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LUCAS POLYNOMIALS AND APPLICATIONS TO AN UNIFIED CLASS OF BI-UNIVALENT FUNCTIONS EQUIPPED WITH (P,Q)-DERIVATIVE OPERATORS

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Abstract. We want to remark explicitly that, by using the $L_n(x)$ functions (essentially linked to Lucas polynomials of the second kind), our methodology builds a bridge, to our knowledge not previously well known, between the Theory of Geometric Functions and that of Special Functions, which are usually considered as very different fields. Thus, also making use of the differential operator $\boldsymbol{I}^k_{p,q}$, we introduce a new class of analytic bi-univalent functions. Coefficient estimates, Fekete-Szegö inequalities and several special consequences of the results are obtained.

Keywords: Lucas polynomials, coefficient bounds, bi-univalent functions, q-calculus, (p,q)-derivative operator.

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1. Introduction

In ([8], [9]), for any variable quantity x, Lucas polynomials $L_n(x)$ are defined recursively as follows:

$$L_n(x) := \begin{cases} 2, & n=0, \\ x, & n=1, \\ xL_{n-1}(x)+L_{n-2}(x), & n\geq 2, \end{cases}$$

from which the first few Lucas polynomials can be found as

$$\begin{gathered} L_0(x)=2,\ L_1(x)=x,\ L_2(x)=x^2+2, \\ L_3(x)=x^3+3x,\ L_4(x)=x^4+4x^2+2,\ \ldots . \end{gathered} \tag{1}$$

By letting $x=1$ in the Lucas polynomials the Lucas numbers are obtained. The ordinary generating function of the Lucas polynomials is

$$G_{\{L_n(x)\}}(z)=\sum_{n=0}^{\infty} L_n(x)z^n=\frac{2-xz}{1-z(x+z)}.$$

Various authors have studied the properties of the Lucas polynomials and obtained many interesting results. It is well known that many number and polynomial sequences can be generated by recurrence relations of second order. Of these important sequences are the celebrated sequences of Lucas. These sequences of polynomials and numbers are of great importance in a variety of

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branches such as physics, engineering, architecture, nature, art, number theory, combinatorics and numerical analysis. These sequences have been studied in several papers from a theoretical point of view (see, [14, 15, 18, 19, 28, 29]).

Fractional calculus is a pivotal branch of mathematical analysis. This kind of calculus deals with derivatives and integrals to an arbitrary order (real or complex). Due to the frequent appearance of differential equations of fractional order in various disciplines such as fluid mechanics, biology, engineering and physics, many researchers have focused on studying them from theoretical and practical points of view. Historically speaking, a firm footing of the usage of the the q-calculus in the context of geometric function theory was actually provided and the basic (or q-) hypergeometric functions were first used in geometric function theory in a book chapter by Srivastava (see, for details, [27]). In fact, the theory of univalent functions can be described by using the theory of the q-calculus. Moreover, in recent years, such q-calculus operators as the fractional q-integral and fractional q-derivative operators were used to construct several subclasses of analytic functions (see, for example, [2, 12, 16, 20, 21]). On the other hand, Mohammed and Darus [17] studied approximation and geometric properties of these q-operators in regard to some subclasses of analytic functions in a compact disk.

Further, the possibility of extension of the q-calculus to post-quantum calculus was denoted by the (p, q)-calculus. The (p, q)-calculus with have many applications in areas of science and engineering was introduced in order to generalize the q-series by Gasper and Rahman [10]. The (p, q)-series are derived as corresponding extensions of q-identities (for example [5, 21]).

We begin by providing some basic definitions and concept details of the (p, q)-calculus used in this paper.

The (p, q)-number is given by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q} \quad (p \neq q),$$

which is a natural generalization of the q-number (see [11]), that is

$$\lim_{p \to 1} [n]_{p,q} = [n]_q = \frac{1 - q^n}{1 - q}, \quad q \neq 1.$$

It is clear that the notation $[n]_{p,q}$ is symmetric, that is,

$$[n]_{p,q} = [n]_{q,p} \, .$$

Let p and q be elements of complex numbers and $D = D_{p,q} \subset \mathbb{C}$ such that $x \in D$ implies $px \in D$ and $qx \in D$. Here, we give the following two definitions which involve a post-quantum generalization of Sofonea's work [24].

