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NEW INTEGRAL OPERATOR FOR SOLUTIONS OF DIFFERENTIAL EQUATIONS

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Abstract. This study is aimed to define general representation of integral transforms for solving differential equations. The Generalized Integral Transform consists of the well-known Laplace transform, Sumudu transform, Tarig transform and Elzaki transform, as a common coverage. Since all these transforms, respectively, promise an effective usage for solving differential equations, their corresponding theories can easily be derived by using Generalized Integral Transform. Moreover, this study shows that Generalized Integral Transform can be easily used to define a new integral operator which will provide the easiest approach to solution of the given differential equation. Some examples discussed in the paper show that the Generalized Integral Transform can be applicable for many differential equations while Laplace transform can not be applicable for the same differential equations.

Keywords: integral transform , Sumudu transform, Elzaki transform, Laplace transform.

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1. Introduction

One of the effective methods to solve differential equations, systems of differential equations, integral equations, system of integral equations is using the integral transform methods. For example a new integral transform, based on well-known Laplace integral transform, alternatively was introduced by Watugala in [11] especially for solving differential equations in control engineering problems. Watagula named this new integral transform as Sumudu transform and it can be given as

$$S\left(u\right)=\frac{1}{u}\int\limits_{0}^{\infty}f(t)e^{-t/u}dt.$$

Later the Sumudu transform was used for solution of some partial differential equations in [12]. As an interesting application, a relationship between Sumudu transform and Laplace transform together with Fourier transform was given in [1]. Moreover, Sumudu transform was used in [1] in an Electromagnetic research. Additionally, many different applications of Sumudu transform were done in different applications. Finally, we refer [2], [7] and [9] papers therein for further applications of Sumudu transform.

Alternatively, another integral transform called ”Elzaki transform” is defined in [4] by

$$E\left[f\left(t\right)\right]=u^{2}\int\limits_{0}^{\infty}f\left(ut\right)e^{-t}dt=T(u).$$

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The author showed in [4] that Elzaki transform can be effectively used for solution of ordinary differential equations and then this transform was used for control engineering problems. Moreover the authors showed in [5] that the Elzaki transform can be applicable for some differential equations where Sumudu transform failed. On the other hand, this also shows that while some of the defined integral transform can not be applicable for certain differential equations, other(s) can be applicable for the same differential equations [6, 8]. Hence, the Generalized Integral Transform can be used for any of the given differential equations for the selection of the most appropriate integral transform.

Recently, Tarig M.Elzaki has introduced another new integral transform and named as Tarig transform in [3], which was defined by the following formula:

$$E(u) = T\left[f\left(t\right)\right] = \frac{1}{u}\int_{0}^{\infty} f(t)e^{-\frac{t}{u^2}}dt, \quad u \neq 0.$$

In [3], the authors note that Tarig transform is a convenient tool for solving differential equations in the time domain without the need for performing an inverse Tarig transform.

As a result when application to a specific problem is required, it is difficult to choose an applicable transform. In this matter, Generalized Integral Transform (operator) and its general theories present the wide range approach to all defined and possibly definable integral transform. It also reveals the advantage of one integral transform to others on the general theories.

After this paper, one can consider a question that is it possible to define an integral transform which works some nonlinear differential equation? The authors of the paper asked the question after reading the interesting paper [10].

2. Generalized Integral Transform

In this section, the Generalized Integral Transform will be defined and then some important theorems and properties of Generalized Integral Transform will be discussed.

Definition 2.1. Given a function $f\left(t\right)$ defined for all $t \geq 0$, the Generalized Integral Transform of f can be defined as

$$G(a(s), b(s), f\left(t\right)) = \frac{1}{a(s)}\int_{0}^{\infty} e^{-a(s)b(s)t}f(t)dt, \tag{1}$$

for all expressions of $a\left(s\right)$ and $b\left(s\right)$ for which the improper integral converges and $a(s)b(s)$ non-constant positive function of s.

With an appropriate choice of $a\left(s\right)$ and $b\left(s\right)$, in (1) we can easily derive the corresponding formulations and theories for any of integral transforms. Exemplarily, the corresponding formulae of some special functions for each defined integral transforms can be easily given by the following Table 1.

Definition 2.2. *The linear transform* G^{-1} :*Range* $[G] \to G$ *defined by* $G^{-1}(a(s), b(s), F\left(s\right)) = f(t)$ *iff* $G(a(s), b(s), f\left(t\right)) = F(s)$ *for all* $t \geq 0$ *is called Inverse Generalized Integral transform.*

Basic theories of Generalized Integral Transform which provide the general approach for each integral transform such as Laplace, Sumudu, Elzaki and etc., will be discussed by the following theorems.

Table 1. Generalized Integral Transform with other well-known integral transforms.

