TWMS J. Pure Appl. Math. V.12, N.1, Special Issue, 2021, pp.29-40

HOW VISIONS OF ZADEH LED TO FORMATION OF NEW MODELS OF NATURAL LANGUAGE

VILÉM NOVÁK[1]

Abstract. This is an overview paper that discusses papers by Lotfi A. Zadeh related to natural language. He was the first who noticed that semantics of the natural language is vague in principle and suggested modeling it using fuzzy sets. He also came with the idea that some phenomena, namely linguistic hedges, can be modeled using special operators. In the paper, we briefly overview some of the essential Zadeh's papers and discuss results following his ideas. We especially mention the theory of evaluative linguistic expressions and fuzzy quantifiers.

Keywords: fuzzy set, fuzzy quantifier, evaluative linguistic expression.

AMS Subject Classification: 03B52, 03B65, 0302, 01A99.

1. Introduction

Lotfi A. Zadeh, without a doubt, belongs among the most influential thinkers of 20^{th} century. His idea of a fuzzy set and the phenomenon of fuzziness has appeared in almost any paper, not only from mathematics or computer science but also from other scientific areas such as biology, psychology, sociology, and even various aspects of engineering. He always emphasized the role of natural language in fuzzy set theory and suggested how fuzzy sets can be used in applications in which natural language plays an important role. His suggestions are based predominantly on the fact that the meaning of most words and expressions of natural language is intrinsically vague. And fuzzy sets are a very appropriate tool for modeling vagueness. Various arguments in favor of this assertion have been given in [24, 25].

The *vagueness phenomenon* raises when trying to *group* together objects that have a certain property φ. The result is an *actualized* grouping of objects

$$X = \{x \mid x \text{ is an object having the property } \varphi\}.$$

The essential fact is that X *cannot be in general taken as a set* since the property φ may be *vague*, i.e., it may not be possible to characterize the grouping X precisely and unambiguously; there can exist *borderline* elements x for which it is unclear whether they have the property φ (and thus, whether they belong to X), or not. On the other hand, it is always possible to characterize, at least some *typical objects* (prototypes), i.e., objects having typically the property in concern, and also, objects that for sure do not have φ. For example, everybody can show a "red apple," but it is impossible to show "all kinds of red apples". At the same time, one, without doubt, says that an orange is not a red apple.

[1] University of Ostrava, Institute for Research and Applications of Fuzzy Modeling, Czech Republic
e-mail: Vilem.Novak@osu.cz

Manuscript received January 2021.

A very good mathematical tool enabling us to grasp such groupings is fuzzy set theory. Its main idea applies a principle called *graded, or fuzzy approach* using which a relation between object and its property is characterized using a scale. Note that for the human mind, it is natural to introduce a scale whenever a vague property is encountered. For example, we often say “almost green strawberry”, “a very strong car”, etc. In all these examples, we introduce “degrees of intensity” of the property in concern that is taken from a scale that must be an ordered set, and it must have the potential to capture the continuity feature of vagueness, i.e., it must be uncountable. Since, at the same time, it must enable us to represent various kinds of operations with the properties, we come to the notion of an *algebra of truth values* (see [11, 30, 32]).

A fuzzy set A is determined by a universe U that is an ordinary set and a membership function which Zadeh denotes by μ_A and which assigns to each element $u \in U$ its membership degree $\mu_A(u) \in [0,1]$. This definition is mathematically insufficient. Therefore, we define a *fuzzy set as a function* $A : U \to [0,1]$ where $A(u)$ is a membership degree of $u \in U$. To emphasize that A is a fuzzy set on U, we can write $A \underset{\sim}{\subset} U$ and explicitly write it as

$$A = \left\{ {}^{\mu_{A(u)}}/u \;\middle|\; u \in U \right\}. \tag{1}$$

We thus put the membership degree $A(u) = \mu_A(u)$. The function μ_A is here a rule how the membership degree of $u \in U$ should be computed.

The interval $[0,1]$ is interpreted as a set of *truth values* and it is a support of an algebra $\langle [0,1], \vee, \wedge, \otimes, \to, 0, 1\rangle$ where $\vee, \wedge$ are lattice operations of supremum and infimum and $\otimes, \to$ are additional operations of multiplication and implication. These operations are used for the definition of operations with fuzzy sets. Note also that $[0,1]$ can be replaced by support of arbitrary algebra suitable as an algebra of truth values; for the details, see, e.g., [32].

Note that fuzzy sets can be taken as a certain *approximation* of vague groupings[1]. We argue that such an approximation is a consequence of an *indiscernibility* relation among objects. For example, a movie is a sequence of pictures. When projected at a sufficient speed, we cannot distinguish them one from another, and the result is a vague phenomenon that we perceive as a continuous movement. Similarly, a shape of a heap of stones is also vague, and when adding or removing one stone, its shape indiscernibly changes. This is the core of the well known *sorites (heap) paradox*:

> *One stone does not form a heap. If one adds a stone to what is not a heap, then the result is not a heap. Consequently, there are no heaps.*

Using the tools of mathematical fuzzy logic, we can demonstrate that this is not a paradox (cf. [12]), and since the corresponding logical theory has a model, it does not lead to a contradiction.