Definition 1.1. *Let* $0 < |q| < |p| \leq 1$. *A given function* $f : D_{p,q} \to \mathbb{C}$ *is called* (p, q)-*differentiable under the restriction that, if* $0 \in D_{p,q}$, then $f'(0)$ *exists.*

Definition 1.2. *Let* $0 < |q| < |p| \leq 1$. *A given function* $f : D_{p,q} \to \mathbb{C}$ *is called* (p, q)-*differentiable of order* n, *if and only if* $0 \in D_{p,q}$, then $f^{(n)}(0)$ *exists.*

Definition 1.3. (*see* [5]) *The* (p, q)-*derivative of a function* f *is defined as*

$$(D_{p,q}f)(x) = \frac{f(px) - f(qx)}{(p - q)x} \quad (x \neq 0),$$

and $(D_{p,q}f)(0) = f'(0)$, *provided* $f'(0)$ *exists.*

Let A be the class of functions f of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots , \tag{2}$$

which are analytic in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and normalized under the conditions below:

$$f(0) = 0,$$
$$f'(0) = 1.$$

Further, by S we shall denote the class of all functions in A which are univalent in Δ.

If f is of the form (2), then

$$(D_{p,q}f)(z) = 1 + \sum_{n=2}^{\infty} [n]_{p,q}\, a_n z^{n-1}.$$

With a view to recalling the principle of subordination between analytic functions, let the functions f and g be analytic in Δ. Given functions $f, g \in A$, f is subordinate to g if there exists a Schwarz function $w \in \Lambda$, where

$$\Lambda = \{w : w(0) = 0,\ |w(z)| < 1,\ z \in \Delta\},$$

such that

$$f(z) = g(w(z)) \qquad (z \in \Delta).$$

We denote this subordination by

$$f \prec g \ \text{ or } \ f(z) \prec g(z) \quad (z \in \Delta).$$

In particular, if the function g is univalent in Δ, the above subordination is equivalent to

$$f(0) = g(0), \quad f(\Delta) \subset g(\Delta).$$

The Koebe-One Quarter Theorem [6] ensures that the image of Δ under every univalent function $f \in A$ contains a disk of radius $1/4$. Thus every univalent function $f \in A$ has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$ and $f\left(f^{-1}(w)\right) = w \ \left(|w| < r_0(f)\ ,\ r_0(f) \geq \frac{1}{4}\right)$, where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{3}$$

A function $f \in A$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ. Let σ denote the class of bi-univalent functions in Δ given by (2). For a brief history and interesting examples in the class σ, see [26] (see et also [1, 3, 4, 13, 18, 25, 30]).

Recently for $f \in A$, Selvaraj et al. [23] defined and discussed (p, q)-analogue of Salagean differential operators as given below:

$$\begin{aligned}
\boldsymbol{I}_{p,q}^{0} f(z) &= f(z),\\
\boldsymbol{I}_{p,q}^{1} f(z) &= z\left(\boldsymbol{I}_{p,q} f(z)\right),\\
&\ \vdots\\
\boldsymbol{I}_{p,q}^{k} f(z) &= z\boldsymbol{I}_{p,q}(\boldsymbol{I}_{p,q}^{k-1} f(z)),\\
\boldsymbol{I}_{p,q}^{k} f(z) &= z + \sum_{n=2}^{\infty} [n]_{p,q}^{k} a_n z^n \quad (k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in \Delta).
\end{aligned}$$

If we let $p = 1$ and $q \to 1^-$, then $\boldsymbol{I}_{p,q}^{k} f(z)$ reduces to the well-known Salagean differential operator [22].