Function	Generalized Integral Transform	Laplace	Sumudu	Elzaki	Tarig
$f(t)$	$G(a(s), b(s), f(t))$	$G(1, s)$	$G(s, \frac{1}{s^2})$	$G(\frac{1}{s}, 1)$	$G(s, \frac{1}{s^3})$
c	$\frac{c}{a^2 b}$	$\frac{c}{s}$	c	cs^2	cs
$\cos nt$	$\frac{b}{n^2+a^2b^2}$	$\frac{s}{n^2+s^2}$	$\frac{1}{1+n^2s^2}$	$\frac{s^2}{s^2n^2+1}$	$\frac{s}{s^4n^2+1}$
$\sin nt$	$\frac{n}{n^2a+a^3b^2}$	$\frac{n}{n^2+s^2}$	$\frac{ns}{s^2n^2+1}$	$\frac{s^3n}{s^2n^2+1}$	$\frac{s^3n}{s^4n^2+1}$
$\sinh nt$	$\frac{n}{a^3b^2-an^2}$	$\frac{n}{s^2-n^2}$	$\frac{ns}{1-s^2n^2}$	$\frac{s^3n}{-s^2n^2+1}$	$\frac{s^3n}{1-s^4n^2}$
$\cosh nt$	$\frac{b}{a^2b^2-n^2}$	$\frac{s}{s^2-n^2}$	$\frac{1}{1-s^2n^2}$	$\frac{s^2}{1-n^2s^2}$	$\frac{s}{1-n^2s^4}$
e^{nt}	$\frac{1}{a(-n+ab)}$	$\frac{1}{s-n}$	$\frac{1}{1-sn}$	$\frac{s^2}{1-ns}$	$\frac{s}{1-ns^2}$
t^n	$\Gamma(n+1)\frac{1}{(ab)^{1+n}a}$	$\frac{n!}{s^{n+1}}$	$n!s^n$	$n!s^{n+2}$	$n!s^{2n+1}$

Theorem 2.1. *(EXISTENCE) Consider a function f which is piecewise continuos for $t \geq 0$ and there exist a $\alpha \geq 0, c \geq 0, T \geq 0$ such that $|f(t)| \leq \alpha e^{ct}$ for all $t \geq T$. Then Generalized Integral Transform of $f(t)$ exists for all $a(s)\, b(s) > c$ and $a(s) > 0$.*

Proof. Since f is piecewise continuos, $|f(t)|$ is bounded on $[0, T]$. Thus there exist M such that $|f(t)| < M$ for all $0 \leq t \leq T$. Thus $|f| < Me^{ct}$ whenever $c, t \geq 0$ and $e^{ct} \geq 1$. Consequently, consider the improper integral

$$\begin{aligned} \frac{1}{a(s)} \lim_{r\to\infty} \int_0^r \left| e^{-a(s)b(s)t} f(t) \right| dt &\leq \frac{1}{a(s)} \lim_{r\to\infty} \int_0^r \left| e^{-a(s)b(s)t} M e^{ct} \right| dt \\ &= \frac{M}{a(s)} \lim_{r\to\infty} \int_0^r e^{-(a(s)b(s)-c)t} dt \leq \frac{M}{a(s)} \frac{1}{a(s)b(s)-c}, \end{aligned}$$

for all $a(s)\, b(s) > c$ and $a(s) > 0$. Hence Generalized Integral Transform converge. □

Theorem 2.2. *(LINEARITY) Let c_1 and c_2 be constants and function $f(t)$ and $g(t)$ have Generalized Integral Transform respectively. Then, Generalized Integral Transform of $(c_1\, f(t) + c_2\, g(t))$ exists and*

$$G(a(s), b(s), c_1 f(t) + c_2 g(t)) = c_1 G(a(s), b(s), f(t)) + c_2 G(a(s), b(s), g(t)).$$

Proof. From the definition of General Integral Transform, we can show that

$$\begin{aligned} G(a(s), b(s), c_1 f(t) + c_2 g(t)) &= \frac{1}{a(s)} \int_0^\infty e^{-b(s)a(s)t} \left[c_1 f(t) + c_2 g(t) \right] dt \\ &= \frac{1}{a(s)} \int_0^\infty e^{-b(s)a(s)t} c_1 f(t) dt + \frac{1}{a(s)} \int_0^\infty e^{-b(s)a(s)t} c_2 g(t) dt \\ &= c_1 G(a(s), b(s), f(t)) + c_2 G(a(s), b(s), g(t)). \end{aligned}$$

□

Theorem 2.3. *(UNIQUENESS) Suppose that the continuous function $f(t)$ and $g(t)$ on $[0, \infty)$ have Generalized Integral Transform respectively. If Generalized Integral Transform of $f(t)$ and $g(t)$ are equal for all $a(s)\, b(s) > c$ and $a(s) > 0$ then $f(t) = g(t)$.*

Proof. Directly obtained from the definition of Generalized Integral Transform. □

