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CLASSIFICATION OF A FAMILY OF THREE DIMENSIONAL REAL EVOLUTION ALGEBRAS

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ABSTRACT. In this paper we classify a family of three-dimensional real evolution algebras. We also consider an evolution operator for an evolution algebra and find fixed points of this operator for two and three-dimensional cases. Then we construct an evolution algebra, the matrix of structural constants of which is Jacobian of the evolution operator at a fixed point. We study isomorphism between these evolution algebras.

Keywords: evolution algebra, evolution operator, isomorphism, Jacobian, fixed point.

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1. Introduction

To study a non-linear function one usually finds the linear approximation to the function at a given point. The linear approximation of a function is the first order Taylor expansion around the point of interest. In the theory of dynamical systems, linearization is a method for assessing the local stability of an equilibrium point. For an algebra (with a fixed multiplication $*$) and the (cubic) matrix $\mathcal{M}$ of structural constants one can define a quadratic (non-linear) operator $F(x) = x * x$, with coefficients given by the matrix $\mathcal{M}$. The Jacobian J of F at a given point, can be considered as a linear approximation of F. Consequently, J generates an evolution algebra as its matrix of structural constants.

Let $(A, \cdot)$ be an algebra over a field K. If it admits a countable basis $e_1, e_2, \dots, e_n, \dots$, such that

$$e_i \cdot e_j = 0, \ if \ i \neq j$$

$$e_i \cdot e_i = \sum_k a_{ik} e_k, \ for \ any \ i$$

then it is called an evolution algebra. This basis is called a natural basis.

We note that to every evolution algebra corresponds a square matrix (a_{ik}) of structural constants of the given evolution algebra.

In [10] the following basic properties of evolution algebras are proved:

1) Evolution algebras are not associative, in general.
2) Evolution algebras are commutative, flexible.
3) Evolution algebras are not power-associative, in general.
4) The direct sum of evolution algebras is also an evolution algebra.

In [7] the dynamics of absolutely nilpotent and idempotent elements in chains generated by two-dimensional evolution algebras are studied. In [2] the authors consider an evolution algebra which has a rectangular matrix of structural constants. This algebra is called evolution algebras of "chicken" population (EACP). The mentioned paper is devoted to the description

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of structure of EACPs. Using the Jordan form of the rectangular matrix of structural constants, a simple description of EACPs over the field of complex numbers is given. The classification of three-dimensional complex EACPs is obtained. Moreover, some $(n+1)$-dimensional EACPs are described. The fundamentals of evolution algebras have been being developed in the last years with no probabilistic restrictions on the stucture constants [4, 5, 8].

In Section 2 we study an approximation of two-dimensional real evolution algebras and isomorphism between these algebras. In Section 3 we will classify a family of three-dimensional real evolution algebras. We show that there are 13 class of such evolution algebras. We consider an approximation of three-dimensional real evolution algebras in Section 4 and also study isomorphism between these algebras.

2. Approximation of two-dimensional real evolution algebras

Let E be a 2-dimensional evolution algebra over the field of real numbers. Such algebras are classified in [6]:

Theorem 2.1. [6] *Any two-dimensional real evolution algebra E is isomorphic to one of the following pairwise non-isomorphic algebras:*
(i) $dim(E^2)=1$:
E_1: $e_1e_1=e_1,\ e_2e_2=0$;
E_2: $e_1e_1=e_1,\ e_2e_2=e_1$;
E_3: $e_1e_1=e_1+e_2,\ e_2e_2=-e_1-e_2$;
E_4: $e_1e_1=e_2,\ e_2e_2=0$;
E_5: $e_1e_1=e_2,\ e_2e_2=-e_2$;
(ii) $dim(E^2)=2$:
$E_6(a_2;a_3)$: $e_1e_1=e_1+a_2e_2,\ e_2e_2=a_3e_1+e_2$; $1-a_2a_3\neq 0,\ a_2,a_3\in\mathbb{R}$. *Moreover* $E_6(a_2;a_3)$ *is isomorphic to* $E_6(a_3;a_2)$.
$E_7(a_4)$: $e_1e_1=e_2,\ e_2e_2=e_1+a_4e_4$, *where* $a_4\in\mathbb{R}$;

For a given evolution algebra $(E,\cdot)$ an evolution operator has the following form $F(x)=x\cdot x=x^2$. If $x=\sum\limits_{i=1}^{n}x_ie_i$ then

$$x^2=\sum_{i=1}^{n}x_i^2e_i^2=\sum_{i=1}^{n}x_i^2\Big(\sum_{k=1}^{n}a_{ik}e_k\Big)=\sum_{k=1}^{n}\Big(\sum_{i=1}^{n}a_{ik}x_i^2\Big)e_k.$$

We denote $x'_k=\sum_{i=1}^{n}a_{ik}x_i^2$. Thus we have the following operator, $F:E\to E$,

$$F: x'_k=\sum_{i=1}^{n}a_{ik}x_i^2, k=\overline{1,n}.$$

Jacobian of the operator F at the point x for two-dimensional case has a form

$$J_F(x)=\begin{pmatrix} 2a_{11}x_1 & 2a_{21}x_2\\ 2a_{12}x_1 & 2a_{22}x_2\end{pmatrix}.$$

Following [9] and [3] we define an evolution algebra $\widetilde{E}$ with matrix $J_F(x)$ as the matrix of structural constants.

