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MEASURE PROBLEM ON CONJUGATION LOGICS

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ABSTRACT. In the paper we give a classification of von Neumann algebras in complex Hilbert space with conjugation operator J, study J-projections from von Neumann J-algebras of type (A) and (B), and discuss some of the measure problems on conjugation logics.

Keywords: Hilbert space, conjugation operator, projection, measure.

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1. Introduction

In [5] (see also [11]) the problem of construction of probability theory for quantum mechanics is posed. One of the basic problems related to the propositional calculus approach to the foundations of quantum mechanics is the description of probability measures (called states in physical terminology) on the set of experimentally verifiable propositions regarding a physical system. The set of propositions form an orthomodular partial ordered set, where the order is induced by the relation of implication, and called a *quantum logic*.

Many papers are devoted to quantum logic. A quantum logic [31] is a set L endowed with a partial order $\leq$ and a unary operation $^{\perp}$ such that the following conditions are satisfied (the symbols $\vee$, $\wedge$ denote the lattice-theoretic operations induced by $\leq$):

(i) L possesses a least and a greatest element, 0 and I, and $0 \neq I$;

(ii) $a \leq b$ implies $b^{\perp} \leq a^{\perp}$ for any a, $b \in L$;

(iii) $(a^{\perp})^{\perp} = a$ for any $a \in L$;

(iv) if $\{a_i\}_{i\in X}$ is a finite subset of L such that $a_i \leq a_j^{\perp}$ for $i \neq j$, then supremum $\vee_{i\in X} a_i$ exists in L.

(v) if a, $b \in L$ and $a \leq b$, then $b = a \vee (b \wedge a^{\perp})$.

Sometimes the axioms (iv) and (v) are replaced by:

(iv') if $a \leq b^{\perp}$ then there exist $a \vee b$;

(v') if a, $b \in L$ and $a \leq b$, then there exist $c \leq a^{\perp}$ such that $b = a \vee c$.

Algebraically, quantum logics are called orthomodular partially ordered sets (or, shortly, orthomodular posets). A logic L is neither distributive nor a lattice in general. Two elements a, $b \in L$ are called orthogonal if $a \leq b^{\perp}$. We will denote the orthogonality of a, b by the symbol $a \perp b$.

Let $(a_i)_{i\in I} \subset L$ be a set of mutually orthogonal elements. Assume that there exists supremum $(=a)$ of the family $\{a_i\}$. We write $a = \sum_i a_i$. The representation $a = \sum_i a_i$ is said to be *decomposition* of a.

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A mapping $\mu : L \to \mathbb{C}$ is said to be a *measure* if $\mu(a) = \sum_{i \in X} \mu(a_i)$ for any decomposition $a = \sum_i a_i$. Here, the convergence of an uncountable family of summands means that there exists a countable set of nonzero terms in the family and the usual series with these summands converges absolutely. A nonnegative measure μ is said to be *probability* measure (=*state*) if $\mu(I) = 1$.

An important example of a quantum logic is the set $B(H)^{or}$ of all orthogonal projections on a Hilbert space H. The problem of the construction of a quantum field theory sometimes leads to an indefinite metric space [6], [30]. In the indefinite case, the set of all $\mathcal{J}$-orthogonal projections serves to be an analog to the logics $B(H)^{or}$. In construction of measure theory on logics of projections it is important to know the properties and the structure of projections. $\mathcal{J}$-projections were extensively studied at [9-25].

In the present article we give a classification of von Neumann algebras in a Hilbert space with conjugation operator J. We study J-projections from von Neumann J-algebra. We discuss some of the measure problems on conjugation logics.

2. Measures on projections

Let us first formulate some known results about the measure on the projections. In the book [5], Chapter XII (see Problem 110, page 371, and Problem 88, page 547, [in Russian]) the problem of describing the measures on quantum logics of projections have been posed (see also [11], p.122 [in Russian]).

Any bounded idempotent $P \in B(H)$ is a projection on PH parallel to $(I - P)H$. Thus P is a skew projection, in general. Let L be a logic of projection with the ordering $P \leq_1 Q$ iff $PQ = QP = P$, orthogonality relation $P \perp Q$ iff $PQ = QP = 0$, and orthocomplementation $P^{\perp} = I - P$.

A measure μ is said to be *regular* measure if there exists trace-class operator A such that $\mu(P) = \mathrm{tr}(AP)$ for all $P \in L$. Let H be a real or complex Hilbert space with the inner product $(\cdot, \cdot)$.

2.1. An orthogonal case.

An important example of quantum logic is the set $B(H)^{or}$ of all orthogonal projections of a Hilbert space H. The remarkable Gleason's theorem says.

Theorem 2.1 [8]. *Let H be a Hilbert space* $\dim H \geq 3$ *and let* $\mu : B(H)^{or} \to [0, 1]$ *be a probability measure* (=*state*). *Then there exist unique nonnegative trace-class operator A such that*

$$\mu(P) = \mathrm{tr}(AP) \text{ for all } P \in B(H)^{or}.$$

By Gleason's theorem, any real or complex measure on $B(H)^{or}$, $\dim H = \infty$, is a linear combination of probability measures. Thus the class of probability measures is a major class of measures on orthogonal projections. First generalization of Gleason's Theorem on von Neumann and on Jordan algebras of bounded operators in a Hilbert space was given in the papers [13]-[15]. Another direction of generalization of Gleason's theorem cam be seen in the paper [3].

2.2. An indefinite case.

The problem of construction of a quantum field theory leads to the indefinite metric spaces (=Krein space, =$\mathcal{J}$-space) [30], [6]. In this case, the set $B(H)^{\mathcal{J}}$ of all $\mathcal{J}$-orthogonal projections serves as an analog to the logic $B(H)^{or}$. In the paper [20] we show that any unitary self-adjoint logic is a sum of logics of the type $B(H)^{or}$ and type $B(H)^{\mathcal{J}}$. We need some definitions.

2.3. **Some general properties on Krein spaces.** The following terms and properties are taken from the book [2, 4]. Let H be a linear space over $\mathbb{L}$ where $\mathbb{L} = \mathbb{R}$ or $= \mathbb{C}$. The function $[\cdot,\cdot] : H \times H \to \mathbb{L}$, x,$y \in H$ is said to be an *indefinite metric* if:

1) $[x, y] = \overline{[y, x]}$, and 2) $[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z]$, x, y, $z \in H$, α, $\beta \in \mathbb{L}$.