Theorem 2.4. *Assume that f has Generalized Integral Transform with $G(a(s), b(s), f(t))$. Then the Generalized integral transform of $f'(t)$, namely $G(a(s), b(s), f'(t))$, exists for all $a(s)b(s) > c$ and $a(s) > 0$ and*

$$G(a(s), b(s), f'(t)) = a(s)b(s)G(a(s), b(s), f(t)) - \frac{1}{a(s)} f(0).$$

Generally, $G(a(s), b(s), f^{(m)}(t))$ exists for all $m \in \mathbb{N}^+$ and

$$G(a(s), b(s), f^{(m)}(t)) = \frac{1}{a(s)}\left[b^m(s)a^{m+1}(s)G(s) + F(s)\right] \tag{2}$$

where $F(s) = -b^{m-1}(s)a^{m-1}(s)f(0) - b^{m-2}(s)a^{m-2}(s)f'(0) - \ ... \ - f^{m-1}(0)$.

Proof. Let $f(t)$ has Generalized Integral Transform. Then,

$$\begin{aligned} G(a(s), b(s), f'(t)) &= \frac{1}{a(s)} \int_0^\infty e^{-a(s)b(s)t} f'(t) dt \\ &= \frac{1}{a(s)}\left[f(t)e^{-a(s)b(s)t}\Big|_0^\infty + a(s)b(s)\int_0^\infty f(t)e^{-a(s)b(s)t} dt\right] \\ &= \frac{-f(0)}{a(s)} + a(s)b(s)G(a(s), b(s), f(t)). \end{aligned}$$

Consequently, the general formula for $G(a(s), b(s), f^{(m)}(t)$ can be easily shown by mathematical induction. □

By the Theorem 2.4., some Generalized Integral Transform of higher order derivatives can be given as

$$\begin{aligned} m &= 3 : G(a(s), b(s), f^{(3)}(t)) = b^3(s)a^3(s)G(a(s), b(s), f(t)) \\ &\quad + \frac{1}{a(s)}\left[-f^{(2)}(0) - b(s)a(s)f^{(1)}(0) - b^2(s)a^2(s)f(0)\right], \\ m &= 2 : G(a(s), b(s), f^{(2)}(t)) = a^2(s)b^2(s)G(a(s), b(s), f(t)) - \frac{1}{a(s)} f^{(1)}(0) - b(s)f(0). \end{aligned}$$

Theorem 2.5. *If $G(a(s), b(s), f(t))$ exist for all $a(s)b(s) > c$ with $a(s) > 0$, then $G\left\{a(s), b(s), e^{Lt}f(t)\right\}$ exist for all $a(s)b(s) > L + c$ and*

$$G(a(s), b(s), e^{Lt}f(t)) = G\left(a(s), \frac{a(s)b(s) - L}{a(s)}, f(t)\right).$$

Proof. Assume that $G(a(s), b(s), f(t))$ exists and represents Generalized Integral Transform for $f(t)$. Then,

$$\begin{aligned} G\left(a(s), \frac{a(s)b(s)-L}{a(s)}, f(t)\right) &= \frac{1}{a(s)}\int_0^\infty e^{-(a(s)(\frac{a(s)b(s)-L}{a(s)})t} f(t)dt \\ &= \frac{1}{a(s)}\int_0^\infty e^{-(a(s)b(s)-L)t} f(t)dt \\ &= \frac{1}{a(s)}\int_0^\infty e^{-(a(s)b(s))t}\left\{e^{Lt}f(t)\right\}dt \\ &= G(a(s), b(s), e^{Lt}f(t)). \end{aligned}$$

Hence $G(a(s), b(s), e^{Lt}f(t))$ is also exist and equal to $G\left(a(s), \frac{a(s)b(s)-L}{a(s)}, f(t)\right)$. □

Exemplarily, if we choose $a(s) = 1$ and $b(s) = s$ and assume that Laplace transform of $f(t)$ exists and equals to $L(s)$, then we can easily derive from Theorem 2.5. that Laplace transform of $\left\{e^{nt}f(t)\right\}$exists and equal to $L(s-n)$.

Theorem 2.6. *Suppose that $f(t)$ and $g(t)$ have Generalized Integral Transform with $G(a(s), b(s), f(t))$ and $G(a(s), b(s), g(t))$ respectively. Then the Generalized Integral Transform of the convolution $\{f(t) * g(t)\}$ exists for $a(s)b(s) > c$. Moreover*

$$G\{a(s), b(s), f(t) * g(t)\} = a(s)\, G(a(s), b(s), f(t)) \cdot G(a(s), b(s), g(t))$$

and

$$G^{-1}(a(s), b(s), a(s)\, G(a(s), b(s), f(t)) \cdot G(a(s), b(s), g(t))) = f(t) * g(t).$$