We will find fixed points of this operator, i.e. solutions of $F(x)=x$:

$$\begin{cases} x_1=a_{11}x_1^2+a_{21}x_2^2,\\ x_2=a_{12}x_1^2+a_{22}x_2^2.\end{cases} \tag{1}$$

Note that $(0,0)$ is one of solutions of system of equations (1), and

$$J_F(0,0)=\begin{pmatrix}0&0\\0&0\end{pmatrix}.$$

So corresponding evolution algebra with matrix $J_F(0,0)$ is trivial.

Trivial evolution algebras are not interesting. So we will find a non-zero solutions denoted by $(x_1^0;x_2^0)$ of (1) for algebras $E_i,\ i=1,2,\dots,7$ mentioned in Theorem 2.1 and study isomorphisms of evolution algebras corresponding to these fixed points with other evolution algebras.

In the following table we give all possibilities for two-dimensional case:

2-dimensional real evolution algebras	Non-zero real fixed points of the operator F	Corresponding evolution algebras to fixed points
$E_1:\begin{pmatrix}1&0\\0&0\end{pmatrix}$	$(1;0)$	$\widetilde{E}_1:\begin{pmatrix}2&0\\0&0\end{pmatrix}$
$E_2:\begin{pmatrix}1&0\\1&0\end{pmatrix}$	$(1;0)$	$\widetilde{E}_2:\begin{pmatrix}2&0\\0&0\end{pmatrix}$
$E_3:\begin{pmatrix}1&1\\-1&-1\end{pmatrix}$	Not exists	Not exists
$E_4:\begin{pmatrix}0&1\\0&0\end{pmatrix}$	Not exists	Not exists
$E_5:\begin{pmatrix}0&1\\0&-1\end{pmatrix}$	$(0;-1)$	$\widetilde{E}_5:\begin{pmatrix}0&0\\0&2\end{pmatrix}$
$E_6(a_2;a_3):\begin{pmatrix}1&a_2\\a_3&1\end{pmatrix}$	$(x_1^0;x_2^0)$	$\widetilde{E}_6(a_2;a_3):\begin{pmatrix}2x_1^0&2a_3x_2^0\\2a_2x_1^0&2x_2^0\end{pmatrix}$
$E_7(a_4):\begin{pmatrix}0&1\\1&a_4\end{pmatrix}$	$(x_1^0;x_2^0)$ if $a_4\geq-\frac{3}{\sqrt[3]{4}}$	$\widetilde{E}_7(a_4):\begin{pmatrix}0&2x_2^0\\2x_1^0&2a_4x_2^0\end{pmatrix}$

We have the following theorem.

Theorem 2.2. *i) Evolution algebras $\widetilde{E}_1$, $\widetilde{E}_2$ and $\widetilde{E}_5$ are isomorphic to E_1;*
ii) $\widetilde{E}_6(a_2;a_3)$ is isomorphic to the evolution algebra $E_6(b_2;b_3)$, where $b_2=a_3\left(\frac{x_2^0}{x_1^0}\right)^2$, $b_3=a_2\left(\frac{x_1^0}{x_2^0}\right)^2$;
iii) $\widetilde{E}_7(a_4)$ is isomorphic to $E_7(b_4)$, where $b_4=a_4\sqrt[3]{(\frac{x_2^0}{x_1^0})^2}$.

Proof. $\widetilde{E}_1\approxeq E_1$: By the change of basis $\widetilde{e}_1=\frac{1}{2}e_1$ we can prove that the evolution algebra $\widetilde{E}_1$ is isomorphic to E_1.
$\widetilde{E}_2\approxeq E_1$: It is similar to the above proof.
$\widetilde{E}_5\approxeq E_1$: By the change of basis $\widetilde{e}_1=\frac{1}{2}e_2$, $\widetilde{e}_2=e_1$ we can prove that the evolution algebra $\widetilde{E}_5$ is isomorphic to E_1.
$\widetilde{E}_6(a_2;a_3)\approxeq E_6(b_2;b_3)$: We can see this by the change of basis $\widetilde{e}_1=\frac{1}{2x_1^0}e_1$ and $\widetilde{e}_2=\frac{1}{2x_2^0}e_2$.
$\widetilde{E}_7(a_4)\approxeq E_7(b_4)$: We can see this by the change of basis $\widetilde{e}_1=2\sqrt[3]{x_1^0(x_2^0)^2}e_1$ and $\widetilde{e}_2=2\sqrt[3]{(x_1^0)^2x_2^0}e_2$. $\square$

3. Three-dimensional real evolution algebras with $dim(E^2)=1$

In [1] three dimensional complex evolution algebras are classified. Now we shall consider classification of three dimensional real evolution algebras.