We want to remark explicitly that, by using the $L_n(x)$, functions (essentially linked to Lucas polynomials of the second kind), our methodology builds a bridge, to our knowledge not previously well known, between the theory of geometric functions and that of special functions, which are usually considered as very different fields. Thus, also making use of the differential operator $\boldsymbol{I}_{p,q}^{k}$, we introduce a new class of analytic bi-univalent functions as follows:

Definition 1.4. *A function $f \in \sigma$ is said to be in the class*

$$S_{\sigma}^{k,p,q}(\mu;x) \quad (0<\mu\leq 1,\ k\in\mathbb{N}_0,\ 0<q<p\leq 1;\ z,w\in\Delta),$$

if the following subordinations are satisfied:

$$\frac{1}{2}\left(\frac{z\left(\boldsymbol{I}_{p,q}^{k}f(z)\right)'}{f(z)}+\left(\frac{z\left(\boldsymbol{I}_{p,q}^{k}f(z)\right)'}{f(z)}\right)^{\frac{1}{\mu}}\right)\prec G_{\{L_n(x)\}}(z)-1,$$

$$\frac{1}{2}\left(\frac{w\left(\boldsymbol{I}_{p,q}^{k}g(w)\right)'}{g(w)}+\left(\frac{w\left(\boldsymbol{I}_{p,q}^{k}g(w)\right)'}{g(w)}\right)^{\frac{1}{\mu}}\right)\prec G_{\{L_n(x)\}}(w)-1,$$

where the function g is given by (3).

Example 1.1. *For $\mu=1$, a function $f\in\sigma$ is said to be in the class*

$$S_{\sigma}^{k,p,q}(x) \quad (k\in\mathbb{N}_0,\ 0<q<p\leq 1;\ z,w\in\Delta),$$

if the following conditions are satisfied:

$$\frac{z\left(\boldsymbol{I}_{p,q}^{k}f(z)\right)'}{f(z)}\prec G_{\{L_n(x)\}}(z)-1,$$

$$\frac{w\left(\boldsymbol{I}_{p,q}^{k}g(w)\right)'}{g(w)}\prec G_{\{L_n(x)\}}(w)-1,$$

where the function g is given by (3).

Example 1.2. *For $\mu=1$ and $k=0$, a function $f\in\sigma$ is said to be in the class*

$$S_{\sigma}(x) \quad (z,w\in\Delta),$$

if the following conditions are satisfied:

$$\frac{zf'(z)}{f(z)}\prec G_{\{L_n(x)\}}(z)-1,$$

and

$$\frac{wg'(w)}{g(w)}\prec G_{\{L_n(x)\}}(w)-1,$$

where the function g is given by (3).

2. Coefficient estimates

We begin this section by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $S_{\sigma}^{k,p,q}(\mu;x)$ proposed by Definition 1.4.

Theorem 2.1. *Let f given by (2) be in the class $S_{\sigma}^{k,p,q}(\mu;x)$. Then*

$$|a_2|\leq\frac{2\mu|x|\sqrt{|x|}}{\sqrt{\left|\left\{2\mu(\mu+1)\left[\left(3[3]_{p,q}^{k}-1\right)-\left(2[2]_{p,q}^{k}-1\right)\right]-\mu(\mu+3)\left(2[2]_{p,q}^{k}-1\right)^2\right\}x^2-2\left(2[2]_{p,q}^{k}-1\right)^2(\mu+1)^2\right|}},$$

and

$$|a_3|\leq\frac{4\mu^2x^2}{\left(2\,[2]_{p,q}^{k}-1\right)^2(\mu+1)^2}+\frac{2\mu\,|x|}{\left(3\,[3]_{p,q}^{k}-1\right)(\mu+1)}.$$