2. Zadeh's papers dealing with various aspects of natural language

2.1. **Fuzzy semantics and precisiated natural language.** Among the first papers on fuzzy sets in natural language are [42, 43] published already in 1972–73. The point of departure in them is the definition of a language L as a fuzzy relation from a set of terms, $T = \{x \mid x \text{ is a term of } L\}$, to a universe of discourse U. The fuzzy relation is given by a membership function $\mu_L : T \times U \to [0,1]$, which associates with each ordered pair $(x,y) \in T \times U$ its grade of membership $\mu_L(x,y) \in [0,1]$. For each $x \in T$, the membership function $\mu_L(x,y)$ defines a fuzzy set, $M(x) \underset{\sim}{\subset} U$, whose membership function is given by $\mu_M(x)(y) = \mu_L(x,y)$. The fuzzy set $M(x)$ is defined to be the *meaning of the term* x, with x playing the role of a name for $M(x)$.

[1] This idea occurred already in [23].

If a term $x \in T$ is a concatenation of other terms in T, that is, $x = x_1 \cdots x_n$, $x_i \in T$, $i = 1, \dots, n$, then the meaning of x can be expressed in terms of the meanings of $x_1 \cdots x_n$ through the use of a lambda-expression, or by solving a system of equations in the membership functions of the x_i which are deduced from the syntax tree of x.

The brilliant Zadeh's idea is to model linguistic expressions using special operations. One constituent of complex expressions are also the so-called, *hedges* which modify the category of membership, or truth of a predicate or a noun phrase. In his papers [42, 44], Zadeh considers linguistic hedges such as *very, more or less, much, essentially, slightly*, etc. and views them as operators which act on the fuzzy set representing the meaning of its operand. For example, in the case of the composite term *very tall man*, the operator *very* acts on the fuzzy meaning of the term *tall man.* Zadeh then defines several elementary operations on fuzzy sets from which more complicated operations may be built up by combination or composition. The concept of hedge has been further elaborated by the linguist Lakoff [15] whose work led to establishing the theory of "hedging" as a part of the classical linguistic research.

Zadeh further extends his ideas in several papers, e.g., [54, 45, 51] where also a particular language called PRUF (Possibilistic Relational Universal Fuzzy) is introduced. It is a meaning representation language for natural languages. Its specific feature is the underlying logic, which is not a two-valued or multivalued logic, but a certain kind of fuzzy logic in which the truth-values are linguistic, that is, are of the form *true, not true, very true, more or less true, not very true*, etc. Let us remark that the idea of using linguistic expressions for truth values instead of values has never obtained greater attention and did not lead to a well-established logical system.

A very interesting and often cited idea is that of *linguistic variable* [44]. Recall that this is a linguistic expression that is a name of a feature of objects, which can be evaluated by various evaluative expressions. A typical example given by Zadeh is *age* which is a feature of living beings (not necessarily people)[2] whose values are *old, very old, young, medium old,* etc. Formal definition of linguistic variable given by Zadeh is

$$\langle \mathscr{X}, T(\mathscr{X}), V, G, M \rangle, \tag{2}$$

where $\mathscr{X}$ is the *name of the variable* (e.g., *age*), $T(\mathscr{X})$ is the set of its values which are special evaluative expressions of natural language. Furthermore, U is the universe, G a syntactical rule using which the expressions $\mathscr{A}, \mathscr{B}, \ldots \in T(\mathscr{X})$ are formed, and M is a semantical rule, using which every evaluative linguistic expression $\mathscr{A} \in T(\mathscr{X})$ is assigned its meaning being a fuzzy set A in the universe V, i.e. $A \underset{\sim}{\subset} V$.

Example 2.1. Let us consider a linguistic variable $\mathscr{X} = Age$. Its term set is

$$T(\mathscr{A}) = \{\text{young, very young, not young, young or middle age, middle age, old, very old, more or less old, rather old, etc.}\}$$

Elements of $T(\mathscr{A})$ are generated using a grammar G. Zadeh suggests it to be a context-free one. This, however, is a problem since such a grammar also generates terms such as *very small and more or less medium or not extremely big* which is not an English expression and has no meaning. Note that in all Zadeh's examples of the term set $T(\mathscr{X})$ are *evaluative linguistic expressions* (see below).

[2]In fact, even to be "living" is unnecessary. For example, it has a good sense to speak about *age of stars*.

Let the universe be $U = [1, 100]$. The semantic rule M assigns to each term $\mathscr{A} \in T(\mathscr{X})$ a fuzzy set $M(\mathscr{A}) \underset{\sim}{\subseteq} U$. For example, the meaning of $\mathscr{A} =$ young can be the fuzzy set

$$M(\text{young}) = \left\{\mu(v)/v \;\middle|\; \mu(v) = \max\{0, (2.7 - 0.07v)\}, v \in [1, 100]\right\} \underset{\sim}{\subseteq} [1, 100]. \tag{3}$$

□

Zadeh's fascinating and helpful idea is that of *precisiated natural language* [53]. If one reads works of classical linguists (cf., e.g., [13, 35, 36, 37]) then he/she realizes that natural language is far too complex with many, many exceptions and so, its full formalization is (at least nowadays) practically impossible. Therefore, Zadeh came with the idea to formalize only part of the language, namely that part necessary for various kinds of applications, without capturing all the neat finenesses of its semantics. This idea seems to be the leading idea for various kinds of artificial intelligence applications connected with natural language.