Fix a three-dimensional real evolution algebra E and a natural basis $B = \{e_1, e_2, e_3\}$. Let M_B be the matrix of structural constants of E relative to B:

$$M_B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

In order to classify three dimensional real evolution algebras with condition $dim(E^2) = 1$ we find a basis of E for which its structure matrix has an expression as simple as possible, where by 'simple' we mean with the maximal number of 0, 1 and -1 in the entries.

Let $dim(E^2) = 1$. Without loss of generality we may assume $e_1^2 \neq 0$. Write $e_1^2 = a_1e_1 + a_2e_2 + a_3e_3$, where $a_i \in \mathbb{R}$ and $a_i \neq 0$ for some i. Note that e_1^2 is basis of E^2.

Since $e_2^2, e_3^2 \in E^2$, there exist $c_1, c_2 \in \mathbb{R}$ such that

$$e_2^2 = c_1e_1^2 = c_1(a_1e_1 + a_2e_2 + a_3e_3),$$
$$e_3^2 = c_2e_1^2 = c_2(a_1e_1 + a_2e_2 + a_3e_3).$$

Then

$$M_B = \begin{pmatrix} a_1 & a_2 & a_3 \\ c_1a_1 & c_1a_2 & c_1a_3 \\ c_2a_1 & c_2a_2 & c_2a_3 \end{pmatrix}.$$

We analyze when E^2 has the extension property. This means that there exists a natural basis $B' = \{e'_1, e'_2, e'_3\}$ of E with

$$\begin{aligned} e'_1 &= e_1^2 = a_1e_1 + a_2e_2 + a_3e_3 \\ e'_2 &= \alpha e_1 + \beta e_2 + \gamma e_3 \\ e'_3 &= \delta e_1 + \nu e_2 + \eta e_3 \end{aligned} \tag{2}$$

for some $\alpha, \beta, \gamma, \delta, \nu, \eta \in \mathbb{R}$ such that $\beta\eta - \gamma\nu \neq 0$. This implies that

$$|P_{B'B}| = \begin{vmatrix} a_1 & a_2 & a_3 \\ \alpha & \beta & \gamma \\ \delta & \nu & \eta \end{vmatrix} \neq 0. \tag{3}$$

By products $e'_1e'_2 = 0, e'_1e'_3 = 0, e'_2e'_3 = 0$, B' is a natural basis if and only if the following conditions are satisfied:

$$\alpha a_1 + \beta a_2c_1 + \gamma a_3c_2 = 0 \tag{4}$$
$$\delta a_1 + \nu a_2c_1 + \eta a_3c_2 = 0 \tag{5}$$
$$\alpha\delta + \beta\nu c_1 + \gamma\eta c_2 = 0.$$

In the above conditions, the structure matrix of E relative to B' is:

$$M_{B'} = \begin{pmatrix} a_1^2 + a_2^2c_1 + a_3^2c_2 & 0 & 0 \\ \alpha^2 + \beta^2c_1 + \gamma^2c_2 & 0 & 0 \\ \delta^2 + \nu^2c_1 + \eta^2c_2 & 0 & 0 \end{pmatrix}.$$

Now, we start with the analysis of possible cases.

Case 1. Suppose that $a_1 \neq 0$.
By changing the basis, we may assume that $e_1^2 = e_1 + a_2e_2 + a_3e_3$. Using (4) we get $\alpha = -(\beta a_2c_1 + \gamma a_3c_2)$ and by (5), $\delta = -(\nu a_2c_1 + \eta a_3c_2)$. If we replace α and δ in (3) we obtain that:

$$|P_{B'B}| = (1 + a_2^2c_1 + a_3^2c_2)(\beta\eta - \gamma\nu).$$

Now we check that $|P_{B'B}|$ is zero or not. This happens depending on $1 + a_2^2c_1 + a_3^2c_2$ being zero or not.

Case 1.1 Assume $1 + a_2^2 c_1 + a_3^2 c_2 = 0$.
In this case E^2 has not the extension property since $|P_{B'B}| = 0$. We will analyze what happens when $1 + a_3^2 c_2 \neq 0$ and when $1 + a_3^2 c_2 = 0$.

Case 1.1.1 If $1 + a_3^2 c_2 \neq 0$.
Note that $a_2^2 c_1 \neq 0$ since otherwise we get a contradiction. Then $c_1 = \frac{-1-a_3^2 c_2}{a_2^2}$. In this case, the structure matrix is:

$$M_B = \begin{pmatrix} 1 & a_2 & a_3 \\ \frac{-1-a_3^2 c_2}{a_2^2} & \frac{-1-a_3^2 c_2}{a_2} & \frac{(-1-a_3^2 c_2) a_3}{a_2^2} \\ c_2 & c_2 a_2 & c_2 a_3 \end{pmatrix}.$$

Case 1.1.1.1 Suppose that $a_3 \neq 0$.
If we take the natural basis $B'' = \{e_1, a_2 e_2, a_3 e_3\}$, then

$$M_{B''} = \begin{pmatrix} 1 & 1 & 1 \\ -1 - a_3^2 c_2 & -1 - a_3^2 c_2 & -1 - a_3^2 c_2 \\ a_3^2 c_2 & a_3^2 c_2 & a_3^2 c_2 \end{pmatrix}. \tag{6}$$

We are going to verify two cases: $c_2 = 0$ and $c_2 \neq 0$.