Proof. Assume that Generalized Integral Transform of $f(t)$ and $g(t)$ exist respectively. Then, Generalized Integral Transform of convolution $\{f(t) * g(t)\}$ can be given as

$$\begin{aligned} G(a(s), b(s), f * g) &= \frac{1}{a(s)}\int_{t=0}^\infty e^{-(a(s)b(s))t}\left[\int_{x=0}^t f(x)g(t-x)dx\right]dt \\ &= \frac{1}{a(s)}\int_{t=0}^\infty\int_{x=0}^t e^{-(a(s)b(s))t} f(x)g(t-x)dxdt \\ &= \frac{1}{a(s)}\int_{x=0}^\infty\int_{t=x}^\infty e^{-(a(s)b(s))t} f(x)g(t-x)dtdx \\ &= \frac{1}{a(s)}\int_{x=0}^\infty\int_{t_1=0}^\infty e^{-(a(s)b(s))(t_1+x)} f(x)g(t_1)dt_1dx (t_1 = t-x) \\ &= a(s)\, G(a(s), b(s), f(t)) \cdot G(a(s), b(s), g(t)). \end{aligned}$$

□

Indeed, if we consider Sumudu transform of $f(t)$ and $g(t)$ exist with $S_1(s)$ and $S_2(s)$ respectively, then Sumudu transform of convolution $f(t)*g(t)$ also exist and according to the Theorem 2.6. with $a(s)=s$ and $b(s)=\frac{1}{s^2}$, it is equal to $s\cdot S_1(s)\cdot S_2(s)$.

Mathematical models of many problems in science and engineering involve discontinuous phenomena. One simple example can be taught of the unit step function at $t=k$ such that

$$u_k(t)=\{\begin{array}{c}0 \text{ if } t<k\\ 1 \text{ if } t\geq k\end{array}. \tag{3}$$

We now discuss Generalized Integral Transform of function (3).

Theorem 2.7. *Let $u_k(t)$ be unit step function in (3). Then the function $u_k(t)$ in (3) has Generalized Integral transform $G(a(s),b(s),u_k(t))$ for all $k\geq 0$ and*

$$G(a(s),b(s),u_k(t))=\frac{e^{-a(s)b(s)k}}{a^2(s)b(s)}$$

Moreover,

$$G(a(s),b(s),u_k(t)f(t-k))=e^{-a(s)b(s)k}\cdot G(a(s),b(s),f(t))$$

where $a(s)b(s)>c+k$.

Proof. Let $u_k(t)$ be unit step function in (3). Then, by the definition of Generalized Integral Transform and $\int\limits_0^k e^{-b(s)a(s)t}u_k(t)\,dt=0$, we have

$$G(a(s),b(s),u_k(t))=\frac{1}{a(s)}\left[\int\limits_0^k e^{-b(s)a(s)t}u_k(t)\,dt+\int\limits_k^\infty e^{-b(s)a(s)t}u_k(t)\,dt\right]=\frac{e^{-a(s)b(s)k}}{a^2(s)b(s)}.$$

Consequently, from the definition of Generalized Integral Transform,

$$e^{-a(s)b(s)k}\cdot G(a(s),b(s),f(t))=\int\limits_0^\infty e^{-b(s)a(s)(x+k)}f(x)\,dx.$$

The substitution $t=x+k$ gives

$$e^{-a(s)b(s)k}\cdot G(a(s),b(s),f(t))=\int\limits_k^\infty e^{-b(s)a(s)t}f(t-k)\,dt=\int\limits_0^\infty e^{-b(s)a(s)t}u_k(t)\cdot f(t-k)\,dt.$$

□

Theorem 2.8. *Suppose that $f(t)$ has Generalized Integral Transform. Then the functions $(tf(t))$ and $\left(t^2f(t)\right)$ have Generalized Integral Transform for all $a(s)b(s)>c$ and*

$$G(a(s),b(s),-tf(t))=\frac{\frac{d}{ds}\left[G(a(s),b(s),f(t))\right]}{\frac{d}{ds}(a(s)b(s))}-\frac{[\frac{d}{ds}\left(\frac{1}{a(s)}\right)]a(s)G(a(s),b(s),f(t))}{\frac{d}{ds}(a(s)b(s))} \tag{4}$$

and

$$G\left(a(s),b(s),t^2f(t)\right)=\frac{1}{\frac{d}{ds}(a(s)b(s))}\{\frac{d}{ds}[G(a(s),b(s),-tf(t))]-[\frac{d}{ds}\left(\frac{1}{a(s)}\right)]a(s)G(a(s),b(s),-tf(t))\}. \tag{5}$$

