decomposable and admit nowhere zero time-like covariantly constant vector field $u_a$ such that $u_{a;b} = 0$ and $u_a u^a = -1$. Expand equation (2) and using $u_a$ is covariantly constant we get

$$X^b_{,a}\, u_b = \alpha\, u_a \Rightarrow X^0_{,a} = \alpha\, u_a. \tag{6}$$

Equation (6) implies that $X^0 = X^0(u)$ and $X^0 = \alpha\, u + \beta$, where $\beta \in R$. The components of the projection tensor $h_{ab}$ are

$$h_{11} = 1, h_{22} = e^{B(x)} \texttt{ and } h_{33} = e^{B(x)}. \tag{7}$$

Expanding equation (3) explicitly and using (7) we get

$$X^1_{,1} = \delta, \tag{8}$$

$$e^{B(x)}X^2_{,1} + X^1_{,2} = 0, \tag{9}$$

$$e^{B(x)}X^3_{,1} + X^1_{,3} = 0, \tag{10}$$

$$B^\bullet(x)X^1 + 2X^2{}_{,2} = 2\delta, \tag{11}$$

$$X^3_{,2} + X^2_{,3} = 0, \tag{12}$$

$$B^{\bullet}(x)X^1 + 2X^3{}_{,3} = 2\delta, \tag{13}$$

where 'dot' represents differentiation with respect to $x$. Equations (8), (9) and (10) give

$$\begin{aligned} X^1 &= \delta\, x + K^1(y,z), \\ X^2 &= -K^1_y(y,z)\int e^{-B(x)}dx + K^2(y,z), \\ X^3 &= -K^1_z(y,z)\int e^{-B(x)}dx + K^3(y,z), \end{aligned} \tag{14}$$

where $K^1(y,z)$, $K^2(y,z)$ and $K^3(y,z)$ are functions of integration. In order to calculate the vector field $X$ we need to find the functions $K^1(y,z)$, $K^2(y,z)$ and $K^3(y,z)$. To avoid lengthy details here we will only presents the results. Here, solutions of equation (8) to (13) are

$$X^1 = \delta\, x + d_1, \;\; X^2 = -z\, d_3 + d_4, \;\; X^3 = y\, d_3 + d_5, \tag{15}$$

where $B(x) = \ln(\delta\, x + d_1)^{2(1-\frac{d_2}{\delta})}$ and $d_1,\, d_2,\, d_3,\, d_4,\, d_5, \delta \in R(\delta \neq 0)$. The case when $\delta = 0$ will be discuss later. It is important to remind the reader that we also have $X^0 = \alpha\, u + \beta$. Now we will discuss when the above space-time admits self similar vector fields in tilted and non tilted cases.

## 3. Tilted cases

There exist following three possibilities which we discuss in turn.

(1) Here we choose $\alpha \neq 0$, $\delta \neq 0$, and $\alpha = \delta$. Self similar vector fields take the form after subtracting Killing vector fields

$$X^0 = \alpha\, u, \qquad X^1 = \alpha\, x + d_1, \quad X^2 = X^3 = 0. \tag{16}$$

The line element in this case takes the form

$$ds^2 = -du^2 + dx^2 + (\alpha\, x + d_1)^{2(1-\frac{d_2}{\alpha})}\left(dy^2 + dz^2\right). \tag{17}$$

Here, the vector field (16) is tilted to the time-like vector field $u^a$ and gives the self similarity of first kind. It is important to note that the above space-time (17) admits proper homothetic vector field which is given in equation (16).

(2) In this case we choose $\alpha \neq 0$, $\delta \neq 0$ and $\alpha \neq \delta$. Self similar vector fields take the form after subtracting Killing vector fields

$$X^0 = \alpha\, u, \qquad X^1 = \delta\, x + d_1, \quad X^2 = X^3 = 0 \tag{18}$$

and the line element takes the form

$$ds^2 = -du^2 + dx^2 + (\delta\, x + d_1)^{2(1-\frac{d_2}{\delta})}\left(dy^2 + dz^2\right). \tag{19}$$

The vector field in (18) is tilted to the time-like vector field $u^a$ and gives the self similarity of second kind.

