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HILFER AND HADAMARD FRACTIONAL DIFFERENTIAL EQUATIONS IN FRÉCHET SPACES

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Abstract. In this paper, by using the fixed point theory and the technique relying on the concept of measure of noncompactness in Fréchet spaces, we prove some existence and Ulam–Hyers–Rassias stability results for some Hilfer and Hilfer–Hadamard fractional differential equations.

Keywords: fractional differential equation, left-sided mixed integral, fractional order, Hilfer fractional derivative, measure of noncompactness, Fréchet space, existence, Ulam stability, fixed point.

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1. Introduction

Fractional calculus is a branch of classical mathematics, which deals with the generalization of operations of differentiation and integration to fractional order. Fractional differential equations have recently been applied in various areas of engineering, mathematics, physics and bio-engineering, and other applied sciences [15, 34]. For some fundamental results in the theory of fractional calculus and fractional differential equations we refer the reader to the monographs [4, 5, 6, 21, 30, 38], the papers [3, 7, 8, 24, 31, 32, 33] and the references therein. Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations with Hilfer fractional derivative; [12, 13, 15, 19, 35, 37], and the references therein.

The stability of functional equations was originally raised by [36] and by [16]. Thereafter, this type of stability is called the Ulam–Hyers stability. In [27] was provided a remarkable generalization of the Ulam–Hyers stability of mappings by considering variables. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Considerable attention has been given to the study of the Ulam–Hyers and Ulam–Hyers–Rassias stability of all kinds of functional equations; one can see the monographs [6, 17], the papers [1, 2, 3, 7, 8, 23, 28, 29] and discussed the Ulam–Hyers stability for operatorial equations and inclusions. More details from historical point of view, and recent developments of such stabilities are reported in [18, 28].

Recently, in [10, 11] the authors applied the measure of noncompactness to some classes of functional integral equations in Fréchet spaces. Motivated by the above papers, we discuss

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the existence and the Ulam stability of solutions for the following problem of Hilfer fractional differential equations of the form

$$\begin{cases} (D_0^{\alpha,\beta}u)(t) = f(t,u(t));\ t \in \mathbb{R}_+ := [0,+\infty), \\ \\ (I_0^{1-\gamma}u)(0) = \phi, \end{cases} \tag{1}$$

where $\alpha \in (0,1),\ \beta \in [0,1],\ \gamma = \alpha + \beta - \alpha\beta,\ T > 0,\ \phi \in E,\ f : \mathbb{R}_+ \times E \to E$ is a given function, E is a real (or complex) Banach space with a norm $\|\cdot\|$, $I_0^{1-\gamma}$ is the left-sided mixed Riemann–Liouville integral of order $1-\gamma$, and $D_0^{\alpha,\beta}$ is the generalized Riemann–Liouville derivative operator of order α and type β, introduced by Hilfer in [15].

Next, we consider the following problem of Hilfer–Hadamard fractional differential equations of the form

$$\begin{cases} ({}^H D_1^{\alpha,\beta}u)(t) = g(t,u(t));\ t \in [1,\infty), \\ \\ ({}^H I_1^{1-\gamma}u)(1) = \phi_0, \end{cases} \tag{2}$$

where $\alpha \in (0,1),\ \beta \in [0,1],\ \gamma = \alpha + \beta - \alpha\beta,\ T > 1,\ \phi_0 \in E,\ g : [1,\infty) \times E \to E$ is a given function, ${}^H I_1^{1-\gamma}$ is the left-sided mixed Hadamard integral of order $1-\gamma$, and ${}^H D_1^{\alpha,\beta}$ is the Hilfer–Hadamard fractional derivative of order α and type β.

This paper initiates the use of the measure of noncompactness in Fréchet spaces for the Ulam stability of problems (1) and (2).

2. Preliminaries

Let C be the Banach space of all continuous functions v from $I := [0,T];\ T > 0$ into E with the supremum (uniform) norm

$$\|v\|_\infty := \sup_{t\in I} \|v(t)\|.$$

As usual, $AC(I)$ denotes the space of absolutely continuous functions from I into E. By $L^1(I)$, we denote the space of Bochner-integrable functions $v : I \to E$ with the norm

$$\|v\|_1 = \int_0^T |v(t)|dt.$$

By $C_\gamma(I)$ and $C_\gamma^1(I)$, we denote the weighted spaces of continuous functions defined by

$$C_\gamma(I) = \{w : (0,T] \to E : t^{1-\gamma}w(t) \in C\},$$

with the norm

$$\|w\|_{C_\gamma} := \sup_{t\in I} |t^{1-\gamma}w(t)|,$$

and

$$C_\gamma^1(I) = \{w \in C : \frac{dw}{dt} \in C_\gamma\},$$

with the norm

$$\|w\|_{C_\gamma^1} := \|w\|_\infty + \|w'\|_{C_\gamma}.$$

Let $C(\mathbb{R}_+)$ be the Fréchet space of all continuous functions v from $\mathbb{R}_+$ into E, equipped with the family of seminorms

$$\|v\|_n = \sup_{t\in[0,n]} \|v(t)\|;\ n \in \mathbb{N}.$$

In what follows, we will work in the weighted Fréchet space $X := C_\gamma(\mathbb{R}_+)$ of continuous functions defined by

$$X = \{w : (0,\infty) \to E : t^{1-\gamma} w(t) \in C(\mathbb{R}_+)\},$$

equipped with the family of seminorms

$$\|v\|_n = \sup_{t\in[0,n]} \|t^{1-\gamma} v(t)\|;\ n \in \mathbb{N},$$

and the distance

$$d(u,v) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|u-v\|_n}{1+\|u-v\|_n};\ u,v \in X.$$