Proof. Let $f(t)$ has Generalized Integral Transform. Then differentiation of Generalized Integral Transform with respect to s gives

$$\begin{aligned}\frac{d}{ds}G\left(a(s),b\left(s\right),f\left(t\right)\right) &= \int_0^\infty \frac{d}{ds}\left[\frac{1}{a(s)}e^{-a(s)b(s)t}f\left(t\right)\right]dt \\ &= \frac{d}{ds}\left(a\left(s\right)b\left(s\right)\right)\int_0^\infty \frac{1}{a(s)}\left[e^{-a(s)b(s)t}\left(-tf\left(t\right)\right)\right]dt \\ &+(\frac{d}{ds}\left(\frac{1}{a\left(s\right)}\right))a\left(s\right)(\frac{1}{a\left(s\right)}\int_0^\infty\left[e^{-a(s)b(s)t}\left(f\left(t\right)\right)\right]).\end{aligned}$$

Furthermore, the second case (5) can easily be shown by differentiating (4) with respect to s. □

Indeed, if we consider Tarig transform of $f(t)$ exist and equal to $T\left(s\right)$, then Tarig transform of new function $tf(t)$ also exist and it is equal to $\frac{1}{2}\left[s^3\frac{d}{ds}T\left(s\right)+s^2T\left(s\right)\right]$ according to the theorem 10 with $a\left(s\right)=s$ and $b\left(s\right)=\frac{1}{s^3}$.

More precisely, the general case $G\left(a(s),b\left(s\right),(-1)^n t^n f\left(t\right)\right)$ for all positive integer $n>2$ can be obtained as

$$\begin{aligned}G\left(a(s),b\left(s\right),(-1)^n t^n f\left(t\right)\right) &= \frac{\frac{d}{ds}\left[\left(G\left(a\left(s\right),b\left(s\right),(-1)^{n-1}t^{n-1}f\left(t\right)\right)\right)\right]}{\frac{d}{ds}\left(a(s)b(s)\right)}- \\ &\frac{[\frac{d}{ds}\left(\frac{1}{a(s)}\right)]a(s)G\left(a(s),b\left(s\right),(-1)^{n-1}t^{n-1}f\left(t\right)\right)}{\frac{d}{ds}\left(a(s)b(s)\right)}. \end{aligned} \tag{6}$$

If we consider the Laplace transform with $a\left(s\right)=1$ and $b\left(s\right)=s$, then second term of formula (6) will be zero. Hence we can easily derive the corresponding formula for Laplace transform as

$$L\left[t^n f\left(t\right)\right]=(-1)^n L^{(n)}\left(s\right) \tag{7}$$

where $L\left(s\right)$ is Laplace transform of $f\left(t\right)$.

We notice that although Laplace transform gives the easiest formulation for (6), it can not be useful for many differential equations.

Theorem 2.9. *Suppose that $f(t)$ has Generalized Integral Transform with $G(s,f)$. Then, the function $(f(t)/t)$ has Generalized Integral Transform for all $a(s)b(s)>c$ and*

$$G(a(s),b(s),\frac{f(t)}{t})=\frac{1}{a(s)}\int_s^\infty a(\tau)(a(\tau)b(\tau))'(\tau)G(\tau)d\tau$$

Proof. Let $f(t)$ has Generalized Integral Transform with $G(s,f)$ and $g(t)=(f(t)/t)$. Then by Theorem 2.6.

$$\begin{aligned}G(a(s),b(s),\frac{f(t)}{t}) &= G(a(s),b(s),t\cdot g(t)) \\ &= \frac{G'(a(s),b(s),g(s))+(\ln a(s))'G(a(s),b(s),g(s))}{(a(s)b(s))'}\end{aligned}$$

Consequently

$$G'(a(s),b(s),g(s))+(\ln a)'G(a(s),b(s),g(s))=(ab)'G(s,f) \tag{8}$$

Hence, first order linear differential equation (8) gives

$$G(a(s), b(s), \frac{f(t)}{t}) = \frac{1}{a(s)} \int_{s}^{\infty} a(\tau)(ab)'(\tau)G(\tau)d\tau.$$

□

Theorem 2.10. *Let $\delta(t)$ be Dirac delta function. Then the function $\delta(t-c)$ has Generalized Integral Transform for all $c > 0$ and*

$$G(a(s), b(s), \delta(t-c)) = \frac{e^{-a(s)b(s)c}}{a(s)}.$$

Proof. Let $\delta(t)$ be Dirac delta function. By the property of Dirac delta

$$\int_{c-\epsilon}^{c+\epsilon} \delta(t-c)f(t)dt = f(c) \text{ for all } \epsilon > 0,$$

we have

$$G(a(s), b(s), \delta(t-c)) = \frac{1}{a(s)} \int_{0}^{\infty} \delta(t-c)e^{-a(s)b(s)t}dt = \frac{e^{-a(s)b(s)c}}{a(s)}.$$

□

3. Applications

Different applications will be considered to show the applicability of proposed integral transform.