Proof. Let $f \in S_{\sigma}^{k,p,q}(\mu;x)$. From Definition 1.4, for some analytic functions Φ, Ψ such that $\Phi(0) = \Psi(0) = 0$,

$$|\Phi(z)| = \left|t_1 z + t_2 z^2 + t_3 z^3 + \cdots\right| < 1,$$

$$|\Psi(w)| = \left|s_1 w + s_2 w^2 + s_3 w^3 + \cdots\right| < 1$$

and

$$|t_k| \leq 1, \qquad |s_k| \leq 1 \quad (k \in \mathbb{N})$$

for all $z, w \in \Delta$, we can write

$$\frac{1}{2}\left(\frac{z\left(\boldsymbol{I}_{p,q}^k f(z)\right)'}{f(z)} + \left(\frac{z\left(\boldsymbol{I}_{p,q}^k f(z)\right)'}{f(z)}\right)^{\frac{1}{\mu}}\right) = G_{\{L(x)\}}(\Phi(z)) - 1,$$

$$\frac{1}{2}\left(\frac{w\left(\boldsymbol{I}_{p,q}^k g(w)\right)'}{g(w)} + \left(\frac{w\left(\boldsymbol{I}_{p,q}^k g(w)\right)'}{g(w)}\right)^{\frac{1}{\mu}}\right) = G_{\{L(x)\}}(\Psi(w)) - 1,$$

or equivalently

$$\frac{1}{2}\left(\frac{z\left(\boldsymbol{I}_{p,q}^k f(z)\right)'}{f(z)} + \left(\frac{z\left(\boldsymbol{I}_{p,q}^k f(z)\right)'}{f(z)}\right)^{\frac{1}{\mu}}\right) = 1 + L_1(x)\Phi(z) + L_2(x)\Phi^2(z) + \cdots, \tag{4}$$

and

$$\frac{1}{2}\left(\frac{w\left(\boldsymbol{I}_{p,q}^k g(w)\right)'}{g(w)} + \left(\frac{w\left(\boldsymbol{I}_{p,q}^k g(w)\right)'}{g(w)}\right)^{\frac{1}{\mu}}\right) = 1 + L_1(x)\Psi(w) + L_2(x)\Psi^2(w) + \cdots. \tag{5}$$

From the equalities (4) and (5), we obtain that

$$\frac{1}{2}\left(\frac{z\left(\boldsymbol{I}_{p,q}^k f(z)\right)'}{f(z)} + \left(\frac{z\left(\boldsymbol{I}_{p,q}^k f(z)\right)'}{f(z)}\right)^{\frac{1}{\mu}}\right) = 1 + L_1(x)t_1 z + \left[L_1(x)t_2 + L_2(x)t_1^2\right] z^2 + \cdots, \tag{6}$$

and

$$\frac{1}{2}\left(\frac{w\left(\boldsymbol{I}_{p,q}^k g(w)\right)'}{g(w)} + \left(\frac{w\left(\boldsymbol{I}_{p,q}^k g(w)\right)'}{g(w)}\right)^{\frac{1}{\mu}}\right) = 1 + L_1(x)s_1 w + \left[L_1(x)s_2 + L_2(x)s_1^2\right] w^2 + \cdots. \tag{7}$$

Thus, upon comparing the corresponding coefficients in (6) and (7), we have

$$\frac{\left(2\,[2]_{p,q}^k - 1\right)(\mu+1)}{2\mu} a_2 = L_1(x)t_1, \tag{8}$$

$$\frac{\mu+1}{2\mu}\left[\left(3\,[3]_{p,q}^k - 1\right)a_3 - \left(2\,[2]_{p,q}^k - 1\right)a_2^2\right] + \frac{1-\mu}{4\mu^2}\left(2\,[2]_{p,q}^k - 1\right)^2 a_2^2 = L_1(x)t_2 + L_2(x)t_1^2, \tag{9}$$

$$-\frac{\left(2\,[2]_{p,q}^k - 1\right)(\mu+1)}{2\mu} a_2 = L_1(x)s_1 \tag{10}$$

and

$$\frac{\mu+1}{2\mu}\left[\left(3\,[3]_{p,q}^k - 1\right)\left(2a_2^2 - a_3\right) - \left(2\,[2]_{p,q}^k - 1\right)a_2^2\right] + \frac{1-\mu}{4\mu^2}\left(2\,[2]_{p,q}^k - 1\right)^2 a_2^2 = L_1(x)s_2 + L_2(x)s_1^2. \tag{11}$$