Definition 2.1. *A nonempty subset $B \subset X$ is said to be bounded if*

$$\sup_{v\in X} \|v\|_n < \infty;\ for\ n \in \mathbb{N}.$$

We recall the following definition of the notion of a sequence of measures of noncompactness [12, 13].

Definition 2.2. *Let $\mathcal{M}_X$ be the family of all nonempty and bounded subsets of a Fréchet space X. A family of functions $\{\mu_n\}_{n\in\mathbb{N}}$ where $\mu_n : \mathcal{M}_X \to [0,\infty)$ is said to be a family of measures of noncompactness in the real Fréchet space X if it satisfies the following conditions for all $B, B_1, B_2 \in \mathcal{M}_X$:*

(a) $\{\mu_n\}_{n\in\mathbb{N}}$ is full, that is: $\mu_n(B) = 0$ for $n \in \mathbb{N}$ if and only if B is precompact,
(b) $\mu_n(B_1) \le \mu_n(B_2)$ for $B_1 \subset B_2$ and $n \in \mathbb{N}$,
(c) $\mu_n(ConvB) = \mu_n(B)$ for $n \in \mathbb{N}$,
(d) If $\{B_i\}_{i=1,\cdots}$ is a sequence of closed sets from $\mathcal{M}_X$ such that $B_{i+1} \subset B_i;\ i = 1,\cdots$ and if $\lim_{i\to\infty} \mu_n(B_i) = 0$, for each $n \in \mathbb{N}$, then the intersection set $B_\infty := \cap_{i=1}^{\infty} B_i$ is nonempty.

Some Properties

(e) We call the family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$ to be homogeneous if $\mu_n(\lambda B) = |\lambda|\mu_n(B)$; for $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$.
(f) If the family $\{\mu_n\}_{n\in\mathbb{N}}$ satisfied the condition $\mu_n(B_1 \cup B_2) \le \mu_n(B_1) + \mu_n(B_2)$, for $n \in \mathbb{N}$, it is called subadditive.
(g) It is sublinear if both conditions (e) and (f) hold.
(h) We say that the family of measures $\{\mu_n\}_{n\in\mathbb{N}}$ has the maximum property if

$$\mu_n(B_1 \cup B_2) = \max\{\mu_n(B_1), \mu_n(B_2)\},$$

(i) The family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$ is said to be regular if if the conditions (a), (g) and (h) hold; (full sublinear and has maximum property).

Example 2.1. F*or $B \in \mathcal{M}_X,\ x \in B,\ n \in \mathbb{N}$ and $\epsilon > 0$, let us denote by $\omega^n(x,\epsilon)$ for $n \in \mathbb{N}$; the modulus of continuity of the function x on the interval $[0,n]$; that is,*

$$\omega^n(x,\epsilon) = \sup\{|x(t) - x(s)| : t,s \in [0,n], |t-s| \le \epsilon\}.$$

Further, let us put

$$\omega^n(B,\epsilon) = \sup\{\omega^n(x,\epsilon) : x \in B\},$$

$$\omega_0^n(B) = \lim_{\epsilon\to 0^+} \omega^n(B,\epsilon),$$

$$\bar{\alpha}^n(B) = \sup_{t\in[0,n]} \alpha(B(t)) := \sup_{t\in[0,n]} \alpha(\{x(t) : x \in B\}),$$

and

$$\beta_n(B) = \omega_0^n(B) + \bar{\alpha}^n(B).$$

The family of mappings $\{\beta_n\}_{n\in\mathbb{N}}$ *where* $\beta_n : \mathcal{M}_X \to [0,\infty)$, *satisfies the conditions (a)-(d) from Definition 2.2.*

Now, we give some results and properties of fractional calculus.

Definition 2.3. [5, 21, 30] *The left-sided mixed Riemann–Liouville integral of order* $r > 0$ *of a function* $w \in L^1(I)$ *is defined by*

$$(I_0^r w)(t) = \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}w(s)ds;\ for\ a.e.\ t \in I,$$

where $\Gamma(\cdot)$ *is the (Euler's) Gamma function defined by*

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1}e^{-t}dt;\ \xi > 0.$$

Notice that for all $r, r_1, r_2 > 0$ and each $w \in C$, we have $I_0^r w \in C$, and

$$(I_0^{r_1} I_0^{r_2} w)(t) = (I_0^{r_1+r_2}w)(t);\ for\ a.e.\ t \in I.$$

Definition 2.4. [5, 21, 30] *The Riemann–Liouville fractional derivative of order* $r \in (0,1]$ *of a function* $w \in L^1(I)$ *is defined by*

$$\begin{aligned}(D_0^r w)(t) &= \left(\frac{d}{dt}I_0^{1-r}w\right)(t)\\ &= \frac{1}{\Gamma(1-r)}\frac{d}{dt}\int_0^t (t-s)^{-r}w(s)ds;\ for\ a.e.\ t \in I.\end{aligned}$$

