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AN INVERSE BOUNDARY VALUE PROBLEM FOR THE BOUSSINESQ-LOVE EQUATION WITH NONLOCAL INTEGRAL CONDITION

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Abstract. The work is devoted to the study of the solvability of an inverse boundary value problem with an unknown time-dependent coefficient for the Boussinesq-Love equation with Nonlocal Integral Condition. The goal of the paper consists of the determination of the unknown coefficient together with the solution. The problem is considered in a rectangular domain. The definition of the classical solution of the problem is given. First, the given problem is reduced to an equivalent problem in a certain sense. Then, using the Fourier method the equivalent problem is reduced to solving the system of integral equations. Thus, the solution of an auxiliary inverse boundary value problem reduces to a system of three nonlinear integro-differential equations for unknown functions. A concrete Banach space is constructed. Further, in the ball from the constructed Banach space by the contraction mapping principle, the solvability of the system of nonlinear integro-differential equations is proved. This solution is also a unique solution to the equivalent problem. Finally, by equivalence, the theorem of existence and uniqueness of a classical solution to the given problem is proved.

Keywords: inverse problems, hyperbolic equations, nonlocal integral condition, classical solution, existence, uniqueness.

AMS Subject Classification: 35R30, 35L10, 35L70, 35A01, 35A02, 35A09.

1. Introduction

There are many cases where the needs of the practice bring about the problems of determining coefficients or the right hand side of differential equations from some knowledge of its solutions. Such problems are called inverse boundary value problems of mathematical physics. Inverse boundary value problems arise in various areas of human activity such as seismology, mineral exploration, biology, medicine, quality control in industry etc., which makes them an active field of contemporary mathematics. Inverse problems for various types of PDEs have been studied in many papers. Among them we should mention the papers of A.N. Tikhonov [8], M.M. Lavrentyev [4, 5], V.K. Ivanov [2] and their followers. For a comprehensive overview, the reader should see the monograph by A.M. Denisov [1]. In this paper, we prove existence and uniqueness of the solution to an inverse boundary value problem for the Boussinesq-Love equation modeling the longitudinal waves in an elastic bar with the transverse inertia.

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2. Problem statement and its reduction to an equivalent problem

Let $T>0$ be some fixed number and denote by $D_T=\{(x,t): 0\le x\le 1, 0\le t\le T\}$. Consider the one-dimensional inverse problem of identifying an unknown pair of functions $\{u(x,t),a(t)\}$ for the following Boussinesq-Love equation [7]

$$u_{tt}(x,t)-u_{ttxx}(x,t)-\alpha u_{txx}(x,t)-\beta u_{xx}(x,t)=a(t)u(x,t)+f(x,t), \tag{1}$$

with the nonlocal initial conditions

$$u(x,0)=\varphi(x),\ u_t(x,0)=\int_0^T p(t)u(x,t)dt+\psi(x),\ x\in[0,1]. \tag{2}$$

Neumann boundary condition

$$u_x(0,t)=0,\ t\in[0,T], \tag{3}$$

nonlocal integral condition

$$\int_0^1 u(x,t)dx=0,\ t\in[0,T], \tag{4}$$

and overdetermination condition

$$u(0,t)=h(t),\ t\in[0,T], \tag{5}$$

where $\alpha>0,\ \beta>0$ are known numbers, $f(x,t),\varphi(x),\psi(x),p(t)$, and $h(t)$ are given sufficiently smooth functions of $x\in[0,1]$ and $t\in[0,T]$.

We introduce the following set of functions

$$\tilde{C}^{(2,2)}(D_T)=\{u(x,t): u(x,t)\in C^2(D_T), u_{ttxx}(x,t)\in C(D_T)\}.$$

Definition 2.1. *The pair $\{u(x,t),a(t)\}$ is said to be a classical solution to the problem (1)-(5), if the functions $u(x,t)\in\tilde{C}^{(2,2)}(D_T)$ and $a(t)\in C[0,T]$ satisfies an equation (1) in the region D_T, the condition (2) on $[0,1]$, and the statements (3)-(5) on the interval $[0,T]$ ordinary meaning.*

In order to investigate the problem (1) - (5), first we consider the following auxiliary problem

$$y''(t)=a(t)y(t),\ t\in[0,T], \tag{6}$$

$$y(0)=0,\ y'(0)=\int_0^T p(t)y(t)dt, \tag{7}$$

where $p(t),a(t)\in C[0,T]$ are given functions, and $y=y(t)$ is desired function. Moreover, by the solution of the problem (6), (7), we mean a function $y(t)$ belonging to $C^2[0,T]$ and satisfying the conditions (6), (7) in the usual sense.