The space H with an indefinite metric is said to be a *Krein space* if there are two nonzero subspaces H^+, $H^- \subset H$ such that:

1) $H = H^+ + H^-$, 2) $[x, y] = 0$, $x \in H^+$, $y \in H^-$, 3) H^+ is a Hilbert space with respect to the inner product $[\cdot,\cdot]$, and H^- is a Hilbert space with respect to $-[\cdot,\cdot]$. Note by 3), $H^+ \cap H^- = \{0\}$.

The representation $H = H^+ + H^-$ is said to be a *canonical* decomposition of H and denoted by $H = H^+[\dot{+}]H^-$. Note that for any Krein space there are infinite set of the canonical decompositions [4]. Let $H = H_1^+[\dot{+}]H_1^-$ be an another decomposition. Then $\dim H^+ = \dim H_1^+$, $\dim H^- = \dim H_1^-$.

The cardinal number $\kappa = \min\{\dim H^+, \dim H^-\}$ is said to be the *indefinite rank* of H. The Krein space H is said to be *Pontryagin space* and denote by Π_κ if $\kappa < +\infty$.

There is another approach to the definition of Krein spaces. Let us present this approach.

Let H be a Hilbert space. Fix a self-adjoint unitary operator $\mathcal{J}$, $\mathcal{J} \neq \pm I$ on H. There exists unique orthogonal projections Q^+, Q^- such that $\mathcal{J} = Q^+ - Q^-$. (Without loss of generality we may assume that $\dim Q^+ \leq \dim Q^-$.) The space H with respect to the product $[x, y] := (\mathcal{J}x, y)$, x, $y \in H$ is an indefinite metric space (=*Krein space*) [4].

Let $B(H)^{pr}$ be the set of all bounded idempotents (=projections) on H. Let $P \in B(H)^{pr}$ and let $[Px, y] = [x, Py]$, x, $y \in H$. The projection P is said to be $\mathcal{J}$-*projection*. Any one-dimensional $\mathcal{J}$-projection has the form $[x, x][\cdot, x]x$, where $x \in H$ and $[x, x] = \pm 1$. A $\mathcal{J}$-projection P is said to be *positive* (*negative*) if $[Px, x] \geq 0$ ($[Px, x] \leq 0$, respectively), $\forall x \in H$. Let us denote by $B(H)^{\mathcal{J}}$ (by $B(H)^{\mathcal{J}}_+$, $B(H)^{\mathcal{J}}_-$) the set of all $\mathcal{J}$-projections (of all positive, negative, respectively, $\mathcal{J}$-projections). Note that any projection $P \in B(H)^{\mathcal{J}}$ is representable (not uniquely!) in the form $P = P_+ + P_-$, where $P_+ \in B(H)^{\mathcal{J}}_+$, $P_- \in B(H)^{\mathcal{J}}_-$.

The logic $B(H)^{\mathcal{J}}$ is said to be hyperbolic logic.

A measure $\mu : B(H)^{\mathcal{J}} \to \mathbb{R}$ is said to be: *indefinite* if μ on $B(H)^{\mathcal{J}}_+$ is non negative and on $B(H)^{\mathcal{J}}_-$ is non positive; *semi-trace measure* if $\mu(P) = c\dim(P_+)$ or $\mu(P) = c\dim(P_-)$, for all P, where $c \in \mathbb{R}$; *Hermitian* if $\mu(P) = \mu(P^*)$ for all P; *skew Hermitian* if $\mu(P) = -\mu(P^*)$ for all P. Note that the function $\mu^*(P) := \mu(P^*)$ for all P is the measure also and μ is the sum $\mu = 1/2(\mu + \mu^*) + 1/2(\mu - \mu^*)$ of Hermitian and skew Hermitian measures. If μ is a probability measure and $\mu^2(P) = \mu(P)$ for all P then μ is said to be *two-valued probability* measure.

Theorem 2.2 [18]. *Let $\mu : B(H)^{\mathcal{J}} \to [0, 1]$ be a probability measure. Then μ is a sum of semi-trace measures.*

Two-valued probability measures are connected with the problem of hidden variables in quantum mechanics (see [10]*). Let us present some result in this direction.*

Corollary 2.1. *Let H, $\dim H \geq 3$ be a real or complex Krein space. A two-valued probability measure on $B(H)^{\mathcal{J}}$ exists if and only if H is the Pontryagin space with the indefinite rank one (i.e $H = \Pi_1$).*

Let $H = \Pi_1$. Any two-valued probability measure $\mu : \mathcal{P} \to \{0, 1\}$ has the following property: if $\dim H^+ = 1$ then $\mu(P) = \mathrm{tr}(P_+)$; if $\dim H^- = 1$ then $\mu(P) = \mathrm{tr}(P_-)$ for all $P \in B(H)^{\mathcal{J}}$.

There is an indefinite version of Gleason's Theorem.

Theorem 2.3. [16]. *Let $\dim H \geq 3$, and let $\mu : B(H)^{\mathcal{J}} \to \mathbb{R}$ be an indefinite measure. Then there exist a $\mathcal{J}$-self-adjoint trace-class operator T and a semi-trace measure μ_0 such that*

$$\mu(P) = \mathrm{tr}(TP) + \mu_0(P) \text{ for all } P \in B(H)^{\mathcal{J}}.$$

Moreover, if $\dim Q^{+}H$ and $\dim Q^{-}H$ are equal to $+\infty$, then $\mu_0(\cdot) \equiv 0$ and the operator T is $\mathcal{J}$-non negative.

Theorem 2.4. generalizes the theorem on the indefinite measures. We will use Theorem 2.4. for the formulation of Problem 2.1.

Theorem 2.4 [17]. *Let $\mu : B(H)^{\mathcal{J}} \to \mathbb{R}$ be a measure on a infinite-dimensional Krein space H, $\dim Q^{+}H = \dim Q^{-}H$. Then there exists a $\mathcal{J}$-self-adjoint trace-class operator T such that*

$$\mu(P) = \mathrm{tr}(TP) \text{ for all } P \in B(H)^{\mathcal{J}}.$$

Note that: i) Let a measure μ from Theorem 1.5 be a Hermitian measure. Then $\mathrm{tr}(T^{*}p) = \mathrm{tr}(Tp^{*}) = \mu(p^{*}) = \mu(p) = \mathrm{tr}(Tp)$ for all $p \in B(H)^{Jc}$. Thus the operator T may be chosen as self-adjoint.

ii) Let a measure μ from Theorem 1.5 be a skew-Hermitian measure. By the analogy, the operator T may be chosen as skew self-adjoint.