Let $r \in (0,1]$, $\gamma \in [0,1)$ *and* $w \in C_{1-\gamma}(I)$. *Then the following expression leads to the left inverse operator as follows.*

$$(D_0^r I_0^r w)(t) = w(t);\ for\ all\ t \in (0,T].$$

Moreover, if $I_0^{1-r}w \in C^1_{1-\gamma}(I)$, *then the following composition is proved in* [30]

$$(I_0^r D_0^r w)(t) = w(t) - \frac{(I_0^{1-r}w)(0^+)}{\Gamma(r)}t^{r-1};\ for\ all\ t \in (0,T].$$

Definition 2.5. [5, 21, 30] *The Caputo fractional derivative of order* $r \in (0,1]$ *of a function* $w \in L^1(I)$ *is defined by*

$$\begin{aligned}({}^cD_0^r w)(t) &= \left(I_0^{1-r}\frac{d}{dt}w\right)(t)\\ &= \frac{1}{\Gamma(1-r)}\int_0^t (t-s)^{-r}\frac{d}{ds}w(s)ds;\ for\ a.e.\ t \in I.\end{aligned}$$

In [15], *Hilfer studied applications of a generalized fractional operator having the Riemann–Liouville and the Caputo derivatives as specific cases (see also* [19, 35].

Definition 2.6. *(Hilfer derivative). Let* $\alpha \in (0,1)$, $\beta \in [0,1]$, $w \in L^1(I)$, $I_0^{(1-\alpha)(1-\beta)} \in AC(I)$. *The Hilfer fractional derivative of order* α *and type* β *of* w *is defined as*

$$(D_0^{\alpha,\beta}w)(t) = \left(I_0^{\beta(1-\alpha)}\frac{d}{dt}I_0^{(1-\alpha)(1-\beta)}w\right)(t);\ for\ a.e.\ t \in I. \tag{3}$$

Some Properties. Let $\alpha \in (0,1),\ \beta \in [0,1],\ \gamma = \alpha + \beta - \alpha\beta$, and $w \in L^1(I)$.

1. The operator $(D_0^{\alpha,\beta} w)(t)$ can be written as

$$(D_0^{\alpha,\beta} w)(t) = \left(I_0^{\beta(1-\alpha)} \frac{d}{dt} I_0^{1-\gamma} w\right)(t) = \left(I_0^{\beta(1-\alpha)} D_0^{\gamma} w\right)(t);\ for\ a.e.\ t \in I.$$

Moreover, the parameter γ satisfies

$$\gamma \in (0,1],\ \gamma \geq \alpha,\ \gamma > \beta,\ 1-\gamma < 1-\beta(1-\alpha).$$

2. The generalization (3) for $\beta = 0$, coincides with the Riemann–Liouville derivative and for $\beta = 1$ with the Caputo derivative.

$$D_0^{\alpha,0} = D_0^{\alpha},\ and\ D_0^{\alpha,1} = {}^cD_0^{\alpha}.$$

3. If $D_0^{\beta(1-\alpha)} w$ exists and in $L^1(I)$, then

$$(D_0^{\alpha,\beta} I_0^{\alpha} w)(t) = (I_0^{\beta(1-\alpha)} D_0^{\beta(1-\alpha)} w)(t);\ for\ a.e.\ t \in I.$$

Furthermore, if $w \in C_\gamma(I)$ and $I_0^{1-\beta(1-\alpha)} w \in C_\gamma^1(I)$, then

$$(D_0^{\alpha,\beta} I_0^{\alpha} w)(t) = w(t);\ for\ a.e.\ t \in I.$$

4. If $D_0^{\gamma} w$ exists and in $L^1(I)$, then

$$(I_0^{\alpha} D_0^{\alpha,\beta} w)(t) = (I_0^{\gamma} D_0^{\gamma} w)(t) = w(t) - \frac{I_0^{1-\gamma}(0^+)}{\Gamma(\gamma)} t^{\gamma-1};\ for\ a.e.\ t \in I.$$

Corollary 2.1. *Let $h \in C_\gamma(I)$. Then the Cauchy problem*

$$\begin{cases} (D_0^{\alpha,\beta} u)(t) = h(t);\ t \in I, \\ \\ (I_0^{1-\gamma} u)(t)|_{t=0} = \phi, \end{cases}$$

has the following unique solution

$$w(t) = \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + (I_0^{\alpha} h)(t).$$

From the above corollary, we conclude with the following lemma.