Assume first $c_2 = 0$. Then $M_{B''} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. By considering another change of basis we find a structure matrix with more zeros. Namely, let $B''' = \{e_2, e_1 + e_3, e_3\}$. Then

$$M_{B'''} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In what follows we will assume that $c_2 \neq 0$. We recall that we are considering the structure matrix given in (6). Take $I :=< (1 + a_3^2 c_2) e_1 + e_2 >$. Then I is a two-dimensional evolution ideal which is degenerate as an evolution algebra.

Now, for B''' the natural basis change is

$$P_{B'''B''} = \begin{pmatrix} \frac{1+D}{2(1+a_3^2 c_2)} & \frac{-1+D}{2(1+a_3^2 c_2)} & \frac{1+D}{2(1+a_3^2 c_2)} \\ \frac{-1+D}{2(1+a_3^2 c_2)} & \frac{1+D}{2(1+a_3^2 c_2)} & \frac{-1+D}{2(1+a_3^2 c_2)} \\ -(a_3^2 c_2) & 0 & 1 \end{pmatrix}$$

where $D = (a_3^2 c_2)^3 + 2(a_3^2 c_2)^2 + (a_3^2 c_2)$ and we obtain:

$$M_{B'''} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Note that $|P_{B'''B''}| = -2(a_3^2 c_2)(1 + a_3^2 c_2)^2 \neq 0$ because $a_3^2 c_2 \neq 0$ and $a_3^2 c_2 \neq -1$.

Case 1.1.1.2 Suppose that $a_3 = 0$. Then $1 + a_2^2 c_1 = 0$ and necessarily $a_2^2 c_1 \neq 0$. In this case,

$$M_B = \begin{pmatrix} 1 & a_2 & 0 \\ \frac{-1}{a_2^2} & \frac{-1}{a_2} & 0 \\ c_2 & c_2 a_2 & 0 \end{pmatrix}. \tag{7}$$

Again we will verify two cases depending on c_2.

Assume $c_2 > 0$. Take $B'' = \{e_1, a_2e_2, \frac{1}{\sqrt{c_2}}e_3\}$. Then $M_{B''} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$, which has already appeared.

Assume $c_2 < 0$. Take $B'' = \{e_1, a_2e_2, \frac{1}{\sqrt{-c_2}}e_3\}$. Then $M_{B''} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ -1 & -1 & 0 \end{pmatrix}$.

Suppose $c_2 = 0$. Then, for $B'' = \{e_1, a_2e_2, e_3\}$ we have $M_{B''} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, matrix that has already appeared.

Case 1.1.2 Suppose that $1 + a_3^2c_2 = 0$.
This implies that $a_3^2c_2 \neq 0$ and $a_2^2c_1 = 0$.

Case 1.1.2.1 Assume $c_1 \neq 0$.
This implies that $a_2 = 0$. Moreover, since $a_3 \neq 0$, necessarily $c_2 = \frac{-1}{a_3^2}$. If we take the natural basis $B'' = \{e_1, e_3, e_2\}$, then $M_{B''} = \begin{pmatrix} 1 & a_3 & 0 \\ \frac{-1}{a_3^2} & \frac{-1}{a_3} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and we are in the Case 1.1.1.2.

Case 1.1.2.2 Suppose $c_1 = 0$ and $a_2 = 0$.
Take $B'' = \{e_1, a_3e_3, e_2\}$. Then $M_{B''} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ as above.

Case 1.1.2.3 Suppose $c_1 = 0$ and $a_2 \neq 0$.
Taking $B'' = \{e_1, e_3, e_2\}$, we are in the same conditions as in the Case 1.1.1.1 with $c_2 = 0$.

Case 1.2 Assume $1 + a_2^2c_1 + a_3^2c_2 \neq 0$.
We will prove that E^2 has the extension property. In any subcase we will provide with a natural basis for E one of its elements gives a natural basis of E^2.

Case 1.2.1 Suppose that $c_1 = c_2 = 0$.
Consider the natural basis $B' = \{e_1^2, e_2 + e_3, 2e_2 + e_3\}$. Then

$$M_{B'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We claim that this evolution algebra does not have a two-dimensional evolution ideal generated by one element. To prove this consider $f = me_1 + ne_2 + pe_3$. Then the ideal I generated by f is the linear span of $\{f\} \cup \{m^ie_i\}_{i\in\mathbb{N}}$. In order for I to have a natural basis with two elements, necessarily $m = 0$, implying that the dimension of I is one, a contradiction.

Case 1.2.2 Assume that $c_1 = 0$ and $c_2 \neq 0$.
Then $1 + c_2a_3^2 \neq 0$. For $B' = \{e_1 + a_2e_2 + a_3e_3, e_2, -a_3c_2e_1 + e_2 + e_3\}$ the structure matrix is

$$M_{B'} = \begin{pmatrix} 1 + c_2a_3^2 & 0 & 0 \\ 0 & 0 & 0 \\ c_2(1 + c_2a_3^2) & 0 & 0 \end{pmatrix}.$$

Note that E^2 has the extension property because the first element in B' is e_1^2, which is a natural basis of E^2.