From the equations (8) and (10), we can easily see that

$$t_1 = -s_1, \tag{12}$$

$$\frac{\left(2[2]_{p,q}^k-1\right)^2(\mu+1)^2}{2\mu^2}a_2^2 = L_1^2(x)\left(t_1^2+s_1^2\right). \tag{13}$$

If we add (9) to (11), we get

$$\left[\frac{\left[\left(3[3]_{p,q}^k-1\right)-\left(2[2]_{p,q}^k-1\right)\right](\mu+1)}{\mu}+\frac{\left(2[2]_{p,q}^k-1\right)^2(1-\mu)}{2\mu^2}\right]a_2^2 = L_1(x)\left(t_2+s_2\right)+L_2(x)\left(t_1^2+s_1^2\right). \tag{14}$$

By using (13) in the equality (14), we have

$$\left[\frac{\left\{2\left[\left(3[3]_{p,q}^k-1\right)-\left(2[2]_{p,q}^k-1\right)\right]\mu(\mu+1)+\left(2[2]_{p,q}^k-1\right)^2(1-\mu)\right\}L_1^2(x)-\left(2[2]_{p,q}^k-1\right)^2(\mu+1)^2L_2(x)}{2\mu^2L_1^2(x)}\right]a_2^2 = L_1(x)\left(t_2+s_2\right), \tag{15}$$

which gives

$$|a_2| \le \frac{2\mu|x|\sqrt{|x|}}{\sqrt{\left|\left\{2\mu(\mu+1)\left[\left(3[3]_{p,q}^k-1\right)-\left(2[2]_{p,q}^k-1\right)\right]-\mu(\mu+3)\left(2[2]_{p,q}^k-1\right)^2\right\}x^2-2\left(2[2]_{p,q}^k-1\right)^2(\mu+1)^2\right|}}.$$

Moreover, if we subtract (11) from (9), we obtain

$$\frac{\left(3[3]_{p,q}^k-1\right)(\mu+1)}{\mu}\left(a_3-a_2^2\right) = L_1(x)\left(t_2-s_2\right)+L_2(x)\left(t_1^2-s_1^2\right). \tag{16}$$

Then, in view of (12) and (13), also (16)

$$a_3 = \frac{2L_1^2(x)\mu^2}{\left(2[2]_{p,q}^k-1\right)^2(\mu+1)^2}\left(t_1^2+s_1^2\right)+\frac{L_1(x)\mu}{\left(3[3]_{p,q}^k-1\right)(\mu+1)}\left(t_2-s_2\right).$$

Then, with the help of (1), we finally deduce

$$|a_3| \le \frac{4\mu^2x^2}{\left(2[2]_{p,q}^k-1\right)^2(\mu+1)^2}+\frac{2\mu|x|}{\left(3[3]_{p,q}^k-1\right)(\mu+1)}.$$

□

Putting $\mu = 1$ in Theorem 2.1, we obtain

Corollary 2.1. *Let f given by (2) be in the class $S_\sigma^{k,p,q}(x)$. Then*

$$|a_2| \le \frac{|x|\sqrt{|x|}}{\sqrt{\left|\left\{\left(3[3]_{p,q}^k-1\right)-2\left(2[2]_{p,q}^k-1\right)[2]_{p,q}^k\right\}x^2-2\left(2[2]_{p,q}^k-1\right)^2\right|}}$$

and

$$|a_3| \le \frac{x^2}{\left(2[2]_{p,q}^k-1\right)^2}+\frac{|x|}{3[3]_{p,q}^k-1}.$$

Putting $\mu = 1$ and $k = 0$ in Theorem 2.1, we obtain

Corollary 2.2. *Let f given by (2) be in the class $S_\sigma(x)$. Then*

$$|a_2| \le |x|\sqrt{\frac{|x|}{2}},$$

and

$$|a_3| \le x^2+\frac{|x|}{2}.$$

3. Fekete-Szegö problem for the function class $S_{\sigma}^{k,p,q}(\mu;x)$

The classical Fekete-Szegö inequality, presented by means of Loewner's method, for the coefficients of $f\in S$ is

$$\left|a_3-\xi a_2^2\right|\le 1+2\exp(-2\xi/(1-\xi)) \quad \text{for } \xi\in[0,1).$$