3.1. Analysis of Non-Homogenous Cauchy- Euler differential equations via generalized Integral Transform.

The higher order differential equations which involve non-constant coefficients need more work in establishing a solution. In many cases, the existence methods are not able to solve them. Some particular type of higher order differential equations can be solved by using integral transforms such as Laplace, Sumudu transforms and others. Generalized Integral Transform can also be applicable for many differential equation with non-constant coefficients. The superiority of the proposed method is that the appropriate $a(s)$ and $b(s)$ can be selected according to the given differential equation. Thus, the proposed method will facilitate the solution and will help us to derive the solution in a simple and short way.

To show the applicability of the Generalized Integral Transform, we now use it to analyze nonhomogeneous Cauchy-Euler differential equation of the form

$$t^n y^{(n)} + a_{n-1}t^{n-1}y^{(n-1)} + \cdots + a_0 y = f(t), \quad y(0) = y_0, ..., y^{n-1}(0) = y_{n-1}, \tag{9}$$

where $a_0, a_1, \cdots, a_{n-1}$. Generally, we can apply the transformation $t = e^x$ which reduces the equation (9) to a linear ODE with constant coefficients in variable x. However, this method contain especially two difficult procedures which are finding the roots of characteristic equation and corresponding particular solutions for $f(e^x)$. In this matter, Generalized Integral Transform with the formula (6) can be effectively used for solving differential equation (9) in a simple way. According to the analysis with Generalized Integral Transform, we can understand which particular selection of $a(s)$ and $b(s)$ can be more appropriate for solution. Taking Generalized integral transform of both side of differential equation (9), we have

$$\sum_{i=0}^{n} G\left(a(s), b(s), t^i y^{(i)}\right) = G(a(s), b(s), f(t)). \tag{10}$$

Since we consider higher order derivatives of solution $y(x)$ in equation (10), we should apply the formulae (2) and (6) together. Therefore very complicated formulae will be obtained for the equation (10) for finding corresponding solutions. Nevertheless, these complicated formulae can be applicable for these type of differential equations where other integral transforms as well as Laplace transform may not be applied. We suggest to use one of the mathematical software such as Matlab, Mathematica to find the corresponding terms of higher order derivatives of $G(a(s), b(s), y(t))$ in the equation (10).

To demonstrate the applicability of introduced method, we now discuss the equation (9) for $n = 2$ with initial conditions $y(0) = y'(0) = 0$. In some applications, Laplace transform can not be useful and it turns another complicate differential equation.

The authors in [5] considered the differential equation

$$t^2 y'' + 4ty' + 2y = 12t^2, \quad y(0) = y'(0) = 0, \tag{11}$$

and showed that the equation (11) can be solved by Elzaki transform where Laplace and Sumudu transforms can not be applicable for the equation (11). In this example the selection $a(s) = 1/s$ and $b(s) = 1$ eliminate the terms $G'(1/s, 1, y)$ and $G(1/s, 1, y)$ together. But if the coefficients of equation (11) are changed, the Elzaki transform cannot be applicable too. Alternatively, we can use Generalized Integral Transform to choose the most useful integral transform for solution.

We now aim to consider the non-homogenous Cauchy Euler differential equation of second order with the following initial conditions

$$x^2 y^{''} - Axy^{'} + By = f(t) \text{ with } y(0) = y^{'}(0) = 0, \tag{12}$$

where A, B are real constants.

If we apply both the formulae (2) and (6) of Generalized Integral Transform, then the differential equation (12) will be represented another second order differential equation. The selection of $a(s)$ and $b(s)$ play a very great importance for the solution of differential equation (12). For comparison with Laplace transform, we are choosing $a(s) = 1$ and then the coefficients of Generalized Integral Transform will be

$$\begin{aligned}
G(a(s), b(s), y) &: \quad A + B + \frac{(b^2)^{''}(b)^{'}}{((b)^{'})^3} - \frac{(b^2)^{'}(b)^{''}}{((b)^{'})^3}, \\
G'(a(s), b(s), y) &: \quad \frac{2(b^2)^{'}(b)^{'}}{((b)^{'})^3} + \frac{Ab}{(b)^{'}} - \frac{b^2(b)^{''}}{((b)^{'})^3}, \\
G''(a(s), b(s), y) &: \quad \frac{b^2(b)^{'}}{((b)^{'})^3}.
\end{aligned}$$

If we consider Laplace transform, we will have second order differential equation

$$(A + B)L(s) + (4 + A)sL'(s) + s^2 L''(s) = L(f(t)). \tag{13}$$

As a result, equation (13) with Laplace transform is turned to another Cauchy-Euler differential equation. We now understand that Laplace transform can not be applicable for differential equation (12). Hence we can choose different $b(s)$ for finding solution. Let us choose $b(s) = s^T$ where T is any constant. Then, coefficients of G, G', G'' can be obtain as

$$
\begin{aligned}
G\left(a(s), b\left(s\right), y\right) &: A+B+2,\\
G\left(a(s), b\left(s\right), y\right)' &: (\frac{3+A}{T}+\frac{1}{T^2})s,\\
G\left(a(s), b\left(s\right), y\right)'' &: \frac{s^2}{T^2}.
\end{aligned}
$$