Case 1.2.2.1 Assume that $c_2 > 0$.
Consider $B'' = \left\{\frac{1}{1+c_2a_3^2}e_1, e_2, \frac{1}{\sqrt{c_2}(1+c_2a_3^2)}e_3\right\}$. Then

$$M_{B''} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Case 1.2.2.2 Assume that $c_2 < 0$.
Consider $B'' = \left\{\frac{1}{1+c_2a_3^2}e_1, e_2, \frac{1}{\sqrt{-c_2}(1+c_2a_3^2)}e_3\right\}$. Then

$$M_{B''} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

We claim that these evolution algebras do not have a two-dimensional evolution ideal generated by one element. Let $f = \alpha e_1 + \beta e_2 + \gamma e_3$. Then the ideal generated by f, say I, is the linear span of $\{f, \gamma e_1, \alpha e_1\} \cup \{(\alpha^2+\gamma^2)\alpha^i e_1\}_{i\in\mathbb{N}\cup\{0\}} \cup \{(\alpha^2+\gamma^2)^2\alpha^i e_1\}_{i\in\mathbb{N}\cup\{0\}}$. After some computations, in order for I to have dimension 2 and to be degenerated implies $\alpha = 0$ or $\gamma = 0$, a contradiction.

Case 1.2.3 If $c_1 > 0$ and $c_2 > 0$.

If B' is the natural basis such that $P_{B'B} = \begin{pmatrix} 1 & a_2 & a_3 \\ -a_2c_1 & 1 & 0 \\ \frac{-a_3c_2}{1+c_1a_2^2} & \frac{-a_3a_2c_2}{1+c_1a_2^2} & 1 \end{pmatrix}$, we obtain that $M_{B'} =$

$$\begin{pmatrix} 1+a_2^2c_1+a_3^2c_2 & 0 & 0 \\ c_1(1+c_1a_2^2) & 0 & 0 \\ \frac{c_2(1+a_2^2c_1+a_3^2c_2)}{(1+c_1a_2^2)} & 0 & 0 \end{pmatrix}.$$

Now, consider the natural basis $B'' = \{f_1, f_2, f_3\}$ such that

$$P_{B''B'} = \begin{pmatrix} \frac{1}{1+a_2^2c_1+a_3^2c_2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{c_1(1+c_1a_2^2)(1+a_2^2c_1+a_3^2c_2)}} & 0 \\ 0 & 0 & \frac{\sqrt{1+c_1a_2^2}}{\sqrt{c_2}(1+a_2^2c_1+a_3^2c_2)} \end{pmatrix}$$

and the structure matrix is

$$M_{B''} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

It is not difficult to show that this evolution algebra does not have a degenerate two-dimensional evolution ideal generated by one element.

Case 1.2.4 If $c_1 > 0$ and $c_2 < 0$.

For the natural basis B' such that $P_{B'B} = \begin{pmatrix} 1 & a_2 & a_3 \\ -a_2c_1 & 1 & 0 \\ \frac{-a_3c_2}{1+c_1a_2^2} & \frac{-a_3a_2c_2}{1+c_1a_2^2} & 1 \end{pmatrix}$, we obtain that $M_{B'} =$

$$\begin{pmatrix} 1+a_2^2c_1+a_3^2c_2 & 0 & 0 \\ c_1(1+c_1a_2^2) & 0 & 0 \\ \frac{c_2(1+a_2^2c_1+a_3^2c_2)}{(1+c_1a_2^2)} & 0 & 0 \end{pmatrix}.$$

Case 1.2.4.1 Assume $1+a_2^2c_1+a_3^2c_2>0$.
Now, consider the natural basis $B''=\{f_1,f_2,f_3\}$ such that

$$P_{B''B'}=\begin{pmatrix} \frac{1}{1+a_2^2c_1+a_3^2c_2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{c_1(1+c_1a_2^2)(1+a_2^2c_1+a_3^2c_2)}} & 0 \\ 0 & 0 & \frac{\sqrt{1+c_1a_2^2}}{\sqrt{-c_2(1+a_2^2c_1+a_3^2c_2)}} \end{pmatrix}$$

and the structure matrix is

$$M_{B''}=\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Case 1.2.4.2 Assume $1+a_2^2c_1+a_3^2c_2<0$.
Now, consider the natural basis $B''=\{f_1,f_2,f_3\}$ such that

$$P_{B''B'}=\begin{pmatrix} \frac{1}{1+a_2^2c_1+a_3^2c_2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{-c_1(1+c_1a_2^2)(1+a_2^2c_1+a_3^2c_2)}} & 0 \\ 0 & 0 & \frac{\sqrt{1+c_1a_2^2}}{\sqrt{-c_2(1+a_2^2c_1+a_3^2c_2)}} \end{pmatrix}$$

and the structure matrix is:

$$M_{B''}=\begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Case 1.2.5 If $c_1<0$ and $c_2>0$.