As $\xi\to 1^-$, we have the elementary inequality $\left|a_3-a_2^2\right|\le 1$. Moreover, the coefficient functional

$$\Gamma_{\xi}(f)=a_3-\xi a_2^2,$$

on the normalized analytic functions f in the unit disk Δ plays an important role in function theory. The problem of maximizing the absolute value of the functional $\Gamma_{\xi}(f)$ is called the Fekete-Szegö problem, see [7].

In this section, we aim to provide Fekete-Szegö inequalities for functions in the class $S_{\sigma}^{k,p,q}(\mu;x)$. These inequalities are given in the following theorem.

Theorem 3.1. *Let f given by (2) be in the class $S_{\sigma}^{k,p,q}(\mu;x)$ and $\vartheta\in\mathbb{R}$. Then*

$$\left|a_3-\vartheta a_2^2\right|\le\begin{cases}\dfrac{2\mu|x|}{\left(3[3]_{p,q}^k-1\right)(\mu+1)}, \\[2ex] |\vartheta-1|\le\left|\dfrac{\left[\left(3[3]_{p,q}^k-1\right)-\left(2[2]_{p,q}^k-1\right)\right]}{3[3]_{p,q}^k-1}-\dfrac{(\mu+3)\left(2[2]_{p,q}^k-1\right)^2}{2(\mu+1)\left(3[3]_{p,q}^k-1\right)}-\dfrac{\left(2[2]_{p,q}^k-1\right)^2(\mu+1)}{\mu\left(3[3]_{p,q}^k-1\right)x^2}\right|, \\[2ex] \dfrac{4\mu^2|1-\vartheta||x|^3}{\left|\left\{2\mu(\mu+1)\left[\left(3[3]_{p,q}^k-1\right)-\left(2[2]_{p,q}^k-1\right)\right]-\mu(\mu+3)\left(2[2]_{p,q}^k-1\right)^2\right\}x^2-2\left(2[2]_{p,q}^k-1\right)^2(\mu+1)^2\right|}, \\[2ex] |\vartheta-1|\ge\left|\dfrac{\left[\left(3[3]_{p,q}^k-1\right)-\left(2[2]_{p,q}^k-1\right)\right]}{3[3]_{p,q}^k-1}-\dfrac{(\mu+3)\left(2[2]_{p,q}^k-1\right)^2}{2(\mu+1)\left(3[3]_{p,q}^k-1\right)}-\dfrac{\left(2[2]_{p,q}^k-1\right)^2(\mu+1)}{\mu\left(3[3]_{p,q}^k-1\right)x^2}\right|,\end{cases}$$

Proof. From (15) and (16)we obtain

$$\begin{aligned}a_3-\vartheta a_2^2 &= \frac{2\mu^2L_1^3(x)(1-\vartheta)(t_2+s_2)}{\left\{2\left[\left(3[3]_{p,q}^k-1\right)-\left(2[2]_{p,q}^k-1\right)\right]\mu(\mu+1)+\left(2[2]_{p,q}^k-1\right)^2(1-\mu)\right\}L_1^2(x)-\left(2[2]_{p,q}^k-1\right)^2(\mu+1)^2L_2(x)} \\ &+\frac{L_1(x)\mu(t_2-s_2)}{\left(3[3]_{p,q}^k-1\right)(\mu+1)} \\ &= L_1(x)\left[\left(h(\vartheta)+\frac{\mu}{\left(3[3]_{p,q}^k-1\right)(\mu+1)}\right)t_2+\left(h(\vartheta)-\frac{\mu}{\left(3[3]_{p,q}^k-1\right)(\mu+1)}\right)s_2\right],\end{aligned}$$

where

$$h(\vartheta)=\frac{2\mu^2L_1^2(x)(1-\vartheta)}{\left\{2\left[\left(3[3]_{p,q}^k-1\right)-\left(2[2]_{p,q}^k-1\right)\right]\mu(\mu+1)+\left(2[2]_{p,q}^k-1\right)^2(1-\mu)\right\}L_1^2(x)-\left(2[2]_{p,q}^k-1\right)^2(\mu+1)^2L_2(x)}.$$