(3) Now we choose $\alpha \neq 0$ and $\delta = 0$. Self similar vector fields after subtracting Killing vector fields become

$$X^0 = \alpha\, u, \qquad X^1 = d_1, \quad X^2 = X^3 = 0 \tag{20}$$

and the line element becomes

$$ds^2 = -du^2 + dx^2 + e^{(ax+c)}\left(dy^2 + dz^2\right)\;, \tag{21}$$

where $a = -2\frac{d_2}{d_1}$ and $d_1, d_2, a, c \in R(d_1 \neq 0)$. Self similar vector fields in this case is tilted to the time-like vector field $u^a$ and gives the self similarity of Infinite kind.

## 4. Non tilted case

In this case we choose $\alpha = 0$ and $\delta \neq 0$. Self similar vector fields after subtracting Killing vector fields take the form

$$X^0 = 0, \;\; X^1 = \delta\, x + d_1, \;\; X^2 = X^3 = 0. \tag{22}$$

The line element in this case takes the form

$$ds^2 = -du^2 + dx^2 + (\delta\, x + d_1)^{2(1-\frac{d_2}{\delta})} \left(dy^2 + dz^2\right). \tag{23}$$

Here the self similar vector field in equation (22) is perpendicular to the timelike vector field $u^a$ and gives the self similarity of zeroth kind.

## References


[1] Cahill, M.E., Taub, A.H., (1971), Spherically symmetric similarity solutions of the Einstein field equations for a perfect fluid, Commun. Math. Physics, 21(1), pp. 1-40.

[2] Carter, B., Henriksen, R. N., (1989), A covariant characterisation of kinematic self-similarity, Annals De Physique, 14, pp. 47-53.

[3] Carter, B., Henriksen, R. N., (1991), A systematic approach to self-similarity in Newtonian space-time, Journal of Math. Physics, 32(10), pp. 2580-2597.

[4] Carr, B. J., Coley, A. A., (1999), Self similarity in general relativity, Class. Quant. Gravity, 16, pp. R31-R71.

[5] Coley, A. A. (1997), Kinematic self similarity, Class. Quant. Gravity, 14, pp. 87-118.

[6] Harada, T., Maeda, H., (2004), Stability criterion for self-similar solutions with a scalar field and those with a stiff fluid in general relativity, Class. Quant. Gravity, 21, pp. 371-389.

[7] Maeda, H., Harada, T., Iguchi, H., Okuyama, N., (2002), A classification of spherically symmetric kinematic self-similar perfect-fluid solutions, Prog. of Theoretical physics, 108(5), pp. 819-851.

[8] Nozawa, M., Maeda, H., (2006), Effects of Lovelock terms on the final fate of gravitational collapse: analysis in dimensionally continued gravity, Class. Quant. Gravity, 21, pp. 1179-1800

[9] Tsamparlis, M., Apostolopoulous, P. S. (2000), Symmetries of Bianchi I space-times, Journal of Math. Physics, 41(1), pp. 7573-7588.

**Ghulam Shabbir** received his M. Sc. & M. Phil. degrees in Mathematics from Quaid-i-Azam University Islamabad Pakistan in 1993 and 1995, and Ph. D. degree in Applied Mathematics from Department of Mathematical Sciences, Aberdeen University, Aberdeen, U. K. in 2001, respectively. He started his teaching career as a Lecturer in the Department of Mathematics, The Islamia University Bahawalpur, Pakistan in 1996. He joined the Faculty of Engineering Sciences, GIK Institute, Pakistan as an Assistant Professor in August 2002. The research area of Dr. Shabbir is Symmetries in General Relativity.

**Suhail Khan** completed his M.Sc. Mathematics from Gomal University D. I. Khan in the year 2000 and started his career as university teacher at University of Peshawar on contract basis. In 2003, he has been selected as lecturer in Higher Education Department by NWFP Public Service Commission. In 2005, Higher Education Commission of Pakistan awarded him Indigenous Scholarship for Ph. D in the field of Mathematics. Soon after accepting the scholarship he joined Ghulam Ishaq Khan (GIK) Institute of Engineering Sciences and Technology as Ph. D scholar. In 2007, he obtained his MS degree from GIK and now pursuing his Ph. D in the field of General relativity.