2.2. **Fuzzy quantifiers.** A very fruitful and exciting concept introduced and elaborated by Zadeh is that of *fuzzy quantifiers* [47]. According to him, a fuzzy quantifier is obtained when interpreting Q in the statement "$Q\,B$ are A" as a fuzzy characterization of the relative cardinality measure by the so-called, sigma-count of B in A. More precisely, let $U = \{x_1, \ldots, x_n\}$ be a finite universe and $A, B \underset{\sim}{\subseteq} U$ be fuzzy sets in U. Then

$$Q\,B \text{ are } A \mapsto \Sigma count(B/A) \text{ is } Q \mapsto \mu_Q(\Sigma count(B/A)), \tag{4}$$

where $Q \underset{\sim}{\subseteq} [0, 1]$ is a fuzzy set determined by a membership function μ_Q that evaluates the relative size (cardinality) of the fuzzy set $B \cap A$ w.r.t. A, and

$$\Sigma count(B/A) = \frac{\sum\limits_{i=1}^{n} \max\{\mu_B(x_i), \mu_A(x_i)\}}{\sum\limits_{i=1}^{n} \mu_A(x_i)}.$$

Note that formula (4) consists of two constituents. The first constituent is the "size" of the fuzzy set $B \cap A$ (possibly related to the size of A). Zadeh speaks about cardinality but considers only fuzzy sets on the finite universe. Hence, in this case, the size of a fuzzy set is well captured by its cardinality.

The second constituent of the formula (4) is the evaluation of the relative size of $B \cap A$ using the fuzzy set Q. Zadeh suggests several basic shapes of Q. Some authors even identify fuzzy quantifiers with it.

2.3. **Commonsense reasoning and computing with words.** Other parts of Zadeh's works related to natural language discuss various aspects of commonsense reasoning [46, 48, 49]. He argues that commonsense knowledge may be regarded as a collection of dispositions, that is, propositions that are preponderantly, but not necessarily always, true. Technically, a disposition may be interpreted as a proposition with implicit fuzzy quantifiers. For example, a disposition such as *Swedes are blond* may be interpreted as *most Swedes are blond.* For purposes of inference from commonsense knowledge, the conversion of a disposition into a proposition with explicit fuzzy quantifiers gives rise to syllogistic reasoning consisting of statements of the form

$$Q\,B \text{ are } A.$$

This is a standard expression studied also in the classical theory of generalized quantifiers (cf. [33, 39, 40]). The difference here consists in the assumption that Q is a fuzzy quantifier.

A typical syllogism discussed by Zadeh in [56] is

$$\frac{\begin{array}{c} Most\ M \text{ are } X \\ Most\ M \text{ are } Y \end{array}}{Q_C\ Y \text{ are } X,} \tag{5}$$

where Q_C is an appropriate (fuzzy) quantifier. Below, we will see that such a syllogism is indeed formally valid.

As a sort of summarization of the results in the papers mentioned above, Zadeh came with the idea of *computing with words* [55, 50, 52]. Recall that computing is classically centered on manipulation with numbers and symbols. In contrast, computing with words should be a methodology in which the objects of computation are words and propositions drawn from a natural language. For example, we may use expressions such as *small, large, far, heavy, not very likely, the price of shares is declining, Prague is not far from Brno, the prices recently significantly increased*, etc. The primary purpose of using words or sentences instead of numbers is that linguistic characterizations are, in general, less specific than numerical ones but much closer to the way that humans express and use their knowledge.

For example, if we say "John's weight is big" is less specific than "John's weight is 110 kg". Despite its less informative nature, the value *big* allows humans to naturally express and deal with information that may be vague or incomplete. Computing with words is inspired by the remarkable human capacity to perform a wide variety of physical and mental tasks without any measurements and any computations. Familiar examples of such tasks are parking a car, driving in heavy traffic, playing golf, riding a bicycle, understanding speech, or summarizing a story.

3. Evaluative linguistic expressions

Zadeh's ideas about fuzzy semantics and linguistic hedges led to the more general and comprehensive concept of *evaluative linguistic expression.* A deeper study of its semantics require the following important concepts that are missing in Zadeh's papers: *possible world*, *intension*, and *extension.*

3.1. Definition of evaluative expression. Note that all examples given by Zadeh in his papers are evaluative (linguistic)[3] expressions. Their formal theory has been published in several papers (see [26, 28] and elsewhere) and so, we will not repeat it here in detail.

We distinguish *simple and compound evaluative expressions.* The former have a general form

$$\langle \text{linguistic hedge} \rangle \langle \text{TE-adjective} \rangle,$$

where ⟨TE-adjective⟩ is a special trichotomic[4] evaluative adjective that is either of the gradable adjectives (*big, cold, deep, fast, friendly, happy, high, hot, important, long, popular, rich, strong, tall, warm, weak, young*), evaluative adjectives (*good, bad, clever, stupid, ugly*, etc.), and specific adjectives such as *left, middle, right.*

The compound evaluative expressions are formed using negation and the standard connectives "and" and "or". We must be careful, however, because semantic rules do not allow us to form arbitrary boolean combinations. For example "very small and more or less medium or not extremely big" has no sense and, so, it is not an English expression.