Then, in view of (1), we conclude that

$$\left|a_3-\vartheta a_2^2\right|\le\begin{cases}\dfrac{2\mu|x|}{\left(3[3]_{p,q}^k-1\right)(\mu+1)}, & 0\le|h(\vartheta)|\le\dfrac{\mu}{\left(3[3]_{p,q}^k-1\right)(\mu+1)} \\[2ex] 2|x||h(\vartheta)|, & |h(\vartheta)|\ge\dfrac{\mu}{\left(3[3]_{p,q}^k-1\right)(\mu+1)}.\end{cases}$$

□

Putting $\mu=1$ in Theorem 3.1, we have

Corollary 3.1. *Let f given by (2) be in the class $S_{\sigma}^{k,p,q}(x)$ and $\vartheta \in \mathbb{R}$. Then*

$$\left|a_3-\vartheta a_2^2\right| \leq \begin{cases} \frac{|x|}{3[3]_{p,q}^k-1}, \\ |\vartheta-1| \leq \left|\frac{\left[\left(3[3]_{p,q}^k-1\right)-2\left(2[2]_{p,q}^k-1\right)[2]_{p,q}^k\right]}{3[3]_{p,q}^k-1}-\frac{2\left(2[2]_{p,q}^k-1\right)^2}{\left(3[3]_{p,q}^k-1\right)x^2}\right|, \\ \frac{|1-\vartheta||x|^3}{\left|\left\{\left(3[3]_{p,q}^k-1\right)-2\left(2[2]_{p,q}^k-1\right)[2]_{p,q}^k\right\}x^2-2\left(2[2]_{p,q}^k-1\right)^2\right|}, \\ |\vartheta-1| \geq \left|\frac{\left[\left(3[3]_{p,q}^k-1\right)-2\left(2[2]_{p,q}^k-1\right)[2]_{p,q}^k\right]}{3[3]_{p,q}^k-1}-\frac{2\left(2[2]_{p,q}^k-1\right)^2}{\left(3[3]_{p,q}^k-1\right)x^2}\right|. \end{cases}$$

Putting $\mu = 1$ and $k = 0$ in Theorem 3.1, we have

Corollary 3.2. *Let f given by (2) be in the class $S_{\sigma}^{k,p,q}(x)$ and $\vartheta \in \mathbb{R}$. Then*

$$\left|a_3-\vartheta a_2^2\right| \leq \frac{|x|}{2}.$$

Putting $\vartheta = 1$ in Theorem 3.1, we have

Corollary 3.3. *If $f \in S_{\sigma}^{k,p,q}(\mu;x)$, then*

$$\left|a_3-a_2^2\right| \leq \frac{2\mu|x|}{\left(3[3]_{p,q}^k-1\right)(\mu+1)}.$$

Putting $\mu = 1$ and $k = 0$ in Corollary 3.3, we have

Corollary 3.4. *Let f given by (2) be in the class $S_{\sigma}(x)$. Then*

$$\left|a_3-a_2^2\right| \leq \frac{|x|}{2}.$$

4. Conclusion

In this investigation, we studied the analytic bi-univalent function class

$$S_{\sigma}^{k,p,q}(\mu;x) \quad \left(0<\mu\leq 1,\ k\in\mathbb{N}_0,\ 0<q<p\leq 1;\ z,w\in\Delta\right),$$

associated with the Lucas polynomials. For functions belonging to this class, we have derived Taylor–Maclaurin coefficient inequalities and the celebrated Fekete–Szegö problem. The geometric properties of the function class $S_{\sigma}^{k,p,q}(\mu;x)$ vary based on to the values according to the parameters included. This approach has been extended to find more examples of bi-univalent functions with the Lucas polynomials.

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