Lemma 2.1. *[6] Assume that $p(t)\in C[0,T],\ a(t)\in C[0,T],\ \|a(t)\|_{C[0,T]}\le R=const$, and the condition*

$$\left(\|p(t)\|_{C[0,T]}+\frac{1}{2}R\right)T^2<1,$$

holds. Then the problem (6)-(7) has a unique trivial solution.

Now, along with the inverse boundary value problem (1)-(5), we consider the following auxiliary inverse boundary value problem: It is required to determine a pair $\{u(x,t),a(t)\}$ of functions $u(x,t)\in\tilde{C}^{(2,2)}(D_T)$ and $a(t)\in C[0,T]$ from relations (1)-(3), and

$$u_x(1,t)=0,\ t\in[0,T], \tag{8}$$

$$h''(t)-u_{ttxx}(0,t)-\alpha u_{txx}(0,t)-\beta u_{xx}(0,t)=a(t)h(t)+f(0,t),\ t\in[0,T]. \tag{9}$$

The following lemma is valid.

Theorem 2.1. *Suppose that* $\varphi(x), \psi(x) \in C^1[0,1],\ \varphi'(1) = 0,\ \psi'(1) = 0,\ p(t) \in C[0,T],\ p(t) \leq 0,\ h(t) \in C^2[0,T],\ h(t) \neq 0,\ t \in [0,T],\ f(x,t) \in C(D_T),\ \int_0^1 f(x,t)dx = 0,\ t \in [0,T],\ \frac{\alpha^2}{4} - \beta > 0,$ *and the compatibility conditions*

$$\int_0^1 \varphi(x)dx = 0,\ \int_0^1 \psi(x)dx = 0, \tag{10}$$

$$h(0) = \varphi(0),\ h'(0) = \int_0^T p(t)h(t)dt + \psi(0) \tag{11}$$

hold. Then the following assertions are valid:

(i) *each classical solution* $\{u(x,t), a(t)\}$ *of the problem (1)-(5) is a solution of problem (1)-(3), (8), (9), as well as;*

(ii) *each solution* $\{u(x,t), a(t)\}$ *of the problem 1)-(3), (8), (9), if*

$$\left(\|p(t)\|_{C[0,T]} + \frac{1}{2}\|a(t)\|_{C[0,T]}\right)T^2 < 1, \tag{12}$$

is a classical solution of problem (1)-(5).

Proof. Let $\{u(x,t), a(t)\}$ be any classical solution to problem (1)-(5). By integrating both sides of equation (1) with respect to x from 0 to 1, we find

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)dx - (u_{ttx}(1,t) - u_{ttx}(0,t)) - \alpha(u_{tx}(1,t) - u_{tx}(0,t)) -$$

$$-\beta(u_x(1,t) - u_x(0,t)) = a(t)\int_0^1 u(x,t)dx + \int_0^1 f(x,t)dx,\ t \in [0,T]. \tag{13}$$

Using the fact that $\int_0^1 f(x,t)dx = 0,\ t \in [0,T]$, and the conditions (3),(4), we find that:

$$u_{ttx}(1,t) + \alpha u_{tx}(1,t) + \beta u_x(1,t) = 0,\ t \in [0,T]. \tag{14}$$

It's obvious that the general solution of equation (14) has the form:

$$u_x(1,t) = c_1 e^{\mu_1 t} + c_2 e^{\mu_2 t}, \tag{15}$$

where c_1, c_2 are the unknown numbers and

$$\mu_1 = -\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} - \beta},\ \ \mu_2 = -\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - \beta}.$$

By (2) and $\varphi'(1) = 0,\ \psi'(1) = 0$ we obtain:

$$u_x(1,0) = \varphi'(1) = 0,\ \ u_{tx}(1,0) - \int_0^T p(t)u_x(1,t)dt = \psi'(1) = 0. \tag{16}$$

Using (15) and (16) we obtain

$$c_1 + c_2 = 0,\ \ c_1\mu_1 + c_2\mu_2 - \int_0^T p(t)(c_1 e^{\mu_1 t} + c_2 e^{\mu_2 t})dt = 0.$$

Hence we find:

$$c_1 = -c_2, \quad c_2 \left(\mu_2 - \mu_1 - \int_0^T p(t)(e^{\mu_2 t} - e^{\mu_1 t})dt \right) = 0.$$

By $p(t) \leq 0,\ \mu_2 - \mu_1 = 2\sqrt{\frac{\alpha^2}{4} - \beta} > 0$, from the latter relations we have $c_1 = c_2 = 0$.