The indefinite version of Gleason's theorem shows that any real or complex measure on $B(H)^{\mathcal{J}}$, $\dim H = \infty$ is a linear combination of indefinite measures. It is clear that on the logics $B(H)^{or}$ and $B(H)^{\mathcal{J}}$ there exist regular real measures.

2.4. **Measure on the skew projections.** The following version of Gleason's theorem is true for the logic $B(H)^{pr}$.

Theorem 2.5. [29]. *Let $\dim H = \infty$ and $\mu : B(H)^{pr} \to \mathbb{R}$ be a measure. Then there exists a trace-class operator T such that $\mu(P) = \Re\mathrm{tr}(TP)$ for all $P \in B(H)^{pr}$.*

The following assertion was formulated without the proof in [29] (for a proof see [27]).

Theorem 2.6. *Probability measure on $B(H)^{pr}$ exists iff $\dim H < \infty$. If $\mu : B(H)^{pr} \to [0,1]$ is a probability measure and $3 \leq \dim H \equiv n$, then $\mu(P) = \frac{1}{n}\mathrm{tr}(P)$ for all P.*

2.5. **Measure on H with conjugation operator.** Let H be a complex Hilbert space and let J be a conjugation operator on H (see [1], Section 50), i.e., 1) $J^2 = I$, 2) $(Jx, Jy) = (y, x)$, for all x, $y \in H$. Note by 1), and 2), $J(\lambda x + \beta y) = \overline{\lambda} Jx + \overline{\beta} Jy$, for all λ, $\beta \in \mathbb{C}$ and for all x, $y \in H$. Put $< x, y > := (Jx, y)$ for all x, $y \in H$. It is clear that $< Ax, y > = < x, A^{\#}y >$ for all x, $y \in H$ iff $A^{\#} = JA^{*}J$. Put $B(H)^{Jc} = \{P \in B(H)^{pr} : \ P = P^{\#}\}$ and $\Pi := B(H)^{Jc} \cap B(H)^{or}$.

Let $\dim H = +\infty$ or $\dim H := 2m < +\infty$. Then for any conjugation operator J there exists (non unique!) orthogonal projection F such that $F + F^{\#} = I$. The logic $B(H)^{Jc}$ ($\mathcal{B}(H)^{Jc} := \{P \in B(H)^{Jc} : \ PF = FP\}$) is said to be *conjugation* logic of type (A) (of type (B), respectively).

At the present time there is no complete description of measures on conjugation logic. Let us present the main known results.

Theorems 2.2, 2.6 and 2.7. shows that the class of probability measures on the logics $B(H)^{\mathcal{J}}$, $B(H)^{pr}$, and on $B(H)^{Jc}$ is extremely simple.

Theorem 2.7. [27]. *i) A probability measure on the conjugation logics $B(H)^{Jc}$ and $\mathcal{B}(H)^{Jc}$ exists if and only if* $\dim \mathrm{H} < +\infty$.

ii) Let $n \equiv \dim \mathrm{H} < +\infty$. On $B(H)^{Jc}$ and on $\mathcal{B}(H)^{Jc}$ there exists a unique probability measure μ and

$$\mu(P) = \frac{1}{n}\mathrm{tr}(P)$$

for all $P \in B(H)^{Jc}$ if $\dim H \geq 3$, and for all $P \in \mathcal{B}(H)^{Jc}$ if $\dim H \geq 6$.

Note that for the conjugation logic of type (A) Theorem 1.8 was proved in [19]. By Theorem 2.5. and by structural properties of the projections from $\mathcal{B}(H)^{Jc}$ we obtain the following

Theorem 2.8. *[27]. Let* $\dim H = \infty$*, and let* $\mu : \mathcal{B}(H)^{Jc} \to \mathbb{R}$ *be a measure. Then there exists a trace-class operator* T *such that* $\mu(P) = \mathrm{tr}(TP)$ *for all* $P \in \mathcal{B}(H)^{Jc}$.

Thus any real measure on $\mathcal{B}(H)^{Jc}$, $\dim H = \infty$ is a regular measure.

Theorem 2.9. *[21]. Let* $\dim H = +\infty$ *and let* $\mu : B(H)^{Jc} \to \mathbb{R}$ *be a Hermitian measure. Then*

$$\mu(P) = \Re\mathrm{tr}(AP) \text{ for all } P \in B(H)^{Jc}.$$

Here A *is an unique* J*-real self-adjoint trace-class operator such that* $\mu(P) = \mathrm{tr}(AP)$*,* $\forall P \in \Pi$.

If $\dim H = +\infty$ then on the logics $B(H)^{pr}$ and $B(H)^{Jc}$ regular real measures do not exist. Note that, for any trace-class operator A there exists a projection $P \in B(H)^{Jc}$ such that $\Im\mathrm{tr}(AP) \neq 0$ (see [24]). Therefore any regular measure on $B(H)^{Jc}$ (and hence on $B(H)^{pr}$) is a complex measure. Note that on the logics $B(H)^{or}$, $B(H)^{pr}$, $B(H)^{Jc}$, $\mathcal{B}(H)^{Jc}$ if $\dim H \geq 3$ there is no two-valued probability measure.

Now we shall formulate problems whose solution can solve Birkhoff's problems on $B(H)^{Jc}$ for skew Hermitian measure. We are interested in the following problem.

Problem 2.1. *Does a certain analog of Theorem 2.9. hold for skew Hermitian measures on* $B(H)^{Jc}$*?*

3. Some additional information

Information contained in this section may be useful for solution of Problem 2.1.