[3] We will usually omit the adjective "linguistic".

[4] They form the *fundamental evaluative trichotomy*: two antonyms and a middle member, for example *low, medium, high*; *stupid, average, clever* , etc.

In relation with linguistic variable, Zadeh considers expressions of the form

$$X \text{ is } \langle\text{evaluative expression}\rangle, \tag{6}$$

where variable X takes values of some measurable *feature* of some noun. They are a simplified from of a special class of verb phrases of the form

$$\langle\text{noun}\rangle \text{ is } \langle\text{evaluative expression}\rangle \tag{7}$$

that are called *evaluative linguistic predications.* The verb "is" in (6) or (7) has the role of a copula joining object and its property. Examples are "temperature is low, this woman is very intelligent, the force is more or less weak, etc.

3.2. **Possible worlds, context.** The concept of a possible world was introduced by Carnap [2] and later studied by many logicians and philosophers. Recall the famous example by Quine "morning star is evening star" which is true for Venus on Earth but can be false on another planet. Informally, the possible world is a state of the world at a given point in time and space. Since such a definition can hardly be formalized, logicians usually take the possible world as a particular index using which we can distinguish various situations that lead to specific extensions (see below). In the case of evaluative expressions considered by Zadeh in his works, the possible world can be defined explicitly as follows.

Evaluative expressions specify a certain part of a nonempty, linearly ordered, and bounded set. Hence, we can model possible world for them as an interval of reals $[v_L, v_R] = [v_L, v_S] \cup [v_S, v_R] \subset \mathbb{R}$, in which three distinguished limit points can be determined: a *left bound* v_L, a *right bound* v_R, and a *central point* v_S. We will formally identify possible world with an ordered triplet

$$w = \langle v_L, v_S, v_R\rangle. \tag{8}$$

This definition has a simple justification: take into account, for example, the predication *small town.* Then, the corresponding possible world (i.e., the number of inhabitants) for the Czech Republic can be

$$\langle 3\ 000, 50\ 000, 1\ 000\ 000\rangle,$$

while for the USA it can be

$$\langle 30\ 000, 200\ 000, 10\ 000\ 000\rangle.$$

Thus, extremely small town in the Czech Republic has 3 000 people, and extremely big town 1 million. In the USA, these numbers are 10 times bigger.

The *set of possible worlds* for evaluative linguistic expressions is the set of triples of numbers

$$W = \{\langle v_L, v_S, v_R\rangle \mid v_L, v_S, v_R \in [0, \infty) \text{ and } v_L < v_S < v_R\}.$$

The term *possible world* comes from general logic and encompasses whatever situation in a broad context. Since for evaluative expressions, this is rather specific, we will replace the term *possible world* with *context.*

3.3. **The concept and semantics of evaluative expressions.**

3.3.1. *Necessity of intension.* Besides possible world, Carnap in [2] also gave many arguments in favor of the concepts of intension and extension (cf. also [5, 9, 38])). Recall that *intension* of a linguistic expression, of a sentence, or a concept, is the property denoted by it. Intension may lead to different truth values in various possible worlds, but it is itself invariant with respect to them. Each concept or a linguistic expression is a name of just one intension which does not change when changing the possible world. Intensions are assumed to keep Frege's compositionality principle: a more complex intension is a function of simpler ones.

Extension of a linguistic expression is a class of elements determined by an intension in a given time and possible world. In terms of fuzzy set theory, an extension is a fuzzy set.

On the basis of Carnap's suggestion, we define intension of a linguistic expression $\mathscr{A}$ as a function

$$\mathrm{Int}(\mathscr{A}) : W \to \mathscr{F}(U), \tag{9}$$

where $\mathscr{F}(U)$ is a set of all fuzzy sets over the universe U.

An example of the necessity to distinguish between the notion of intension and extension has been given in [29]: let us identify the meaning of a linguistic expression with a specific fuzzy set of elements. For example, take the meaning of $\mathscr{A}$ = young from (3) and consider the age 30. Then the membership degree of 30 in the fuzzy set M(young) is M(Young)(30) = 0.6. Let us now apply this definition to the meaning of the following sentence:

If Berta is young in the degree 0.6, then she is about 30.

If Berta is a woman, then the sentence "If a woman is young in the degree 0.6, then she is about 30" has a well-defined meaning. However, if Berta is a dog, then the sentence "If a dog is young in the degree 0.6, then it is about 30" has no meaning since dogs cannot live more than 20 years. Of course, we can solve this problem by defining a different fuzzy set of dogs' ages. But, this is just the implementation of the context in a sense above. Note that the intension (9) remains the same independently on whether "Berta is young" concerns woman or dog.

3.3.2. *Forming extension.* Zadeh's idea to model linguistic hedges using special operators on *extensions* (i.e., on fuzzy sets modeling them) has been further developed and modified by various authors. The problem pointed out already by Lakoff [15] is that hedge acts not only on the fuzzy set but also on the elements of the universe. The first modification has been suggested by the author of this paper in [22] where the hedge is modeled using two functions: a shift over the universe and modification of the membership function. Other suggestions came from Bouchon-Meunier, Jia and Ying [1, 41] whose model of the modifier also includes shift over the universe. A related to it is the suggestion of DeCock and Kerre [3]. The hedge in the above works is essentially modeled using the formula

$$m(A) = r \circ A \circ t,$$

where $A \underset{\sim}{\subseteq} U$ is a fuzzy set being an extension of some linguistic expression, $t : U \to U$ is called *premodifier* and $r : [0,1] \to [0,1]$ a *postmodifier*.