Putting the value of $c_1 = c_2 = 0$ in (15), we get that the problem (14), (16) has only the trivial solution, i.e. we conclude that the statement (8) is true.

Setting $x = 0$ in equation (1), we find

$$u_{tt}(0,t) - u_{ttxx}(0,t) - \alpha u_{txx}(0,t) - \beta u_{xx}(0,t) = a(t)u(0,x) + f(0,t),\ t \in [0,T]. \tag{17}$$

Taking into consideration $h(t) \in C^2[0,T]$ and twice differentiating (5) we have

$$u_{tt}(0,t) = h''(t),\ t \in [0,T]. \tag{18}$$

From (17), taking into account (5) and (18), we conclude that the relation (9) is fulfilled.

Now, assume that $\{u(x,t), a(t)\}$ is the solution to problem (1)-(3), (8), (9). Then from (13), taking into account the condition $\int_0^1 f(x,t)dx = 0,\ t \in [0,T]$ and relations (3), (8) we have

$$\frac{d^2}{dt^2} \int_0^1 u(x,t)dx = a(t) \int_0^1 u(x,t)dx,\ t \in [0,T]. \tag{19}$$

Furthermore, from (2) and (10) it is easy to see that

$$\int_0^1 u(x,0)dx = \int_0^1 \varphi(x)dx = 0,$$
$$\int_0^1 u_t(x,0)dx - \int_0^T p(t) \left(\int_0^1 u(x,t)dx \right) dt = \int_0^1 \left(u_t(x,0) - \int_0^T p(t)u(x,t)dt \right) dx = \int_0^1 \psi(x)dx = 0. \tag{20}$$

Since, by Lemma 2.1., problem (19), (20) has only a trivial solution. It means that

$$\int_0^1 u(x,t)dx = 0,\ t \in [0,T],$$

i.e. the condition (4) is satisfied.

Next, from (9) and (17), we obtain

$$\frac{d^2}{dt^2}(u(0,t) - h(t)) = a(t)(u(0,t) - h(t)),\ 0 \leq t \leq T. \tag{21}$$

By virtue of (2) and the compatibility conditions (11), we have

$$u(0,0) - h(0) = \varphi(0) - h(0) = 0,$$

$$u_t(0,0) - h'(0) - \int_0^T p(t)(u(0,t) - h(t))dt = u_t(0,0) - \int_0^T p(t)u(0,t)dt -$$
$$- \left(h'(0) - \int_0^T p(t)h(t)dt \right) = \psi(0) - \left(h'(0) - \int_0^T p(t)h(t)dt \right) = 0. \tag{22}$$

Using Lemma 2.1., and relations (21), (22), we conclude that condition (5) is satisfied. $\square$

3. Existence and uniqueness of the classical solution

We seek the first component $u(x,t)$ of classical solution $\{u(x,t), a(t)\}$ of the problem (1)-(3), (8), (9) in the form

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t) \cos \lambda_k x, \ \ \lambda_k = k\pi, \tag{23}$$

where

$$u_k(t) = m_k \int_0^1 u(x,t) \cos \lambda_k x dx, \ \ k = 0, 1, 2, \dots,$$

and

$$m_k = \begin{cases} 1, & k = 0, \\ 2, & k = 1, 2, \dots . \end{cases}$$

Then applying the formal scheme of the Fourier method, from (1) and (2) we have

$$(1+\lambda_k^2)u_k''(t) + \alpha\lambda_k^2 u_k'(t) + \beta\lambda_k^2 u_k(t) = F_k(t; u, a), \ \ k = 0, 1, 2 \dots ; \ \ 0 \le t \le T, \tag{24}$$

$$u_k(0) = \varphi_k, u_k'(0) = \psi_k + \int_0^T p(t) u_k(t) dt, \ \ k = 0, 1, 2 \dots , \tag{25}$$

where

$$F_k(t; u, a) = f_k(t) + a(t) u_k(t), \ \ f_k(t) = m_k \int_0^1 f(x,t) \cos \lambda_k x dx,$$

$$\varphi_k = m_k \int_0^1 \varphi(x) \cos \lambda_k x dx, \ \ \psi_k(t) = m_k \int_0^1 \psi(x) \cos \lambda_k x dx, \ \ k = 0, 1, 2, \dots .$$