For simplicity we aim to eliminate the terms $G\left(a(s), b\left(s\right), y\right)$ and $G\left(a(s), b\left(s\right), y\right)'$. Thus we take $T=\frac{-1}{3+A}$ and $A+B=-2$. Hence, the coefficients of G and G' will be zeros and the given differential equation representing (12) can be solved by the following differential equations:

$$G''\left(1, s^{\frac{-1}{3+A}}, y\left(t\right)\right)=\frac{(\frac{-1}{3+A})^2}{s^2}G(1, s^{\frac{-1}{3+A}}, f\left(t\right)). \tag{14}$$

Hence the solution of equation (12) can be obtained by using equation (14) as

$$y\left(t\right)=(\frac{-1}{3+A})^2G^{-1}(1, s^{\frac{-1}{3+A}}, \int\limits_{x=s}\int\limits_{s=x}\frac{G(1, s^{\frac{-1}{3+A}}, f\left(t\right))}{s^2}dxds). \tag{15}$$

We underline that when $A+B\neq-2$, the selection $a(s)=1$ is not suitable to solution. Therefore, Laplace transform can not be applicable for these type of differential equations. Hence, we should select different $a\left(s\right)$ and $b\left(s\right)$ and use Generalized Integral Transform.

Example 3.1. Let us consider the following IVP

$$t^2y''+ty'-y=t^2,\ y\left(0\right)=y'\left(0\right)=0. \tag{16}$$

The equation (16) is a special form of differential equation (12) with $A=-1$ and $B=-1$. Since, $A+B=-2$, we can consider the Generalized Integral Transform with $a\left(s\right)=1$ and $b\left(s\right)=s^T$ where $s>0$ and $T=\frac{-1}{2}$. Consequently, we can use the formula (15) for solution which gives

$$y\left(t\right)=\left(\frac{-1}{2}\right)^2G^{-1}\left(1, s^{\frac{-1}{2}}, \int\limits_{x=s}\int\limits_{s=x}\frac{G(1, s^{\frac{-1}{2}}, t^2)}{s^2}dxds\right)=\frac{1}{4}G^{-1}\left(1, s^{\frac{-1}{2}}, \int\limits_{x=s}(\int\frac{2}{\sqrt{x}}dx)ds\right)$$

$$=\frac{1}{4}G^{-1}\left(1, s^{\frac{-1}{2}}, \frac{8}{3}s^{\frac{3}{2}}\right)=\frac{1}{3}t^2.$$

We underline that Laplace transform as well as Sumudu and Elzaki transforms can not work well for the given differential equation (16) .

3.2. **Some numerical examples via generalized Integral Transform.** In this subsection, two different examples will be considered to demonstrate the applicability of proposed integral transform. Firstly, a system of simultaneous differential equations will be solved by using Generalized Integral Transform. Consequently, a Volterra integral equation will be discuss. It is also shown that Laplace transform can not work well for these examples.

Example 3.2. Consider the following system of differential equations

$$
\begin{aligned}
(t^2-1)y+x' &= t^2e^t, \\
y'+x &= 2e^t-1,
\end{aligned}
\tag{17}
$$

under the initial conditions $y(0)=1$ and $x(0)=0$. Applying Generalized Integral Transform to the system (17) and then multiplying second equation by $(-a\left(s\right)b\left(s\right))$ and adding it to first equation, we have

$$\begin{aligned}
G\left(a(s), b\left(s\right), y\right) &: \frac{\left(\ln a\right)''}{\left(ab\right)'^{2}} - \frac{\left(ab\right)''\left(\ln a\right)'}{\left(ab\right)'^{3}} + \frac{\left(\ln a\right)'^{2}}{\left(ab\right)'^{2}} - 1 - s^{2},\\
G'\left(a(s), b\left(s\right), y\right) &: 2\frac{\left(\ln a\right)'}{\left(ab\right)'^{2}} - \frac{\left(ab\right)''}{\left(ab\right)'^{3}},\\
G''\left(a(s), b\left(s\right), y\right) &: \frac{1}{\left(ab\right)'^{2}},\\
G\left(a(s), b\left(s\right), x\right) &: 0,\\
s\text{terms} &: \frac{2a}{(ab-1)^{3}} - \frac{2b}{(ab-1)} + \frac{1}{a} - b.
\end{aligned}$$

According to the Generalized Integral Transform, we will consider two integral transforms which are Laplace and one of new integral transform with $a\left(s\right)=e^{\frac{s^2}{2}}$ and $a\left(s\right)b\left(s\right) = s$.