It is important that due to the analysis of Lakoff [15], the relation

$$\text{very small} \subset \text{small} \subset \text{roughly small},$$

and similarly for the other expressions must hold. This means that the extensions must look as in Fig.1. They are obtained by a simple procedure using a function $\nu_{a,b,c} : [0,1] \to [0,1]$ determined by three parameters a, b, c using which is each hedge uniquely characterized. This model of the semantics of simple evaluative expressions is justified and formally described in [26]. Specific formulas for computation of all the extensions can be found in [31].

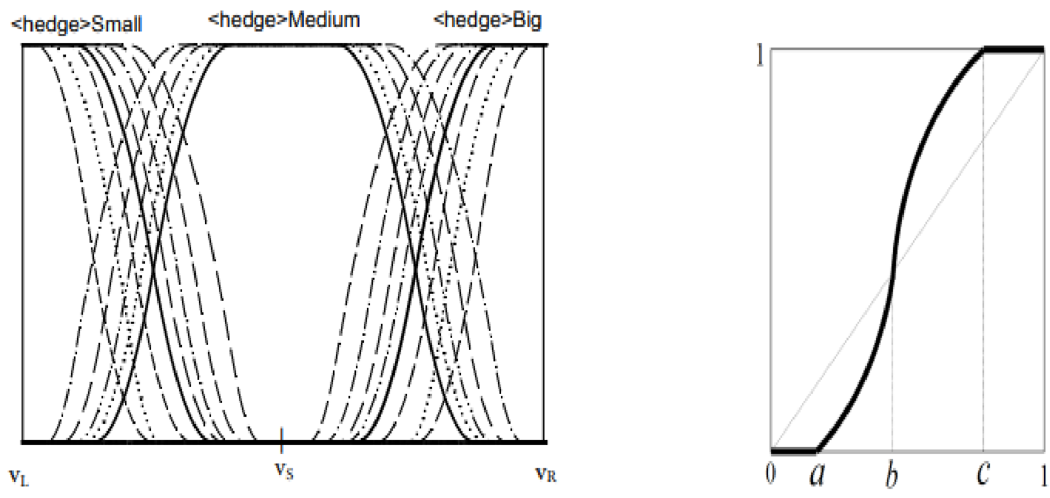

Figure 1. Shapes of fuzzy sets modeling extensions of simple evaluative expressions. **Right:** A function $\nu_{a,b,c}$ using which we obtain all the shapes on the left. Each hedge is assigned unique combination of the parameters a, b, c.

With respect to the concept of intension, a modified definition of linguistic variable is the following (cf. [29]).

Definition 3.1. *Let φ be a feature of objects referred to by a noun phrase $\mathscr{X}$. Then the linguistic variable is a tuple*

$$\langle \mathscr{X}, T(\mathscr{X}), G, V, M, W\rangle,$$

where

(a) $T(\mathscr{X})$ is a set of evaluative linguistic expressions.
(b) G is a syntactic rule generating expressions from $\mathscr{A} \in T(\mathscr{X})$.
(c) W is the set of possible worlds.
(d) M is a semantical rule assigning to each evaluative expression $\mathscr{A} \in T(\mathscr{X})$ its intension (9).

4. Quantifiers in natural language

Quantifiers form a wide class of expressions occurring in natural language. Typical examples are, except for the classical "for all" and "exists", also "most, many, a lot of, a few, several, almost all", and other ones. In logic, their theory is called the *theory of generalized quantifiers*. This theory was initiated by Mostowski [19] and since then studied by many authors (cf. [14, 16, 33, 40] and the citations therein).

The classical theory, however, does not take into account the fact that these quantifiers have vague semantics. This was noticed only by Zadeh and inspired him to introduce the concept of *fuzzy quantifier*. His theory has been further developed by various authors. For example, in [10], Glöckner generalized the above-cited works and developed a theory of semi-fuzzy quantifiers, i.e., fuzzy quantifiers defined on crisp sets, and suggested a method for how semifuzzy quantifiers can be transferred to their fuzzy analogs. Many results continuing Zadeh's theory have been summarized by Liu and Kerre in [17, 18] as well as Delgado et al. in [4] where a lot of further citations can be found.

When specifying the universe of quantification we must take into account that it can also be infinite. For classical quantifiers $\forall$ and $\exists$, we face no problem. The situation, however, changes when considering non-classical ones. Zadeh was aware of it, and, therefore, he considers only finite cardinality of the support of the fuzzy universe of quantification. However, then definition (4) will not work for statements such as "this lemonade contains a lot of sugar", "a little of sauce remained in the can", "the bottle is almost full", where the corresponding amounts of lemonade (sugar, sauce) are represented by uncountable sets whose cardinality is infinite. A solution is to replace cardinality by measure[5]. The role of measure in the theory of fuzzy quantifiers was recognized by Holčapek, and Dvořák, who developed a sophisticated mathematical theory of measure-based fuzzy quantifiers [6, 7, 8].