Lemma 2.1. *Let $f : I \times E \to E$ be such that $f(\cdot, u(\cdot)) \in C_\gamma$ for any $u \in C_\gamma$. Then problem (1) is equivalent to the problem of the solutions of the integral equation*

$$u(t) = \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + (I_0^{\alpha} f(\cdot, u(\cdot)))(t).$$

Now, we consider the Ulam stability for the problem (1). Let $\epsilon > 0$ and $\Phi_n : [0,n] \to [0,\infty)$ for $n \in \mathbb{N}$ be a continuous function. We consider the following inequalities

$$\|(D_0^{\alpha,\beta} u)(t) - f(t, u(t))\| \leq \epsilon_n;\ t \in [0,n]. \tag{4}$$

$$\|(D_0^{\alpha,\beta} u)(t) - f(t, u(t))\| \leq \Phi_n(t);\ t \in [0,n]. \tag{5}$$

$$\|(D_0^{\alpha,\beta} u)(t) - f(t, u(t))\| \leq \epsilon_n \Phi_n(t);\ t \in [0,n]. \tag{6}$$

Definition 2.7. [5, 28] *The problem* (1) *is Ulam–Hyers stable if there exists a real number $c_{n,f} > 0;\ n \in \mathbb{N}$ such that for each $\epsilon > 0$ and for each solution $u \in X$ of the inequality* (4) *there exists a solution $v \in X$ of* (1) *with*

$$\|u(t) - v(t)\| \leq \epsilon c_{n,f};\ t \in [0,n].$$

Definition 2.8. [5, 28] *The problem* (1) *is generalized Ulam–Hyers stable if there exists* $c_{n,f} \in C([0,\infty),[0,\infty))$ *with* $c_{n,f}(0) = 0$ *such that for each* $\epsilon_n > 0$ *and for each solution* $u \in X$ *of the inequality* (4) *there exists a solution* $v \in X$ *of* (1) *with*

$$\|u(t) - v(t)\| \leq c_{n,f}(\epsilon_n);\ t \in [0,n].$$

Definition 2.9. [5, 28] *The problem* (1) *is Ulam–Hyers–Rassias stable with respect to* Φ_n *if there exists a real number* $c_{n,f,\Phi_n} > 0$ *such that for each* $\epsilon_n > 0$ *and for each solution* $u \in X$ *of the inequality* (6) *there exists a solution* $v \in X$ *of* (1) *with*

$$\|u(t) - v(t)\| \leq \epsilon_n c_{n,f,\Phi_n} \Phi_n(t);\ t \in [0,n].$$

Definition 2.10. [5, 28] *The problem (1) is generalized Ulam–Hyers–Rassias stable with respect to* Φ_n *if there exists a real number* $c_{n,f,\Phi_n} > 0$ *such that for each solution* $u \in X$ *of the inequality* (5) *there exists a solution* $v \in X$ *of* (1) *with*

$$\|u(t) - v(t)\| \leq c_{n,f,\Phi_n} \Phi_n(t);\ t \in [0,n].$$

Remark 2.1. *It is clear that*

(i) Definition 2 $\Rightarrow$ Definition 2,
(ii) Definition 2 $\Rightarrow$ Definition 2,
(iii) Definition 2 for $\Phi(\cdot) = 1$ $\Rightarrow$ Definition 2.

One can have similar remarks for the inequalities (4) and (6).

Lemma 2.2. [9] *If* Y *is a bounded subset of Fréchet space* X, *then for each* $\epsilon > 0$, *there is a sequence* $\{y_k\}_{k=1}^{\infty} \subset Y$ *such that*

$$\mu_n(Y) \leq 2\mu_n(\{y_k\}_{k=1}^{\infty}) + \epsilon;\ \textit{for } n \in \mathbb{N}.$$

Lemma 2.3. [22] *If* $\{u_k\}_{k=1}^{\infty} \subset L^1(I)$ *is uniformly integrable, then* $\mu_n(\{u_k\}_{k=1}^{\infty})$ *is measurable for* $n \in \mathbb{N}$, *and*

$$\mu_n\left(\left\{\int_0^t u_k(s)ds\right\}_{k=1}^{\infty}\right) \leq 2\int_0^t \mu_n(\{u_k(s)\}_{k=1}^{\infty})ds,$$

for each $t \in [0,n]$.

Definition 2.11. *Let* Ω *be a nonempty subset of a Fréchet space* X, *and let* $A : \Omega \to X$ *be a continuous operator which transforms bounded subsets of onto bounded ones. One says that* A *satisfies the Darbo condition with constants* $(k_n)_{n\in\mathbb{N}}$ *with respect to a family of measures of noncompactness* $\{\mu_n\}_{n\in\mathbb{N}}$, *if*

$$\mu_n(A(B)) \leq k_n \mu_n(B)$$

for each bounded set $B \subset \Omega$ *and* $n \in \mathbb{N}$.
If $k_n < 1;\ n \in \mathbb{N}$ *then* A *is called a contraction with respect* to $\{\mu_n\}_{n\in\mathbb{N}}$.

In the sequel we will make use of the following generalization of the classical Darbo fixed point theorem for Fréchet spaces.

Theorem 2.1. [10, 11] *Let* Ω *be a nonempty, bounded, closed, and convex subset of a Fréchet space* F *and let* $V : \Omega \to \Omega$ *be a continuous mapping. Suppose that* V *is a contraction with respect to a family of measures of noncompactness* $\{\mu_n\}_{n\in\mathbb{N}}$. *Then* V *has at least one fixed point in the set* Ω.