Case 1.2.5.1 Assume $1+a_2^2c_1>0$.

For the natural basis B' such that $P_{B'B}=\begin{pmatrix} 1 & a_2 & a_3 \\ -a_2c_1 & 1 & 0 \\ \frac{-a_3c_2}{1+c_1a_2^2} & \frac{-a_3a_2c_2}{1+c_1a_2^2} & 1 \end{pmatrix}$, we obtain that $M_{B'}=$

$$\begin{pmatrix} 1+a_2^2c_1+a_3^2c_2 & 0 & 0 \\ c_1(1+c_1a_2^2) & 0 & 0 \\ \frac{c_2(1+a_2^2c_1+a_3^2c_2)}{(1+c_1a_2^2)} & 0 & 0 \end{pmatrix}.$$

Case 1.2.5.1.1 Assume $1+a_2^2c_1+a_3^2c_2>0$.
Now, consider the natural basis $B''=\{f_1,f_2,f_3\}$ such that

$$P_{B''B'}=\begin{pmatrix} \frac{1}{1+a_2^2c_1+a_3^2c_2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{-c_1(1+c_1a_2^2)(1+a_2^2c_1+a_3^2c_2)}} & 0 \\ 0 & 0 & \frac{\sqrt{1+c_1a_2^2}}{\sqrt{c_2}(1+a_2^2c_1+a_3^2c_2)} \end{pmatrix}$$

and the structure matrix is:

$$M_{B''} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

It is not difficult to show that this evolution algebra does not have a degenerate two-dimensional evolution ideal generated by one element.

Case 1.2.5.1.2 Assume $1 + a_2^2 c_1 + a_3^2 c_2 < 0$.
Now, consider the natural basis $B'' = \{f_1, f_2, f_3\}$ such that

$$P_{B''B'} = \begin{pmatrix} \frac{1}{1+a_2^2c_1+a_3^2c_2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{c_1(1+c_1a_2^2)(1+a_2^2c_1+a_3^2c_2)}} & 0 \\ 0 & 0 & \frac{\sqrt{1+c_1a_2^2}}{\sqrt{c_2}(1+a_2^2c_1+a_3^2c_2)} \end{pmatrix}$$

and the structure matrix is: $M_{B''} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, which has already appeared.

Case 1.2.5.2 Assume $1 + a_2^2 c_1 < 0$.

For B' the natural basis such that $P_{B'B} = \begin{pmatrix} 1 & a_2 & a_3 \\ -a_2c_1 & 1 & 0 \\ \frac{-a_3c_2}{1+c_1a_2^2} & \frac{-a_3a_2c_2}{1+c_1a_2^2} & 1 \end{pmatrix}$, we obtain that $M_{B'} =$

$$\begin{pmatrix} 1+a_2^2c_1+a_3^2c_2 & 0 & 0 \\ c_1(1+c_1a_2^2) & 0 & 0 \\ \frac{c_2(1+a_2^2c_1+a_3^2c_2)}{(1+c_1a_2^2)} & 0 & 0 \end{pmatrix}.$$

Case 1.2.5.2.1 Assume $1 + a_2^2 c_1 + a_3^2 c_2 > 0$.
Now, consider the natural basis $B'' = \{f_1, f_2, f_3\}$ such that

$$P_{B''B'} = \begin{pmatrix} \frac{1}{1+a_2^2c_1+a_3^2c_2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{c_1(1+c_1a_2^2)(1+a_2^2c_1+a_3^2c_2)}} & 0 \\ 0 & 0 & \frac{\sqrt{-(1+c_1a_2^2)}}{\sqrt{c_2}(1+a_2^2c_1+a_3^2c_2)} \end{pmatrix}$$

and the structure matrix is: $M_{B''} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, which has already appeared.

Case 1.2.5.2.2 Assume $1 + a_2^2 c_1 + a_3^2 c_2 < 0$.
Now, consider the natural basis $B'' = \{f_1, f_2, f_3\}$ such that

$$P_{B''B'} = \begin{pmatrix} \frac{1}{1+a_2^2c_1+a_3^2c_2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{-c_1(1+c_1a_2^2)(1+a_2^2c_1+a_3^2c_2)}} & 0 \\ 0 & 0 & \frac{\sqrt{-(1+c_1a_2^2)}}{\sqrt{c_2}(1+a_2^2c_1+a_3^2c_2)} \end{pmatrix}$$

and the structure matrix is: $M_{B''} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, which has already appeared.

Case 1.2.6 If $c_1 < 0$ and $c_2 < 0$.

Case 1.2.6.1 Assume $1 + a_2^2 c_1 > 0$.

For the natural basis B' such that $P_{B'B} = \begin{pmatrix} 1 & a_2 & a_3 \\ -a_2c_1 & 1 & 0 \\ \frac{-a_3c_2}{1+c_1a_2^2} & \frac{-a_3a_2c_2}{1+c_1a_2^2} & 1 \end{pmatrix}$, we obtain that $M_{B'} = \begin{pmatrix} 1+a_2^2c_1+a_3^2c_2 & 0 & 0 \\ c_1(1+c_1a_2^2) & 0 & 0 \\ \frac{c_2(1+a_2^2c_1+a_3^2c_2)}{(1+c_1a_2^2)} & 0 & 0 \end{pmatrix}$.