The Zadeh's idea of fuzzy quantifiers is closely related to the concept *intermediate quantifiers* introduced and studied in detail by Peterson in [34]. Intermediate quantifiers are expressions such as *most, a lot of, many, a few, a great deal of, large part of, small amount of*, whose semantics lays between the above classical $\forall$ and $\exists$. Peterson also studied generalized Aristotle's syllogisms and the square of opposition in which intermediate quantifiers occur. He, besides others, proved validity of 105 syllogisms with intermediate quantifiers. However, he did not leave the realm of classical two-valued logic.

Formalization of Peterson's theory in mathematical fuzzy logic has been suggested by the author of this paper in [27]. Since "quantifier" is a logical concept, it is natural to plunge it in some formal logical language. The intermediate quantifiers are thus modeled by special formulas of higher-order fuzzy logic (fuzzy type theory). The main idea is that intermediate quantifiers are classical general or existential quantifiers for which the universe of quantification is modified and the modification can be imprecise. Hence, an intermediate quantifier is either of the formulas

$$(Q^{\forall}_{Ev}\, x)(B, A) \equiv (\exists z)[(\forall x)((B|z)\, x \Rightarrow Ax) \wedge Ev((\mu B)(B|z))], \tag{10}$$

$$(Q^{\exists}_{Ev}\, x)(B, A) \equiv (\exists z)[(\exists x)((B|z)x \wedge Ax) \wedge Ev((\mu B)(B|z))]. \tag{11}$$

Both kinds of quantifiers (10) or (11) construe the sentence

"$\langle$Quantifier$\rangle$ B are A".

The formula B represents a *universe of quantification.* The symbol $B|z$ denotes *cut* of a fuzzy set B by z which takes from B only those singletons that occur also in z. The formula μ represents a measure that is evaluated by an evaluative linguistic expression Ev, for example, *extremely big, very big, not small.*

The following special intermediate quantifiers can be introduced:

A: All B are $A := (Q^{\forall}_{Bi\boldsymbol{\Delta}} x)(B, A) \equiv (\forall x)(Bx \Rightarrow Ax)$,

P: Almost all B are $A := (Q^{\forall}_{Bi\,Ex}\, x)(B, A)$,

T: Most B are $A := (Q^{\forall}_{Bi\,Ve}\, x)(B, A)$,

K: Many B are $A := (Q^{\forall}_{\neg\, Sm}\, x)(B, A)$,

I: Some B are $A := (Q^{\exists}_{Bi\boldsymbol{\Delta}} x)(B, A) \equiv (\exists x)(Bx \wedge Ax)$.

Computation of these quantifiers is simple and straightforward. For example,

$$(\text{Most } x)(B, A) = \bigvee \left\{ \bigwedge_{x \in U} ((B|Z)(x) \to A(x)) \wedge Bi\,Ve(\mu(B, B|Z)) \,\middle|\, Z \in \mathscr{F}(U) \right\}, \tag{12}$$

[5]It should be noted that Zadeh in [47] also mentions that the concept of cardinality considered by him is related to measure.

where $B, A, Z \mathrel{\underset{\sim}{\subseteq}} U$, $\rightarrow$ is a fuzzy implication[6], *BiVe* is extension of the expression *very big* and $\mu(B, B|Z)$ is a measure of a fuzzy set $B|Z$ relative w.r.t. B. Formula (12) provides rules how a truth value of the statement

"Most B are A"

can be computed. Stating informally, we take in (12) the highest (supremum of) truth value of the statements that each element x of $B|Z \mathrel{\underset{\sim}{\subseteq}} U$ for any $Z \mathrel{\underset{\sim}{\subseteq}} U$ has a property A and, at the same time, size (measure) of $B|Z$ w.r.t. B is *very big*. Example of such statement is "Most students (B) are diligent (A)".

The power of the formal theory of intermediate quantifiers consists in its wide generality because its results hold in any model and we can apply its strong formalism in proving further properties. It is also possible to prove formally validity of over 100 generalized syllogisms. A deep analysis of them has been given in papers [20, 21]. Recall that a syllogism is a triplet of formulas $\langle P_1, P_2, C\rangle$ where P_1 is a *major premise*, P_2 a *minor premise* and C is a *conclusion*. This syllogism is *valid* in a theory T if $T \vdash P_1 \,\&\, P_2 \Rightarrow C$ where $\&$ is a strong conjunction interpreted here by Łukasiewicz conjunction $a \otimes b = \max\{0, a + b - 1\}$, $a, b \in [0, 1]$. Then the syllogism (5) suggested by Zadeh, e.g., in [56] is valid with the quantifier $Q_C :=$ Some.

5. Conclusion

In this paper, we focused on L. A. Zadeh's contribution to modeling natural language semantics and using his models in applications. He was the first who noticed that the semantics of the natural language is vague in principle and suggested modeling it using fuzzy sets. He also came with the idea that some phenomena can be modeled using special operators. This concerns especially the concept of *linguistic hedge* that has also been adopted by linguists. His suggestion has been further extended, and now it seems to be well established. A suggestion to develop a model of linguistic semantics taking up a wider part of natural language semantics than that considered by Zadeh (this pertains mostly among evaluative linguistic expressions and quantifiers), including nouns and verbs, has been discussed in [28] and also in the book [23].