We recall Gronwall's lemma for singular kernels.

Lemma 2.4.(Gronwall lemma) [Lemma 7.1.1, [14]] *Let $\upsilon : I \to [0,\infty)$ be a real function and $\omega(t)$ be a measurable, nonnegative and locally integrable function on I. If there are constants $c > 0$ and $0 < \alpha < 1$ such that*

$$\upsilon(t) \leq \omega(t) + c\int_0^t \frac{\upsilon(s)}{(t-s)^\alpha}ds,$$

then there exists a constant $\delta = \delta(\alpha)$ such that

$$\upsilon(t) \leq \omega(t) + \delta c\int_0^t \frac{\omega(s)}{(t-s)^\alpha}ds,$$

for every $t \in I$.

3. Hilfer fractional differential equations in Fréchet spaces

In this section, we are concerned with the existence and the generalized Ulam–Hyers–Rassias stability of our problem (1).

Definition 3.1. *By a solution of the problem (1) we mean a measurable function $u \in X$ that satisfies the condition $(I_0^{1-\gamma}u)(0^+) = \phi$, and the equation $(D_0^{\alpha,\beta}u)(t) = f(t,u(t))$ on $\mathbb{R}_+$.*

The following hypotheses will be used in the sequel.

(H_1) The function $t \mapsto f(t,u)$ is measurable on I for each $u \in E$, and the function $u \mapsto f(t,u)$ is continuous on E for a.e. $t \in \mathbb{R}_+$,

(H_2) There exists a continuous function $p : \mathbb{R}_+ \to [0,\infty)$ such that

$$\|f(t,u) - f(t,v)\| \leq \frac{p(t)\|u-v\|}{1+\|u-v\|};\ \text{for a.e. } t \in \mathbb{R}_+,\ \text{and each } u,v \in E,$$

(H_3) For each bounded and measurable set $B \subset E$ and for each $t \in \mathbb{R}_+$, we have

$$\mu(f(t,B)) \leq p(t)\mu(B).$$

(H_4) For any $n \in \mathbb{N}$, there exists $\lambda_{\Phi_n} > 0$ such that for each $t \in [0,n]$, we have

$$(I_0^\alpha \Phi_n)(t) \leq \lambda_{\Phi_n}\Phi_n(t).$$

For each $n \in \mathbb{N}$, we set

$$p_n^* = \sup_{t\in[0,n]} p(t),\ f_n^* = \sup_{t\in[0,n]} |f(t,0)|.$$

Theorem 3.1. *Assume that the hypotheses $(H_1)-(H_3)$ hold. If for $n \in \mathbb{N}$, we have*

$$\ell_n := \frac{4p_n^* n^\alpha}{\Gamma(1+\alpha)} < 1, \tag{7}$$

where

$$p_n^* = \sup_{t\in[0,n]} p(t),$$

then the problem (1) has at least one solution in X. Furthermore, if the hypothesis (H_4) holds, then the problem (1) is generalized Ulam–Hyers–Rassias stable.

Proof. Consider the operator $N : X \to X$ defined by:

$$(Nu)(t) = \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + \int_0^t (t-s)^{\alpha-1} \frac{f(s,u(s))}{\Gamma(\alpha)} ds. \tag{8}$$

Clearly, the fixed points of the operator N are solution of the problem (1).

For each $n \in \mathbb{N}$, we set

$$f_n^* = \sup_{t\in[0,n]} |f(t,0)|.$$

For any $n \in \mathbb{N}$, and each $u \in X$ and $t \in [0,n]$ we have

$$\|t^{1-\gamma}(Nu)(t)\| \leq \frac{\|\phi\|}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s,u(s))\| ds$$

$$\leq \frac{\|\phi\|}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s,0)\| ds + \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s,u(s)) - f(s,0)\| ds$$

$$\leq \frac{\|\phi\|}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (\|f(s,0)\| + p(s)) ds \leq \frac{\|\phi\|}{\Gamma(\gamma)} + \frac{(f_n^* + p_n^*)T^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds$$

$$\leq \frac{\|\phi\|}{\Gamma(\gamma)} + \frac{(f_n^* + p_n^*)n^{1-\gamma+\alpha}}{\Gamma(1+\alpha)}.$$

Thus

$$\|N(u)\|_n \leq \frac{\|\phi\|}{\Gamma(\gamma)} + \frac{(f_n^* + p_n^*)n^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} := R_n. \tag{9}$$

This proves that N transforms the ball $B_{R_n} := B(0,R_n) = \{w \in X : \|w\|_n \leq R_n\}$ into itself. We shall show that the operator $N : B_R \to B_R$ satisfies all the assumptions of Theorem 2.1. The proof will be given in several steps.