Now let H be a complex Hilbert space with the inner product $(\cdot,\cdot)$ and let S be the unite sphere in H. Let us denote by $\mathrm{lin}_{\mathbb{C}}\{\mathcal{N}\}$ (by $\mathrm{lin}_{\mathbb{R}}\{\mathcal{N}\}$) the complex (the real, respectively) linear subspace generated by a subset $\mathcal{N} \subseteq H$. Let $\mathcal{N}_1, \mathcal{N}_2 \subset H$. We write $\mathcal{N}_1 \perp \mathcal{N}_2$ if $(x, y) = 0$ for all $x \in \mathcal{N}_1$, $y \in \mathcal{N}_2$.

It is clear that on H there exists an infinite set of conjugation operators. Let J be a *conjugation* operator on H. A vector $x \in H$ is said to be J*-real* if $Jx = x$. The vectors $x_{\Re} := \frac{1}{2}(x + Jx)$ and $x_{\Im} := \frac{1}{2i}(x - Jx) = -\frac{1}{2}(ix + Jix)$ are J-real, $\forall x \in H$ and $x = x_{\Re} + ix_{\Im}$. Let $H_{\Re}$ be the set of all J-real vectors. It is clear that $(x, y) = (Jx, Jy) = (y, x)$ for all x, $y \in H_{\Re}$ and $H_{\Re}$ is a real Hilbert space with respect to the inner product $(\cdot,\cdot)$.

An operator $A \in B(H)$ is said to be J*-real* if $JAJ = A$. Note that A is a J-real iff $AH_{\Re} \subseteq H_{\Re}$ iff A^* is a J-real operator. The set of all J-real operators is a real algebra.

Put $< x, y >:= (Jx, y)$. An operator $A \in B(H)$ is said to be J*-self-adjoint*, if $< Ax, y >=< x, Ay >$, $\forall x$, $y \in H$. Hence $A = A^{\#}$ iff $AJ = JA^*$. Any bounded J-self-adjoint idempotent (=projection) P is called J*-projection*. Note that for any idempotent $P \in B(H)^{pr}$ there exists a such conjugation operator J_0 that P is a J_0-projection [25].

3.1. Hyperbolic sub-logics of the logic $B(H)^{Jc}$.

Let $P \in B(H)^{Jc}$. It is clear that:

$P \in \Pi$ iff $PH_{\Re} \subseteq H_{\Re}$ iff the restriction of J on PH is a conjugation operator on the Hilbert space PH.

Let $E \in \Pi$ $(0 \neq E \neq I)$. Let us denote $\mathcal{H}_E \equiv \mathrm{lin}_{\mathbb{R}}\{EH_{\Re} \oplus iE^{\perp}H_{\Re}\}$. The set $\mathcal{H}_E$ is a real Hilbert space with respect to the product $(\cdot,\cdot)$. It is clear that H is equal to the direct sum $\mathcal{H}_E + i\mathcal{H}_E$. Consequently, we have

Proposition 3.1. *Any* $B \in B(\mathcal{H}_E)$ *can be uniquely extended to a linear bounded operator* B_H *on* H*,* $(B_H)^* = (B^*)_H$*, and if* P *is a projection in* $\mathcal{H}_E$*, then* P_H *is a projection, too.*

Denote by $\overline{J}$ the restriction of J to $\mathcal{H}_E$. Clearly $\overline{J} = (E - E^{\perp})/\mathcal{H}_E$ is a symmetry (i.e. $\overline{J}^2 = I$, $\overline{J} = \overline{J}^*$ in $\mathcal{H}_E$. With respect to the product $[x, y] \equiv (Jx, y)$, $\forall x, y \in \mathcal{H}_E$, the set $\mathcal{H}_E$ is a real Krein space, and $\overline{J}$ is a canonical symmetry with respect to the canonical decomposition

$\mathcal{H}_E = \mathcal{H}_E^+[\dot{+}]\mathcal{H}_E^-$, where $\mathcal{H}_E^+ \equiv EH_{\Re}$ and $\mathcal{H}_E^- \equiv iE^{\perp}H_{\Re}$ (see definitions [4]). The indefinite rank of $\mathcal{H}_E$ is equal to $\min\{\dim EH, \dim E^{\perp}H\}$.

Denote by $\mathcal{P}_E$ the set of all bounded projections $P \in B(\mathcal{H}_E)$ such that $[Px, y] = [x, Py]$, $\forall x, y \in \mathcal{H}_E$. Clearly $P \in \mathcal{P}_E$ implies $P_H \in B(H)^{Jc}$. Conversely, if $Q \in B(H)^{Jc}$ and $Q\mathcal{H}_E \subseteq \mathcal{H}_E$, then $Q/\mathcal{H}_E$ is a $\overline{J}$-projection in the Krein space $\mathcal{H}_E$ i.e. $Q/\mathcal{H}_E = \overline{J}(Q/\mathcal{H}_E)^*\overline{J}$.

In [16] the logic $\mathcal{P}_E$ is called a hyperbolic logic. Denote by $\mathcal{P}_E^+$ ($\mathcal{P}_E^-$) the set of all projections $P \in \mathcal{P}_E$ for which the subspace $P\mathcal{H}_E$ is positive (i.e., $\forall z \in P\mathcal{H}_E$, $z \neq 0$, $[z, z] > 0$) (respectively, negative, i.e., $\forall z \in P\mathcal{H}_E$, $z \neq 0$, $[z, z] < 0$). Note, that $P \in \mathcal{P}_E^+$ iff $\overline{J}P \geq 0$ in $\mathcal{H}_E$, and $P \in \mathcal{P}_E^-$ iff $\overline{J}P \leq 0$. For instance, $(., \overline{J}f)f \in \mathcal{P}_E^+$, where $f = \alpha z + i\beta y$, $\alpha, \beta \in R$, $\alpha^2 - \beta^2 = 1$, and $z \in E\mathcal{H}_E \cap S$, $y \in E^{\perp}\mathcal{H}_E \cap S$. It is clear that $-(., \overline{J}g)g \in \mathcal{P}_E^-$, where $g = \beta z + i\alpha y$. Any projection $P \in \mathcal{P}_E$ is representable (not uniquely!) in the form $P = P_- + P_+$, where $P_- \in \mathcal{P}_E^-$, $P_+ \in \mathcal{P}_E^+$ [4].

It is easy to prove the following

Proposition 3.2. *For any projection $P \in B(H)^{Jc}$, $0 \neq P \neq I$ there is a hyperbolic logic $\mathcal{P}_E$ such that $P \in \mathcal{P}_E$.*

3.2. **On a classification of von Neumann J-algebras in the space with conjugation operator.** In the paper [28] a classification of von Neumann algebras in space with an indefinite metric was given. We offer a similar classification of von Neumann algebras in Hilbert space with conjugation operator J [32].