Case 1.2.6.1.1 Assume $1 + a_2^2c_1 + a_3^2c_2 > 0$.
Now, consider the natural basis $B'' = \{f_1, f_2, f_3\}$ such that

$$P_{B''B'} = \begin{pmatrix} \frac{1}{1+a_2^2c_1+a_3^2c_2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{-c_1(1+c_1a_2^2)(1+a_2^2c_1+a_3^2c_2)}} & 0 \\ 0 & 0 & \frac{\sqrt{1+c_1a_2^2}}{\sqrt{-c_2}(1+a_2^2c_1+a_3^2c_2)} \end{pmatrix}$$

and the structure matrix is: $M_{B''} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, which has appeared above.

Case 1.2.6.1.2 Assume $1 + a_2^2c_1 + a_3^2c_2 < 0$.
Now, consider the natural basis $B'' = \{f_1, f_2, f_3\}$ such that

$$P_{B''B'} = \begin{pmatrix} \frac{1}{1+a_2^2c_1+a_3^2c_2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{c_1(1+c_1a_2^2)(1+a_2^2c_1+a_3^2c_2)}} & 0 \\ 0 & 0 & \frac{\sqrt{1+c_1a_2^2}}{\sqrt{-c_2}(1+a_2^2c_1+a_3^2c_2)} \end{pmatrix}$$

and the structure matrix is: $M_{B''} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, which has already appeared.

Case 1.2.6.2 Assume $1 + a_2^2c_1 < 0$.

For the natural basis B' such that $P_{B'B} = \begin{pmatrix} 1 & a_2 & a_3 \\ -a_2c_1 & 1 & 0 \\ \frac{-a_3c_2}{1+c_1a_2^2} & \frac{-a_3a_2c_2}{1+c_1a_2^2} & 1 \end{pmatrix}$, we obtain that $M_{B'} = \begin{pmatrix} 1+a_2^2c_1+a_3^2c_2 & 0 & 0 \\ c_1(1+c_1a_2^2) & 0 & 0 \\ \frac{c_2(1+a_2^2c_1+a_3^2c_2)}{(1+c_1a_2^2)} & 0 & 0 \end{pmatrix}$.

Case 1.2.6.2.1 Assume $1 + a_2^2c_1 + a_3^2c_2 > 0$.

Now, consider the natural basis $B'' = \{f_1, f_2, f_3\}$ such that

$$P_{B''B'} = \begin{pmatrix} \frac{1}{1+a_2^2c_1+a_3^2c_2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{c_1(1+c_1a_2^2)(1+a_2^2c_1+a_3^2c_2)}} & 0 \\ 0 & 0 & \frac{\sqrt{-(1+c_1a_2^2)}}{\sqrt{-c_2}(1+a_2^2c_1+a_3^2c_2)} \end{pmatrix}$$

and the structure matrix is: $M_{B''} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, which has already appeared.

Case 1.2.6.2.2 Assume $1 + a_2^2c_1 + a_3^2c_2 < 0$.
Now, consider the natural basis $B'' = \{f_1, f_2, f_3\}$ such that

$$P_{B''B'} = \begin{pmatrix} \frac{1}{1+a_2^2c_1+a_3^2c_2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{-c_1(1+c_1a_2^2)(1+a_2^2c_1+a_3^2c_2)}} & 0 \\ 0 & 0 & \frac{\sqrt{-(1+c_1a_2^2)}}{\sqrt{-c_2}(1+a_2^2c_1+a_3^2c_2)} \end{pmatrix}$$

and the structure matrix is:

$$M_{B''} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Case 1.2.7 Suppose that $c_1 \neq 0$, $c_2 \neq 0$ and $1 + a_2^2c_1 = 0$.
Then $a_2a_3c_1c_2 \neq 0$ and so $c_1 = -\frac{1}{a_2^2}$. For B' we have

$$M_{B'} = \begin{pmatrix} a_3^2c_2 & 0 & 0 \\ \frac{c_2}{a_3^2} & 0 & 0 \\ -a_3^2c_2 & 0 & 0 \end{pmatrix}.$$

Considering natural basis $B'' = \{\frac{1}{a_3^2c_2}e_1, \frac{1}{c_2}e_2, \frac{1}{a_3^2c_2}e_3\}$ we obtain $M_{B''} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$. Which has already appeared.

Case 1.2.8 Suppose that $c_1 \neq 0$, and $c_2 = 0$.

Considering the natural basis $B'' = \{e_1, e_3, e_2\}$ we obtain $M_{B''} = \begin{pmatrix} 1 & a_3 & a_2 \\ 0 & 0 & 0 \\ c_1 & a_3c_1 & a_2c_1 \end{pmatrix}$, and we are in the same conditions as in Case 1.1.1.2.