Step 1. *$N : B_R \to B_R$ is continuous.*
Let $\{u_k\}_{k\in N}$ be a sequence such that $u_k \to u$ in B_{R_n}. Then, for each $t \in [0,n]$, we have

$$\|t^{1-\gamma}(Nu_k)(t) - t^{1-\gamma}(Nu)(t)\| \leq \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s,u_k(s)) - f(s,u(s))\| ds$$

$$\leq \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s) \|u_k(s) - u(s)\| ds \leq \frac{p_n^* T^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|u_k(s) - u(s)\| ds.$$

Hence

$$\|t^{1-\gamma}(Nu_k)(t) - t^{1-\gamma}(Nu)(t)\| \leq \frac{p_n^* n^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|u_k(s) - u(s)\| ds. \tag{10}$$

Since $u_k \to u$ as $k \to \infty$, then equation (10) implies

$$\|N(u_k) - N(u)\|_n \to 0 \quad \text{as } k \to \infty.$$

Step 2. *$N(B_{R_n})$ is bounded.*
Since $N(B_{R_n}) \subset B_{R_n}$ and B_{R_n} is bounded, then $N(B_{R_n})$ is bounded.

Step 3. *For each bounded subset D of B_{R_n}, $\mu_n(N(D)) \leq \ell_n \mu_n(D)$.*
From Lemmas 2.2., 2.3., for any $D \subset B_{R_n}$ and any $\epsilon > 0$, there exists a sequence $\{u_k\}_{k=0}^{\infty} \subset D$, such that for all $t \in [0, n]$, we have

$$\mu((ND)(t)) = \mu\left(\left\{\frac{\phi}{\Gamma(\gamma)}t^{\gamma-1} + \int_0^t (t-s)^{\alpha-1}\frac{f(s,u(s))}{\Gamma(\alpha)}ds;\ u \in D\right\}\right)$$

$$\leq\ 2\mu\left(\left\{\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u_k(s))ds\right\}_{k=1}^{\infty}\right) + \epsilon \leq 4\int_0^t \mu\left(\left\{\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u_k(s))\right\}_{k=1}^{\infty}\right) ds + \epsilon$$

$$\leq\ 4\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\mu\left(\{f(s,u_k(s)\}_{k=1}^{\infty}\right) ds + \epsilon \leq 4\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)\mu\left(\{u_k(s)\}_{k=1}^{\infty}\right) ds + \epsilon$$

$$\leq\ \left(4\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)ds\right)\mu\left(\{u_k\}_{k=1}^{\infty}\right) + \epsilon \leq \left(4\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)ds\right)\mu_n(D) + \epsilon$$

$$\leq\ \frac{4p_n^* n^\alpha}{\Gamma(1+\alpha)}\mu_n(D) + \epsilon \leq \ell_n \mu_n(D) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, then

$$\mu_n(N(B)) \leq \ell_n \mu_n(B).$$

As a consequence of steps 1 to 3 together with Theorem 2.1, we can conclude that N has at least one fixed point in B_{R_n} which is a solution of problem (1).

Step 4. *The generalized Ulam–Hyers–Rassias stability.*
Let u be a solution of the inequality (5), and let us assume that v is a solution of problem (1). Thus, we have

$$v(t) = \frac{\phi}{\Gamma(\gamma)}t^{\gamma-1} + \int_0^t (t-s)^{\alpha-1}\frac{f(s,v(s))}{\Gamma(\alpha)}ds.$$

From the inequality (5), for any $n \in \mathbb{N}$ and each $t \in [0, n]$, we have

$$\left\|u(t) - \frac{\phi}{\Gamma(\gamma)}t^{\gamma-1} - \int_0^t (t-s)^{\alpha-1}\frac{f(s,u(s))}{\Gamma(\alpha)}ds\right\| \leq (I_0^\alpha \Phi)(t).$$

From hypotheses (H_2) and (H_4), for each $t \in [0, n]$, we get

$$\|u(t) - v(t)\| \leq \left\|u(t) - \frac{\phi}{\Gamma(\gamma)}t^{\gamma-1} - \int_0^t (t-s)^{\alpha-1}\frac{f(s,u(s))}{\Gamma(\alpha)}ds\right\|$$

$$+\int_0^t (t-s)^{\alpha-1}\frac{\|f(s,u(s)) - f(s,v(s))\|}{\Gamma(\alpha)}ds \leq (I_0^\alpha \Phi)(t) + \int_0^t (t-s)^{\alpha-1}\frac{p(s)\|u(s) - v(s)\|}{\Gamma(\alpha)}ds$$

$$\leq \lambda_\phi \Phi(t) + \frac{p_n^*}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|u(s) - v(s)\|ds.$$

From Lemma 2.4., there exists a constant $\delta = \delta(\alpha)$ such that

$$\|u(t) - v(t)\| \leq \lambda_\phi[\Phi(t) + \frac{\delta p_n^*}{\Gamma(\alpha)} + \int_0^t (t-s)^{\alpha-1}\Phi(s)ds] \leq [1 + \delta p_n^* \lambda_\Phi]\lambda_\phi \Phi(t) := c_{n,f,\Phi}\Phi(t).$$

Hence, the problem (1) is generalized Ulam–Hyers–Rassias stable.

4. Hilfer–Hadamard fractional differential equations

Now, we are concerned with the existence and the generalized Ulam–Hyers–Rassias stability of our problem (2).