Solution 3.1. Applying Laplace transform to the system (26) and then eliminating x terms, we have second order non-homogeneous variable coefficient differential equations with respect to s :

$$L''(y) - (1+s^2)L(y) = \frac{2}{(s-1)^3} - \frac{2s}{(s-1)} + 1 - s. \tag{18}$$

The differential equation (18) is a complicated equation and hence Laplace transform can not work well.

Solution 3.2. Consider a new integral transform with $a\left(s\right)=e^{\frac{s^2}{2}}$ and $a\left(s\right)b\left(s\right) = s$. Applying this transform to the system (17) and then eliminating x terms, we have

$$\begin{aligned}
G''\left(y\right) + 2sG'\left(y\right) &= -se^{\frac{-S^2}{2}} + e^{\frac{-S^2}{2}} - \frac{2se^{\frac{-S^2}{2}}}{s-1} + \frac{2e^{\frac{-S^2}{2}}}{(s-1)^3}\\
&= \left[e^{\frac{-S^2}{2}}\left(-s+1-\frac{2s}{s-1}+\frac{2}{(s-1)^3}\right)\right].
\end{aligned} \tag{19}$$

If $w = G'\left(y\right)$ is considered, the equation (19) will reduce to first order differential equation. Consequently,

$$(e^{s^2}w)' = e^{\frac{s^2}{2}}\left(-s+1-\frac{2s}{s-1}+\frac{2}{(s-1)^3}\right).$$

Letting $w = e^{-\frac{s^2}{2}}A\left(s\right)$, then $G'\left(y\right) = w = -\left[\frac{e^{\frac{-s^2}{2}}}{(-1+s)^2} - \frac{e^{\frac{-s^2}{2}}S}{(s-1)}\right] = \left(\frac{e^{\frac{-S^2}{2}}}{s-1}\right)'$, Hence, $G = \frac{e^{\frac{-S^2}{2}}}{s-1}$ and $y\left(t\right) = e^{t}$. Consequently, $x\left(t\right) = e^{t} - 1$.

The above example demostrates that Generalized Integral Transform provides easiest and simple solution for many simultaneous differential equations.

Example 3.3. Consider the following Volterra integral equation

$$ty' = t + 1 - \int_0^t \frac{\delta(t-\tau)}{\tau}y(\tau)d\tau, \tag{20}$$

with $y(0) = 0$.

Solution 3.3. Let $y(t)$ has Laplace Transform $F(s)$. The Laplace transform of equation (20) with its convulation theorem gives

$$-sF''(s) - F(s) = \frac{1}{s^2} + \frac{1}{s} - L\left\{\delta(t)\right\}L\left\{\frac{y(t)}{t}\right\}$$

which implies

$$-sF'(s) - F(s) = \frac{1}{s^2} + \frac{1}{s} - \int_s^\infty F(\tau)d\tau. \tag{21}$$

The equation (21) involves second order variable coefficient non-homogenous differential equations. Hence Laplace transform does not work well for this integral equation.

Solution 3.4. Let $y(t)$ has Generalized Integral Transform $G(s)$. According to the Theorems 2.9. and 2.10 the Generalized integral transform of (20) with general convolution theorem, gives

$$-G(s) - \frac{(ab)G'(s)}{(ab)'} - \frac{(\ln a)'abG(s)}{(ab)'} = \frac{1}{a^3b^2} + \frac{1}{a^2b} - \frac{1}{a}\int_s^\infty a(\tau)(ab)'(\tau)G(\tau)d\tau. \tag{22}$$

We aim to eliminate $G(s)$ in the equation (22). Setting $a(s) = e^{-s}$and $b(s) = e^{2s}$ give

$$-G' = \frac{1}{e^s} + 1 + e^s \int_s^\infty G(\tau)d\tau,$$

which is simplest representation of original Volterra integral equation (20). Consequently,

$$-T'' = \frac{1}{e^s} + 1 + e^s T$$

where $G(s) = T'(s)$. Then $T = -e^{-s}$ and $G(s) = e^{-s}$.By table 1 with $a(s) = e^{-s}$and $b(s) = e^{2s}$, $y(t) = t$.

4. Conclusion

The Generalized Integral Transform is defined and used for solutions of ordinary differential equations, simultaneous differential equations and Volterra integral equations. The selected applications in the paper demonstrate the applicability of this newly proposed integral transform. The theories given in the paper can provide a better selection to functions $a(s)$ and $b(s)$ for Generalized Integral Transform according to the given differential equation and this integral transform with selected $a(s)$ and $b(s)$ gives the easiest approach to corresponding solutions. Additionally, this paper generalizes the theories of each defined integral transforms such as Laplace, Sumudu, Elzaki and etc. Researchers can easily select the most convenient integral transform to solve the differential equations appear in science and engineering. Further studies can be conducted to derive simple and short solutions by rehandling existence differential equations.

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