Let us emphasize that all the papers cited here (and even many more ones) would probably do not appear if not being Zadeh's visionariness and courage to consider areas that had been left mostly untouched by classical mathematics.

References

[1] Bouchon-Meunier, B., Jia, Y., (1992), Linguistic modifiers and imprecise categories, Int. J. Intell. Syst., 7, pp.25-36.

[2] Carnap, R., (1947), Meaning and Necessity: a Study in Semantics and Modal Logic, University of Chicago Press, Chicago, 266p.

[3] De Cock, M., Kerre, E., (2002), A context-based approach to linguistic hedges, Int. J. Appl. Math. Comput. Sci., 12, pp.371-382.

[4] Delgado, M., Ruiz, M., Sanchez, D., Vila, M., (2014), Fuzzy quantification: a state of the art, , Fuzzy Sets Syst., 242, pp.1-30.

[5] Duží, M., Jespersen, B., Materna, P., (2010), Procedural Semantics for Hyperintensional Logic, Springer, Dordrecht, 550p.

[6] Dvořák, A., Holčapek, M., (2014), Type $\langle 1, 1\rangle$ fuzzy quantifiers determined by fuzzy measures on residuated lattices, part I, Fuzzy Sets Syst., 242, pp.31-55.

[6]We usually use the Łukasiewicz one $a \rightarrow b = \min\{1, 1 - a + b\}$, $a, b \in [0, 1]$.

[7] Dvořák, A., Holčapek, M., (2014), Type $\langle 1, 1\rangle$ fuzzy quantifiers determined by fuzzy measures on residuated lattices part II, Fuzzy Sets Syst., 242, pp.56-88.

[8] Dvořák, A., Holčapek, M., (2015), Type $\langle 1, 1\rangle$ fuzzy quantifiers determined by fuzzy measures on residuated lattices, part III, Fuzzy Sets Syst., 271, pp.133-155.

[9] Gallin, D. (ed.), (1975), Intensional and Higher-Order Modal Logic (With Applications to Montague Semantics), North-Holland, Amsterdam, 158p.

[10] Glöckner, I., (2006), Fuzzy Quantifiers: A Computational Theory. Springer, Berlin, 460p.

[11] Hájek, P., (1998), Metamathematics of Fuzzy Logic, Kluwer, Dordrecht, 299p.

[12] Hájek, P., Novák, V., (2003), The sorites paradox and fuzzy logic. International, Int. J. Gen. Syst., 32, pp.373-383.

[13] Hajičová, E., Partee, B.H., Sgall, P., (1998), Topic-Focus Articulation, Tripartite Structures, and Semantics Content, Kluwer, Dordrecht, 218p.

[14] J. Barwise, R.C., (1981), Generalized quantifiers and natural language, Linguist. Philos., 4, pp.159-219.

[15] Lakoff, G.: Hedges, (1973), A study in meaning criteria and logic of fuzzy concepts, J. Philos. Log., 2, pp.458-508.

[16] Lindström, P., (1966), First order predicate logic with generalized quantifiers, Theoria, 32, pp.186-195.

[17] Liu, Y., Kerre, E., (1998), An overview of fuzzy quantifiers: Part I interpretations, Fuzzy Sets Syst., 95, pp.135-146.

[18] Liu, Y., Kerre, E., (1998), An overview of fuzzy quantifiers: Part II reasoning and applications, Fuzzy Sets Syst., 96, pp.135-146.

[19] Mostowski, A., (1957), On a generalization of quantifiers, Fundamenta Mathematicae, 44, pp.12-36.

[20] Murinová, P., Novák, V., (2012), A formal theory of generalized intermediate syllogisms, Fuzzy Sets Syst., 186, pp.47-80.

[21] Murinová, P., Novák, V., (2014), The structure of generalized intermediate syllogisms, Fuzzy Sets Syst., 247, pp.18-37.

[22] Novák, V., (1989), Fuzzy Sets and Their Applications, Adam Hilger, Bristoll, 248p.

[23] Novák, V., (1992), The Alternative Mathematical Model of Linguistic Semantics and Pragmatics, Plenum, New York, 204p.

[24] Novák, V., (2005), Are fuzzy sets a reasonable tool for modeling vague phenomena? Fuzzy Sets Syst., 156, pp.341-348.

[25] Novák, V., (2006), Fuzzy sets as a special mathematical model of vagueness phenomenon, In: Reusch, B.(ed.) Computational Intelligence, Theory and Applications, Springer, Heidelberg, pp.683-690.

[26] Novák, V., (2008), A comprehensive theory of trichotomous evaluative linguistic expressions, Fuzzy Sets Syst., 159(22), pp.2939-2969.

[27] Novák, V., (2008), A formal theory of intermediate quantifiers. Fuzzy Sets Syst., 59(10), pp.1229-1246.

[28] Novák, V., (2015), Fuzzy Natural Logic: Towards Mathematical Logic of Human Reasoning. In: Seising, R., Trillas, E., Kacprzyk, J. (eds.) Fuzzy Logic: Towards the Future, Springer, pp. 137–165

[29] Novák, V., (2020), The Concept of Linguistic Variable Revisited. In: Sugeno, M., Kacprzyk, J., Shabazova, S. (eds.) Recent Developments in Fuzzy Logic and Fuzzy Sets, Springer, Berlin, pp.105-118.