Let $\mathcal{M}$ be a von Neumann algebra on H. (The definition and properties of von Neumann algebras see, for instance [7]). Let us denote by $\mathcal{M}^{pr}$ (by $\mathcal{M}^{or}$, $\mathcal{M}^{Jc}$, Π) the set of all bounded (orthogonal, J-self-adjoint, orthogonal and J-self-adjoint (and, hence, J-real), respectively) projections from $\mathcal{M}$. Any one-dimensional projection: 1) from $B(H)^{pr}$ has the form $(\cdot, x)y$, x, $y \in H$, where $(x, y) = 1$; 2) from Π has the form $(\cdot, x)x$, $x \in H_{\Re}$, $(x, x) = 1$. A von Neumann algebra $\mathcal{M}$ in H is said to be a von Neumann J-algebra if $A \in \mathcal{M}$ implies $A^{\#} \in \mathcal{M}$. Note that for any conjugation operator J_0 exists a von Neumann algebra, which is not a von Neumann J_0-algebra. Let $\mathcal{M}$ be von Neumann J-algebra. Then: *i*) its center $\mathcal{Z}$ ($:= \mathcal{M} \cap \mathcal{M}'$) and $\mathcal{M}'$ are von Neumann J-algebras to; *ii*) the set $\Re\mathcal{M}$ of all J-real operators of $\mathcal{M}$ is a real von Neumann algebra, i.e., $\Re\mathcal{M} \cap i\Re\mathcal{M} = \{0\}$, $\Re\mathcal{M} + i\Re\mathcal{M} = \mathcal{M}$. The logic $\mathcal{M}^{Jc}$ is said to be conjugation logic.

Let $P, Q \in B(H)^{pr}$. Put $P <_1 Q$ if $PQ = QP = P$. Let us denote by E_p the orthogonal projection on PH. We will denote by P_{or} the orthogonal projection on $PH \cap P^*H$. It is clear that $P_{or} \leq_1 E_p$. Note, that the projection P_{or} is the greatest orthogonal projection with $P_{or} \leq_1 P$. If $P \in B(H)^{Jc}$ then $P_{or} \in \Pi$ (see [19]).

The orthogonal projection P_A on $\overline{\{AH + A^*H\}}$ is said to be the *cover* of A, $A \in B(H)$. A projection P, $P \neq 0$ is said to be a *proper skew* projection if $P_{or} = 0$. Note that if $P \neq P^*$ then $P_s := P - P_{or}$ is the proper skew projection.

Lemma 3.1. *Let $\mathcal{M}$ be a commutative von Neumann J-algebra. Then there exists a unique maximal (orthogonal) projection $E \in \mathcal{M}^{or}$ such that $P \leq_1 E$, $P \in \mathcal{M}^{or}$ implies $P = P^{\#}$. In addition there exists (non unique, in general!) a projection $F \in \mathcal{M}^{or}$ such that $F + F^{\#} = I - E$.*

Proof. Note first that $F + F^{\#} = I - E$ implies $FF^{\#} = 0$. The equality $P = P^{\#}$ for all $P \in \mathcal{M}^{or}$ implies $E = I$.

1) Let us suppose that $P \neq P^{\#}$ for some $P \in \mathcal{M}^{or}$. Put $R := PP^{\#}$ ($<_1 P$). Then $R^{\#} = (JPJ)P = PJPJ = R$. In addition $P^{\#} - R = J(P - R)J \neq 0$ and $(P - R)(JPJ - R) = PJPJ - R - R + R = 0$, i.e. $(P - R) \perp (J(P - R)J)$.

2) Let us denote by Δ the set of all systems $\{P_i\} \subset \mathcal{M}^{or}$ of mutually orthogonal projections such that $\sum_i P_i \perp \sum_i P_i^{\#}$. With respect to the inclusion Δ is a partially ordered set. By the Zorn's lemma, Δ contains a maximal element $\{Q_i\}$. Put $F := \sum_i Q_i$ and $E := I - (F + F^{\#})$. By the definition, $E = E^{\#}$.

It is clear that if $P \leq_1 E$, $P \in B(H)^{or}$ then $P^{\#} \leq_1 E$. Let us suppose for the moment that $PP^{\#} \neq P$ for some $P \leq_1 E$. Then by step 1), there exist two orthogonal projections P, $P^{\#}$ with $P + P^{\#} \leq_1 E$. We have a contradiction with the definition of the family $\{Q_i\}$.

A commutative von Neumann J-algebra $\mathcal{Z}$ is said to be a type (A) algebra if $P = P^{\#}$ for all $P \in \mathcal{Z}^{or}$. Note that in this case any self-adjoint operator A is J-real operator.

A commutative von Neumann J-algebra $\mathcal{Z}$ is said to be a type (B) algebra if $\mathcal{Z}$ contains a pair F, $F^{\#} \in \mathcal{Z}^{or}$ such that $F + F^{\#} = I$. Note that $F + F^{\#} = I$ implies $FF^{\#} = 0$, and $F^{\#} = F^{\perp}$.

A von Neumann J-algebra $\mathcal{M}$ is said to be of type (A) (type (B)) if its center $\mathcal{Z}$ is of type (A) algebra (of type (B) algebra, respectively). Let F be the projection from definition of type (B) algebra. Set $\mathcal{B}(H) := FB(H)F + F^{\#}B(H)F^{\#}$. It is clear that $B(H)$ ($\mathcal{B}(H)$) is the type (A) (the type (B), respectively) algebra. Note that $P \in \mathcal{B}(H)^{Jc}$ iff $P \in B(H)^{Jc}$ and $PF = FP$.

3.3. **Conjugation logics of projections.** All J-projections from $B(H)^{Jc}$ are called J-projections of type (A).