Set $C := C([1,T])$, and denote the weighted space of continuous functions defined by

$$C_{\gamma,\ln}([1,T]) = \{w(t) : (\ln t)^{1-\gamma} w(t) \in C\},$$

with the norm

$$\|w\|_{C_{\gamma,\ln}} := \sup_{t\in[1,T]} \|(\ln t)^{1-r} w(t)\|.$$

Let $C([1,\infty))$ be the Fréchet space of all continuous functions v from $[1,\infty)$ into E, equipped with the family of seminorms

$$\|v\|_n = \sup_{t\in[1,n]} \|v(t)\|; \ n \in \mathbb{N}^*.$$

Now, we will work in the weighted Fréchet space $F := C_{\gamma,\ln}$ of continuous functions defined by

$$F = \{w : (1,\infty) \to E : (\ln t)^{1-\gamma} w(t) \in C([1,\infty))\},$$

equipped with the family of seminorms

$$\|v\|_n = \sup_{t\in[1,n]} \|(\ln t)^{1-\gamma} v(t)\|; \ n \in \mathbb{N}^*,$$

Let us recall some definitions and properties of Hadamard fractional integration and differentiation. We refer to [21] for a more detailed analysis.

Definition 4.1. [21] *(Hadamard fractional integral). The Hadamard fractional integral of order $q > 0$ for a function $g \in L^1([1,T])$, is defined as*

$$({}^H I_1^q g)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\ln \frac{x}{s}\right)^{q-1} \frac{g(s)}{s} ds,$$

provided the integral exists.

Example 4.1. Let $0 < q < 1$. *Then*

$${}^H I_1^q \ln t = \frac{1}{\Gamma(2+q)} (\ln t)^{1+q}; \ for \ a.e. \ t \in [0,e].$$

Set

$$\delta = x\frac{d}{dx}, \ q > 0, \ n = [q] + 1,$$

and

$$AC_\delta^n := \{u : [1,T] \to E : \delta^{n-1}[u(x)] \in AC(I)\}.$$

Analogous to the Riemann–Liouville fractional calculus, the Hadamard fractional derivative is defined in terms of the Hadamard fractional integral in the following way:

Definition 4.2. [21] *(Hadamard fractional derivative). The Hadamard fractional derivative of order $q > 0$ applied to the function $w \in AC_\delta^n$ is defined as*

$$({}^H D_1^q w)(x) = \delta^n ({}^H I_1^{n-q} w)(x).$$

In particular, if $q \in (0, 1]$, then

$$({}^H D_1^q w)(x) = \delta({}^H I_1^{1-q} w)(x).$$

Example 4.2. Let $0 < q < 1$. *Then*

$${}^H D_1^q \ln t = \frac{1}{\Gamma(2-q)} (\ln t)^{1-q}, \ for \ a.e. \ t \in [0, e].$$

It has been proved (see e.g. Kilbas [[20], Theorem 4.8]) that in the space $L^1(I)$, the Hadamard fractional derivative is the left-inverse operator to the Hadamard fractional integral, i.e.

$$({}^H D_1^q)({}^H I_1^q w)(x) = w(x).$$

From Theorem 2.3 of [21], we have

$$({}^H I_1^q)({}^H D_1^q w)(x) = w(x) - \frac{({}^H I_1^{1-q} w)(1)}{\Gamma(q)} (\ln x)^{q-1}.$$

Analogous to the Hadamard fractional calculus, the Caputo–Hadamard fractional derivative is defined in the following way:

Definition 4.3. *(Caputo-Hadamard fractional derivative). The Caputo–Hadamard fractional derivative of order $q > 0$ applied to the function $w \in AC_\delta^n$ is defined as*

$$({}^{Hc} D_1^q w)(x) = ({}^H I_1^{n-q} \delta^n w)(x).$$

In particular, if $q \in (0, 1]$, then

$$({}^{Hc} D_1^q w)(x) = ({}^H I_1^{1-q} \delta w)(x).$$

From the Hadamard fractional integral, the Hilfer–Hadamard fractional derivative (introduced for the first time in [25]) is defined in the following way:

Definition 4.4. *(Hilfer–Hadamard fractional derivative). Let $\alpha \in (0, 1)$, $\beta \in [0, 1]$, $\gamma = \alpha + \beta - \alpha\beta$, $w \in L^1(I)$, and ${}^H I_1^{(1-\alpha)(1-\beta)} w \in AC^1(I)$. The Hilfer–Hadamard fractional derivative of order α and type β applied to the function w is defined as*

$$\begin{aligned} ({}^H D_1^{\alpha,\beta} w)(t) &= \left({}^H I_1^{\beta(1-\alpha)} ({}^H D_1^\gamma w)\right)(t) \\ &= \left({}^H I_1^{\beta(1-\alpha)} \delta ({}^H I_1^{1-\gamma} w)\right)(t); \ for \ a.e. \ t \in [1, T]. \end{aligned} \tag{11}$$

This new fractional derivative (11) may be viewed as interpolating the Hadamard fractional derivative and the Caputo–Hadamard fractional derivative. Indeed for $\beta = 0$ this derivative reduces to the Hadamard fractional derivative and when $\beta = 1$, we recover the Caputo–Hadamard fractional derivative.