[30] Novák, V., Perfilieva, I. (eds.), (2000), Discovering the World With Fuzzy Logic, Studies in Fuzziness and Soft Computing, 57, Springer-Verlag, Heidelberg, 556p.

[31] Novák, V., Perfilieva, I., Dvořák, A., (2016), Insight into Fuzzy Modeling, Wiley & Sons, Hoboken, New Jersey, 272p.

[32] Novák, V., Perfilieva, I., Močkoř, J., (1999), Mathematical Principles of Fuzzy Logic, Kluwer, Boston, 320p

[33] Peters, S., Westerståhl, D., (2006), Quantifiers in Language and Logic, Clarendon Press, Oxford, 560p.

[34] Peterson, P., (2000), Intermediate Quantifiers. Logic, linguistics, and Aristotelian semantics, Ashgate, Aldershot, 283p.

[35] Portner, S., Partee, B., (eds.), (2002), Formal Semantics. The essential readings, Blackwell Publishing, Oxford, 570p.

[36] Rotstein, C., Winter, Y., (2004), Total adjectives vs. partial adjectives: Scale structure and higher-order modifiers, Nat. Lang. Semant., 12, pp.259-288.

[37] Sgall, P., Hajičová, E., Panevová, J., (1986), The Meaning of the Sentence in Its Syntactic and Pragmatic Aspects, D. Reidel, Dordrecht, 353p.

[38] Tichý, P., (1988), The Foundations of Frege's Logic, De Gruyter, Berlin, 320p.

[39] Westerståhl, D., (1989), Aristotelian syllogisms and generalized quantifiers, Studia Logica, 48, pp.577-585.

[40] Westerståhl, D., (1989), Quantifiers in formal and natural languages. In: Gabbay, D., Guenthner, F. (eds.) Handbook of Philosophical Logic, IV, D. Reidel, Dordrecht, pp.1-131.
[41] Ying, M., Bouchon-Meunier, B., (1997), Quantifiers, modifiers and qualifiers in fuzzy logic, J. Appl. Non-Class. Log., 7, pp.335-342.
[42] Zadeh, L.A., (1972), A fuzzy-set-theoretic interpretation of linguistic hedges, Journal of Cybernetics, 2-3, pp.4-34.
[43] Zadeh, L.A., (1973), Quantitative fuzzy semantics, Inf. Sci., 3, pp.159-176.
[44] Zadeh, L.A., (1975), The concept of a linguistic variable and its application to approximate reasoning I, II, III, I, Inf. Sci., 8-9, pp.199-257, pp.301-357, pp.43-80.
[45] Zadeh, L.A., (1978), PRUF- a meaning representation language for natural languages, Int. J. Man Mach. Stud., 10, pp.395-460.
[46] Zadeh, L.A., (1983), Commonsense knowledge representation based on fuzzy logic, IEEE Computer Magazine, 16, pp.61-65.
[47] Zadeh, L.A., (1983), A computational approach to fuzzy quantifiers in natural languages, Comput. Math. with Appl., 9, pp.149-184.
[48] Zadeh, L.A., (1985), Syllogistic reasoning in fuzzy logic and its applications to usuality and reasoning with dispositions, IEEE Trans. Syst. Man Cybern., 15, pp.754-765.
[49] Zadeh, L.A., (1996), A Formalization of Commonsense Reasoning Based on Fuzzy Logic, World Scientific Publishing Co., Inc., USA, pp.653-657.
[50] Zadeh, L.A., (1996), Fuzzy logic = computing with words. IEEE Trans, Fuzzy Systems, 4, pp.103-111.
[51] Zadeh, L.A., (1996), Precisiation of Meaning via Translation into PRUF, World Scientific Publishing Co., Inc., USA, pp.614-642.
[52] Zadeh, L.A., (2002), From computing with numbers to computing with words & from manipulation of measurements to manipulation of perceptions, Int. J. Appli. Math. Comp. Sci., 12, pp.307-324.
[53] Zadeh, L.A., (2004), Precisiated natural language, AI Magazine, 25, pp.74-91.
[54] Zadeh, L.A., (1982), Test-score semantics for natural languages, In: COLING'82, pp.425-430, Prague.
[55] Zadeh, L.A., Kacprzyk, J.E. (eds.), (1999), Computing with Words in Information/Intelligent Systems 1, Springer, Heidelberg, 518p.
[56] Zadeh, L.A., (1988), A computational theory of dispositions, In: Turksen, I.B., Asai, K., Ulusoy, G. (eds.) Computer Integrated Manufacturing, Springer Berlin Heidelberg, Berlin, Heidelberg, pp.215-241.

Vilém Novák - is a Professor, is a founder and former director of the Institute for Research and Applications of Fuzzy Modeling of the University of Ostrava, Czech Republic. He obtained a Ph.D. in mathematical logic at Charles University, Prague in 1988; D.Sc. (Doctor of Sciences) in computer science in the Polish Academy of Sciences, Warsaw in 1995; full professor at Masaryk University, Brno in 2001. His research activities include mathematical fuzzy logic, approximate reasoning, mathematical modeling of linguistic semantics, fuzzy control, analysis and forecasting of time series, and various kinds of fuzzy modeling applications. He belongs among pioneers of the fuzzy set theory.