Proposition 3.3. *Let $(.,x)y$ be a nonzero projection. Then $(.,x)y \in B(H)^{Jc}$ iff $(.,x)y = (.,Jy^*)y^*$, where $y^* = (y,Jy)^{-\frac{1}{2}}y$.*

Proof. Let $(.,x)y \in B(H)^{Jc}$. Then

$$(.,x)y = J((\cdot,x)y)^*J = J((.,y)x)J = \overline{(J.,y)}Jx = (y,J.)Jx = (.,Jy)Jx,$$

where $y = \alpha Jx$. Hence $x = (\overline{\alpha})^{-1}Jy$. We have $1 = (y,x) = \alpha^{-1}(y,Jy)$. Hence $(y,Jy) \neq 0$ and $(.,x)y = \alpha^{-1}(.,Jy)y = (y,Jy)^{-1}(.,Jy)y = (.,Jy^*)y^*$.

Conversely, let $(.,x)y = (.,Jy^*)y^*$. Then

$$J((.,x)y)^*J = J((.,y*)Jy*)J = \overline{(J.,y^*)}JJy^* = (y^*,J.)y^* = (.,Jy^*)y^* = (.,x)y.$$

Hence $(.,x)y \in B(H)^{Jc}$. □

A vector $y \in H$ is said to be a projection type vector if $(.,Jy^*)y^* \in B(H)^{Jc}$. Let us denote by H_p the set of all projection type vectors. Note that $H_{\Re}\backslash\{0\} \subset H_p$. Denote by p_y the projection $(.,Jy^*)y^*$ $(= (y,Jy)^{-1}(.,Jy)y)$, $\forall y \in H_p$.

Remark 3.1. *Let $y \in H$. Then*

i) $y \notin H_p$ iff $(y,Jy) = 0$ iff $(y_{\Re}, y_{\Im}) = 0$ and $\| y_{\Re} \| = \| y_{\Im} \|$;

ii) $(\cdot,Jy^*)y^* \in \Pi$ iff $y^* \in H_{\Re}$;

iii) $(\cdot,Jy)y \in B(H)^{Jc}$ iff $(y_{\Re}, y_{\Im}) = 0$ and $\| y_{\Re} \|^2 - \| y_{\Im} \|^2 = 1$;

iv) Let $p_y = (\cdot,Jy)y \in B(H)^{Jc}$. Then $p_{Jy} = (p_y)^*$ and $p_y \neq p_{Jy}$ iff $y_{\Im} \neq 0$;

v) $(x,Jy) = 0$ iff $(x_{\Re}, y_{\Re}) = (x_{\Im}, y_{\Im})$ and $(x_{\Re}, y_{\Im}) = -(x_{\Im}, y_{\Re})$.

Proof. The properties $i)$ and $ii)$ are obvious. $iii)$ Let $(\cdot,Jy)y \in B(H)^{Jc}$. Then $1 = (y,Jy) = \| y_{\Re} \|^2 + 2i(y_{\Re}, y_{\Im}) - \| y_{\Im} \|^2$. Hence $(y_{\Re}, y_{\Im}) = 0$ and $\| y_{\Re} \|^2 - \| y_{\Im} \|^2 = 1$.

Conversely, let $(y_{\Re}, y_{\Im}) = 0$ and $\| y_{\Re} \|^2 - \| y_{\Im} \|^2 = 1$. Then $(y,Jy) = 1$. Hence $(.,Jy)y$ is a projection. Furthermore, $J((\cdot,Jy)y)^*J = J((\cdot,y)Jy)J = (\cdot,Jy)y$. Thus $(\cdot,Jy)y \in B(H)^{Jc}$.

Put $B := (\cdot,Jy)y - (\cdot,y)Jy$. A simple calculation shows that $B^*B = 2[\|y_{\Im}\|^2(\cdot,y_{\Re})y_{\Re} + \|y_{\Re}\|^2(\cdot,y_{\Im})y_{\Im}]$. By $iii)$, $B \neq 0$ iff $y_{\Im} \neq 0$. This prove $iv)$.

The property $v)$ is verified directly. □. □

Let $x \in H$ be such that $(\cdot, Jx)x \in B(H)^{Jc}$. The vector x is said to be *generator* vector for p_x.

Proposition 3.4. *For any $P \in B(H)^{Jc}$ $(P \neq 0)$ we have $PH \cap H_p \neq \{\emptyset\}$, and for any $y \in PH \cap H_p$ the inequality $(\cdot, Jy^*)y^* \leq_1 P$ holds true.*

Proof. Obviously the proposition is true if P is a one-dimensional projection. Let $P \in B(H)^{Jc}$ and $\dim PH \geq 2$. Assume on the contrary that $PH \subset H \backslash H_p$. Let $x, y \in PH$. By Remark 2.5 i),

$$0 = (x + y, J(x + y)) = (x, Jx) + (y, Jx) + (x, Jy) + (y, Jy) =$$
$$= (y, Jx) + (x, Jy) = 2(y, Jx).$$

Hence the subspaces PH and JPH $(= JJP^*JH = P^*JH = P^*H)$ are orthogonal. Thus $0 = (Px, P^*z) = (Px, z) = (x, z), \forall z \in H$. This means that $P = 0$. We have got a contradiction. Hence $PH \cap H_p \neq \{\emptyset\}$. The inequality $(., Jy^*)y^* <_1 P$ is obvious. $\square$. $\square$

Corollary 3.1. *The logic $B(H)^{Jc}$ is atomic.*

The following result was proved in [25].

Proposition 3.5. *Let $P \in B(H)^{pr}$, $P \neq 0$, and let Q^+ (Q^-) be the cover of the positive $= (P + P^*)_+$ (the negative $= (P + P^*)_-$, respectively) part of $P + P^*$. Then:*

1) $P \leq_1 (Q^+ + Q^-)$.

2) $2Q^+PQ^+ = (P + P^*)_+$, $Q^+PQ^+ \geq Q^+$ and $2Q^-PQ^- = -(P + P^*)_-$.

Let $Q^-PQ^+ = U|Q^-PQ^+|$ be the polar decomposition of Q^-PQ^+. Put $X := Q^+P_sQ^+$, $Y := Q^-P_sQ^-$, $P(X) := \{a_0P_X + \sum_{i=1}^n a_iX^i : n \in \mathbb{N}, a_i \in \mathbb{C}\, \forall i\}$, $P^h(X) := \{P \in P(X) : P = P^*\}$, $P(Y) := \{a_0P_Y + \sum_{i=1}^n a_iY^i : n \in \mathbb{N}, a_i \in \mathbb{C}\, \forall i\}$, $P^h(Y) := \{P \in P(Y) : P = P^*\}$. By $X \geq Q^+$, we have $P_X = Q^+$.