$${}^H D_1^{\alpha,0} = {}^H D_1^\alpha, \ and \ {}^H D_1^{\alpha,1} = {}^{Hc} D_1^\alpha.$$

From Theorem 21 in [26], we concluded the following lemma

Lemma 4.1. *Let $g : [1, T] \times E \to E$ be such that $g(\cdot, u(\cdot)) \in C_{\gamma,\ln}([1, T])$ for any $u \in C_{\gamma,\ln}([1, T])$. Then Then problem (2) is equivalent to the following Volterra integral equation*

$$u(t) = \frac{\phi_0}{\Gamma(\gamma)} (\ln t)^{\gamma-1} + ({}^H I_1^\alpha g(\cdot, u(\cdot)))(t).$$

Definition 4.5. *By a solution of the problem (2) we mean a measurable function $u \in C_{\gamma,\ln}$ that satisfies the condition $({}^H I_1^{1-\gamma} u)(1^+) = \phi_0$, and the equation $({}^H D_1^{\alpha,\beta} u)(t) = g(t, u(t))$ on $[1, T]$.*

Now we give (without proof) similar existence and Ulam stability results for problem (2). Let us introduce the following hypotheses:

(H_1') The function $t \mapsto g(t, u)$ is measurable on $[1, \infty)$ for each $u \in E$, and the function $u \mapsto g(t, u)$ is continuous on E for a.e. $t \in [1, \infty)$,

(H_2') There exists a continuous function $q : [1, \infty) \to [0, \infty)$ such that

$$\|g(t, u) - g(t, v)\| \leq \frac{q(t)\|u - v\|}{1 + \|u - v\|}; \text{ for a.e. } t \in [1, \infty), \text{ and each } u, v \in E,$$

(H_3') For each bounded and measurable set $B \subset F$ and for each $t \in [1, \infty)$, we have

$$\mu(g(t, B)) \leq q(t)\mu(B),$$

(H_4') For any $n \in \mathbb{N}^*$, there exists $\lambda_{\Phi_n} > 0$ such that for each $t \in [1, n]$, we have

$$({}^H I_1^{\alpha} \Phi_n)(t) \leq \lambda_{\Phi_n} \Phi_n(t).$$

Theorem 4.1. *Assume that the hypotheses $(H_1') - (H_3')$ hold. If*

$$\ell_n^* := \frac{4q_n^*(\ln n)^{\alpha}}{\Gamma(1 + \alpha)} < 1, \tag{12}$$

where $q_n^* = \sup_{t \in [1,n]} q(t)$, then the problem (2) has at least one solution defined on I. Furthermore, if the hypothesis (H_4') holds, then the problem (2) is generalized Ulam–Hyers–Rassias stable.

5. An example

Let

$$E = l^1 = \left\{ u = (u_1, u_2, \ldots, u_k, \ldots), \sum_{k=1}^{\infty} |u_k| < \infty \right\}$$

be the Banach space with the norm

$$\|u\|_E = \sum_{k=1}^{\infty} |u_k|.$$

Consider the Hilfer fractional differential equation of the form

$$\begin{cases} (D_0^{\frac{1}{2}, \frac{1}{2}} u_k)(t) = f_k(t, u(t)); \ t \in \mathbb{R}_+, \\ \\ (I_0^{\frac{1}{4}} u_k)(t)|_{t=0} = (1, 0, \ldots, 0, \ldots), \end{cases} \tag{13}$$

where

$$\begin{cases} f_k(t, u) = \dfrac{t^{\frac{-1}{4}} u_k(t) \sin t}{64(1 + \sqrt{t})(1 + \|u\|_E)}; \ t \in (0, \infty), \\ f_k(0, u) = 0, \end{cases}$$

with

$$f = (f_1, f_2, \ldots, f_k, \ldots), \ u = (u_1, u_2, \ldots, u_k, \ldots), \ c := \frac{e^3}{8}\Gamma\left(\frac{1}{2}\right).$$

Set $\alpha = \beta = \frac{1}{2}$, then $\gamma = \frac{3}{4}$. The hypothesis (H_2) is satisfied with

$$\begin{cases} p(t) = \dfrac{t^{\frac{-1}{4}}|\sin t|}{64(1+\sqrt{t})};\ t \in (0,\infty), \\ p(0) = 0. \end{cases}$$

Hence, Theorem 3.1 implies that the problem (13) has at least one solution defined on $\mathbb{R}_+$. Also, the hypothesis (H_4) is satisfied with

$$\Phi_n(t) = e^3, \quad \lambda_{\Phi_n} = \frac{1}{\Gamma(1+\alpha)};\ n \in \mathbb{N}.$$

Indeed, for each $t \in [0, n]$ we get

$$\begin{aligned} (I_0^\alpha \Phi_n)(t) &\leq \frac{e^3}{\Gamma(1+\alpha)} \\ &= \lambda_{\Phi_n}\Phi_n(t). \end{aligned}$$

Consequently, the problem (13) is generalized Ulam–Hyers–Rassias stable.

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