Case 2 Suppose that $a_1 = 0$.
The structure matrix of the evolution algebra is

$$M_B = \begin{pmatrix} 0 & a_2 & a_3 \\ 0 & a_2c_1 & a_3c_1 \\ 0 & a_2c_2 & a_3c_2 \end{pmatrix}.$$

Necessarily there exists $i \in \{2, 3\}$ such that $a_i \neq 0$. Without loss in generality we assume $a_2 \neq 0$.

Case 2.1 Assume $c_1 \neq 0$. Consider the natural basis $B'' = \{e_2, e_3, e_1\}$. Then $M_{B''} =$

$\begin{pmatrix} a_2c_1 & a_3c_1 & 0 \\ a_2c_2 & a_3c_2 & 0 \\ 1 & a_3 & 0 \end{pmatrix}$ and we are in the same conditions as in Case 1.

Case 2.2 If $c_1 = 0$.

Case 2.2.1 Assume $c_2a_3 \neq 0$.

Taking the natural basis $B'' = \{e_3, e_2, e_1\}$, we obtain $M_{B''} = \begin{pmatrix} a_3c_2 & a_2c_2 & 0 \\ 0 & 0 & 0 \\ a_3 & a_2 & 0 \end{pmatrix}$ and we are in the same conditions as in the Case1.

Case 2.2.2 Suppose that $c_2a_3 = 0$.

Case 2.2.2.1 Assume $c_2 = 0$.
Take the natural basis $B' = \{a_2e_2 + a_3e_3, \frac{1}{a_2}e_3, e_1\}$. Then

$$M_{B'} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Note that E^2 has the extension property.

Case 2.2.2.2 Assume $c_2 > 0$.
Then $a_3 = 0$. For $B' = \{a_2e_2, e_1, \frac{1}{\sqrt{c_2}}e_3\}$ we have

$$M_{B'} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Case 2.2.2.3 Assume $c_2 < 0$.
Then $a_3 = 0$. For $B' = \{a_2e_2, e_1, \frac{1}{\sqrt{-c_2}}e_3\}$ we have

$$M_{B'} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Thus we have proved the following theorem.

Theorem 3.1. *Any three dimensional real evolution algebra E with $dim(E^2) = 1$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$E_1 : \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_2 : \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, E_3 : \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ -1 & -1 & 0 \end{pmatrix}, E_4 : \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_5 : \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, E_6 : \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, E_7 : \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$E_8 : \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, E_9 : \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, E_{10} : \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$E_{11} : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, E_{12} : \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, E_{13} : \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Remark 3.1. *One can classify real three-dimensional evolution algebras in case* $dim(E^2) \neq 1$. *But it will contain too long cases and subcases.*

4. Approximation of three-dimensional evolution algebras $(dim(E^2) = 1)$

In this section for the evolution algebras $E_i, i = \overline{1, 13}$ from Theorem 3.1 we will construct evolution algebras corresponding to fixed points of the operator F.

Let E be three dimensional evolution algebra with the matrix $(a_{ij}), i, j \in \{1, 2, 3\}$. We will rewrite the operator F for this evolution algebra as:

$$F : \begin{cases} x_1' = a_{11}x_1^2 + a_{21}x_2^2 + a_{31}x_3^2, \\ x_2' = a_{12}x_1^2 + a_{22}x_2^2 + a_{32}x_3^2, \\ x_3' = a_{13}x_1^2 + a_{23}x_2^2 + a_{33}x_3^2. \end{cases}$$

Jacobian of the operator F at the point x has a form

$$J_F(x) = \begin{pmatrix} 2a_{11}x_1 & 2a_{21}x_2 & 2a_{31}x_3 \\ 2a_{12}x_1 & 2a_{22}x_2 & 2a_{32}x_3 \\ 2a_{13}x_1 & 2a_{23}x_2 & 2a_{33}x_3 \end{pmatrix}.$$

Following [9] and [3] we define an evolution algebra $\widetilde{E}$ with matrix $J_F(x)$ as the matrix of structural constants.

There is no non-zero fixed point of the operator F for the evolution algebras E_i, $i \in \{1-3, 11-13\}$ and $(1; 0; 0)$ is the unique fixed point of the operator F for the evolution algebras $E_i, i = \overline{4, 10}$. So

$$J_F(1; 0; 0) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that the evolution algebra with the matrix $J_F(1; 0; 0)$ is isomorphic to the evolution algebra E_4.

5. Conclusions

One of the best methods in the studying of dynamics of nonlinear functions is to linearize this function on the open neighborhood of a certain point. We notice that the linear approximation of a function is the first order Taylor expansion around the point of interest. Our aim is a linearization of algebras with well known algebras which have some good properties. In this paper for the test algebras we have chosen evolution algebras.

We have classified three dimensional real evolution algebras with condition $dim(E^2) = 1$. Then we constructed evolution algebras corresponding to idempotent elements of two and three dimensional evolution algebras and we studied isomorphism such algebras with given evolution algebras.

Clearly it would be of interest to study the evolution algebras whose matrix of structural constants is Jacobian of evolution operator defined on the finite dimensional algebra. The development of such theory will provide the necessary tools to deal with the general situation. Future research should also include to consider the notion of approximation of finite dimensional algebras with the evolution algebras then we will study some properties of such algebras.

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