Put $V := \frac{U}{i}$. In [19] it is proved that: *If $P \in \mathcal{M}^{Jc}$ then Q^+, Q^-, X and V are J-real operators from $\mathcal{M}$.*

The following assertion is also true:

Let $P = P_s$ then $UP(X)U^* = P(Y)$. (Hence $UU^* = Q^-$, $U^*U = Q^+$, and $P_Y = Q^-$).

Thus, we have the useful equality (see [25])

$$P = P_s = X + i(V(X^2 - X)^{1/2} + (X^2 - X)^{1/2}V^*) + V(Q^+ - X)V^*.$$

3.4. **J-projections of type (B).** All J-projections from type (B) von Neumann J-algebra $\mathcal{M}$ are called J-projections of type (B). Type (B) projections were studied in [26]. Let F be the projection from definition of type (B) algebra. We have already mentioned that $\mathcal{B}(H)$ is a von Neumann algebra of type (B). It is clear that $\mathcal{B}(H)^{Jc} \subset B(H)^{Jc}$. Let us denote by $\mathcal{L}_F^{\mathcal{M}}$ the set of all projections from $F\mathcal{M}F$. Then $\mathcal{M}^{Jc} = \{q + Jq^*J : q \in \mathcal{L}_F^{\mathcal{M}}\}$.

Theorem 3.1. [26]. *Let R be a J-projection of type (B). Then there is a J-projection $D \in B(H)^{Jc}$ such that $R = D + JDJ$ and covers of D and JDJ are mutually orthogonal.*

4. Reduction of the measure problem 2.1. to three-dimensional spaces

Note that the main part of the proof of Theorem 2.1. and Theorems 2.3. and 2.4. is to prove the theorems for the real three-dimensional case. In this section, we show how the Problem 2.1. can be reduced to the real three-dimensional case.

Let H, $\dim H = \infty$, be a Hilbert space with conjugation operator J, let $S = \{x \in H : \|x\| = 1\}$ be the unite sphere from H, and let $\mu : B(H)^{Jc} \to \mathbb{R}$ be a skew Hermitian measure. Let $p_x = (\cdot, Jx)x \in B(H)^{Jc}$ be a one-dimensional skew projection (see Proposition 2.4). Thus $x_{\Re} \neq 0$, $x_{\Im} \neq 0$ and $(x_{\Re}, x_{\Im}) = 0$ and $\| x_{\Re} \|^2 - \| x_{\Im} \|^2 = 1$ (see Remark 2.5 ii), iii)). Let $E \in \Pi$ be such that *i)* $x_{\Re} \in EH_{\Re}$, $x_{\Im} \in E^{\perp}H_{\Re}$ and *ii)* $\dim EH = \infty$, $\dim E^{\perp}H = \infty$. Let $\mathcal{P}_E$ be the hyperbolic sub logic of projection from Section 1.§2. Let μ_E be the restriction of the measure μ on to $\mathcal{P}_E$. It is clear that μ_E is the skew Hermitian measure on $\mathcal{P}_E$. By Theorem 1.5, there is a skew-adjoint and J-self-adjoint trace-class operator T such that $\mu_E(p) = \mathrm{tr}(Tp)\ \forall p \in P_E$. Thus

$$\mu((\cdot, Jx)x) = \mu_E((\cdot, Jx)x)$$
$$= \mathrm{tr}(T[(\cdot, x_{\Re})x_{\Re} + i(\cdot, x_{\Re})x_{\Im} + i(\cdot, x_{\Im})x_{\Re} - (\cdot, x_{\Im})x_{\Im}])$$
$$= \mathrm{tr}(T[i(\cdot, x_{\Re})x_{\Im} + i(\cdot, x_{\Im})x_{\Re}]).$$

It is clear that $(\cdot, x_{\Re})x_{\Im} + (\cdot, x_{\Im})x_{\Re})$ is the skew Hermitian part $(=p_x^s)$ of the projection p_x. Put $F((\cdot, x_{\Re})x_{\Im} + (\cdot, x_{\Im})x_{\Re}) := \mathrm{tr}(T[i(\cdot, x_{\Re})x_{\Im} + i(\cdot, x_{\Im})x_{\Re}])$ $(=\mu((\cdot, Jx)x))$. By the definition,

$$F(-[(\cdot, x_{\Re})x_{\Im} + (\cdot, x_{\Im})x_{\Re}]) = \mu(((\cdot, Jx)x)^*) \tag{1}$$
$$= -\mu((\cdot, Jx)x) = -F((\cdot, x_{\Re})x_{\Im} + (\cdot, x_{\Im})x_{\Re}).$$

Put $a^2 = \|x_{\Re}\|^2$, $b^2 = \|x_{\Im}\|^2$ and $e = x_{\Re}/\|x_{\Re}\|$, $e^{\perp} = x_{\Im}/\|x_{\Im}\|$. Thus $(\cdot, x_{\Re})x_{\Im} + (\cdot, x_{\Im})x_{\Re}) = ab[(\cdot, e)e^{\perp} + (\cdot, e^{\perp})e]$. It is clear that $\{ab :\ a^2 - b^2 = 1\} = \mathbb{R}$ and there exists $z \in H$ such that $(\cdot, Jz)z \in B(H)^{Jc}$, $p_z^s = [(\cdot, e)e^{\perp} + (\cdot, e^{\perp})e]$. For any self-adjoint operator $c[(\cdot, e)e^{\perp} + (\cdot, e^{\perp})e]$ there exist vectors ϕ, $\phi^{\perp} \in S \cap H_{\Re}$, $(\phi, \phi^{\perp}) = 0$ such that $[(\cdot, e)e^{\perp} + (\cdot, e^{\perp})e] = [(\cdot, \phi)\phi - (\cdot, \phi^{\perp})\phi^{\perp}]$. Put $F(0) = 0$. Thus by (1), we have the equality

$$F(c[(\cdot, \phi)\phi - (\cdot, \phi^{\perp})\phi^{\perp}]) = cF([(\cdot, \phi)\phi - (\cdot, \phi^{\perp})\phi^{\perp}])\ \ \forall c \in \mathbb{R} \tag{2}$$

for any vectors ϕ, $\phi^{\perp} \in S \cap H_{\Re}$, $(\phi, \phi^{\perp}) = 0$.

Let x, y be generators. Direct calculations show that $p_x p_y = 0$ (i.e. $p_x \perp p_y$) iff $(x, Jy) = 0$. In this case we say that the generators x, y are J-orthogonal. Fix $R \in \Pi$, $\dim RH = 3$. We have already noted that the restriction of J onto RH is also a conjugation operator. Let x_1, x_2, $x_3 \in RH$ be mutually J-orthogonal generators in RH. Taking into account that $p_{x_1} + p_{x_2} + p_{x_3} = R$ we have

$$\mu(p_{x_1}) + \mu(p_{x_2}) + \mu(p_{x_3}) = \mu(R) = 0 \ \text{ and } \ p_{x_1}^s + p_{x_2}^s + p_{x_3}^s = 0.$$

Let $p_{x_i}^s = a_i b_i[(\cdot, \phi_i)\phi_i - (\cdot, \phi_i^{\perp})\phi_i^{\perp}]$ $(= a_i b_i[(\cdot, e_i)e_i^{\perp} + (\cdot, e_i^{\perp})e_i])$, $i = 1, 2, 3$. Then

$$a_1 b_1[(\cdot, \phi_1)\phi_1 - (\cdot, \phi_1^{\perp})\phi_1^{\perp}] + a_2 b_2[(\cdot, \phi_2)\phi_2 - (\cdot, \phi_2^{\perp})\phi_2^{\perp}]$$
$$= -a_3 b_3[(\cdot, \phi_3)\phi_3 - (\cdot, \phi_3^{\perp})\phi_3^{\perp}]. \tag{3}$$

By (3),

$$a_1 b_1 F([(\cdot, \phi_1)\phi_1 - (\cdot, \phi_1^{\perp})\phi_1^{\perp}]) + a_2 b_2 F([(\cdot, \phi_2)\phi_2 - (\cdot, \phi_2^{\perp})\phi_2^{\perp}]) =$$
$$\mu(p_{x_1}) + \mu(p_{x_2}) = -\mu(p_{x_3}) = \mu(p_{x_3}^*) = a_3 b_3 F([(\cdot, \phi_3^{\perp})\phi_3^{\perp} - (\cdot, \phi_3)\phi_3]) =$$
$$= F(a_1 b_1[(\cdot, \phi_1)\phi_1 - (\cdot, \phi_1^{\perp})\phi_1^{\perp}] + a_2 b_2[(\cdot, \phi_2)\phi_2 - (\cdot, \phi_2^{\perp})\phi_2^{\perp}]).$$

Thus

$$a_1 b_1 F([(\cdot, \phi_1)\phi_1 - (\cdot, \phi_1^{\perp})\phi_1^{\perp}]) + a_2 b_2 F([(\cdot, \phi_2)\phi_2 - (\cdot, \phi_2^{\perp})\phi_2^{\perp}]) =$$
$$= F(a_1 b_1[(\cdot, \phi_1)\phi_1 - (\cdot, \phi_1^{\perp})\phi_1^{\perp}] + a_2 b_2[(\cdot, \phi_2)\phi_2 - (\cdot, \phi_2^{\perp})\phi_2^{\perp}]) \tag{4}$$
$$\text{if } \ a_1 a_2(e_1, e_2) = b_1 b_2(e_1^{\perp}, e_2^{\perp}) \text{ and } a_1 b_2(e_1, e_2^{\perp}) = -b_1 a_2(e_1^{\perp}, e_2).$$

Note that, by Remark 2.5.v), the assertion: "The generators x_1, x_2, where $x_j = a_j e_j + i b_j e_j^{\perp}$, $j = 1, 2$ are J-orthogonal" is equivalent the following: "$a_1 a_2(e_1, e_2) = b_1 b_2(e_1^{\perp}, e_2^{\perp})$ and $a_1 b_2(e_1, e_2^{\perp}) = -b_1 a_2(e_1^{\perp}, e_2)$".

We may identify RH with $\mathbb{C}^3$. Here $\mathbb{C}^3$ is the unitary (= Euclidean) space with the usual scalar product $(\cdot, \cdot)$. For any vector $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{C}^3$ put $\theta_{\Re} := (\Re\theta_1, \Re\theta_2, \Re\theta_3)$ and $\theta_{\Im} := (\Im\theta_1, \Im\theta_2, \Im\theta_3)$. It is clear that $\theta = \theta_{\Re} + i\theta_{\Im}$. We may identify the conjugation operator J with the *conjugation* operator $J_1 : \mathbb{C}^3 \to \mathbb{C}^3$. Here $J_1(\phi_1, \phi_2, \phi_3) := (\overline{\phi_1}, \overline{\phi_2}, \overline{\phi_3})$, $\forall \phi_1, \phi_2, \phi_3 \in \mathbb{C}$. Thus $RH_{\Re} = \mathbb{R}^3$. Let $S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$ be the unit sphere from $\mathbb{R}^3$. Let us denote by B

the set of all self-adjoint operators of the form $c[(\cdot, e)e - (\cdot, e^{\perp})e^{\perp}]$, here $c \in \mathbb{R}$, and e, $e^{\perp} \in S^2$, $(e, e^{\perp}) = 0$.

Problem 4.1. *Does the function $F(\cdot)$ on B with the properties (2), (4) admit an extension to a linear functional on the set of all real self-adjoint operators on $\mathbb{R}^3$?*

The positive answer to the Problem 2 would imply the solution of Problem 4.1. in the infinite dimensional space. It is interesting to proof the following weak version of Problem 4.1.

Problem 4.2. *Does the equality*

$$F((\cdot, e_1)e_1 - (\cdot, e_2)e_2) + F((\cdot, e_2)e_2 - (\cdot, e_3)e_3) = F((\cdot, e_1)e_1 - (\cdot, e_3)e_3)$$

hold for any mutually orthogonal vectors e_1, e_2, $e_3 \in S^2$?

To solve the Problem 2.1. in continuous von Neumann J-algebras it is sufficient to give a positive answer to Problem 4.2.

Problem 4.3. *Under what necessary and sufficient conditions for the skew hermitian measure μ the formula $\mu(p) := \Re\mathrm{tr}(Ap)$, $\forall p$ will be true? Here A is an appropriate trace-class operator.*

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