It is obvious that $\lambda_k^2 < 1 + \lambda_k^2 < 2\lambda_k^2$ $(k = 1, 2, ...)$. Therefore

$$\frac{\alpha^2}{8} - \beta < \frac{\alpha^2 \lambda_k^2}{4(1+\lambda_k^2)} - \beta < \frac{\alpha^2}{4} - \beta \ \ (k = 1, 2, ...).$$

Now, suppose that $\frac{\alpha^2}{8} - \beta > 0$. Solving the problem (24)-(25), we find

$$u_0(t) = \varphi_0 + t \left(\psi_0 + \int_0^T p(t) u_0(t) dt \right) + \int_0^t (t-\tau) F_0(\tau; u, a) d\tau, \tag{26}$$

$$u_k(t) = \frac{1}{\gamma_k} \Bigg[\left(\mu_{2k} e^{\mu_{1k} t} - \mu_{1k} e^{\mu_{2k} t} \right) \varphi_k + \left(e^{\mu_{2k} t} - e^{\mu_{1k} t} \right) \left(\psi_k + \int_0^T p(t) u_k(t) dt \right) +$$

$$+ \frac{1}{1+\lambda_k^2} \int_0^t F_k(\tau; u, a) \left(e^{\mu_{2k}(t-\tau)} - e^{\mu_{1k}(t-\tau)} \right) d\tau \Bigg] \ \ (0 \le t \le T; \ k = 1, 2, ...), \tag{27}$$

where

$$\mu_{1k} = -\frac{\alpha\lambda_k^2}{2(1+\lambda_k^2)} - \lambda_k \sqrt{\frac{\alpha^2\lambda_k^2}{4(1+\lambda_k^2)^2} - \frac{\beta}{1+\lambda_k^2}},$$

$$\mu_{2k} = -\frac{\alpha\lambda_k^2}{2(1+\lambda_k^2)} + \lambda_k \sqrt{\frac{\alpha^2\lambda_k^2}{4(1+\lambda_k^2)^2} - \frac{\beta}{1+\lambda_k^2}},$$

$$\gamma_k = \mu_{2k} - \mu_{1k} = 2\lambda_k\sqrt{\frac{\alpha^2\lambda_k^2}{4(1+\lambda_k^2)^2} - \frac{\beta}{1+\lambda_k^2}}.$$

Differentiating (27) twice, we get:

$$u_k'(t) = \frac{1}{\gamma_k}\left[\mu_{1k}\mu_{2k}\left(e^{\mu_{1k}t} - e^{\mu_{2k}t}\right)\varphi_k + \left(\mu_{2k}e^{\mu_{2k}t} - \mu_{1k}e^{\mu_{1k}t}\right)\left(\psi_k + \int_0^T P(t)u_k(t)dt\right) + \right.$$

$$\left. + \frac{1}{1+\lambda_k^2}\int_0^t F_k(\tau;u,a)\left(\mu_{2k}e^{\mu_{2k}(t-\tau)} - \mu_{1k}e^{\mu_{1k}(t-\tau)}\right)d\tau\right] \ (0 \le t \le T;\ k = 1,2,...), \tag{28}$$

$$u_k''(t) = \frac{1}{\gamma_k}\left[\mu_{1k}\mu_{2k}\left(\mu_{1k}e^{\mu_{1k}t} - \mu_{2k}e^{\mu_{2k}t}\right)\varphi_k + \right.$$

$$+ \left(\mu_{2k}^2e^{\mu_{2k}t} - \mu_{1k}^2e^{\mu_{1k}t}\right)\psi_k + \frac{1}{1+\lambda_k^2}\int_0^t F_k(\tau;u,a)\left(\mu_{2k}^2e^{\mu_{2k}(t-\tau)} - \right.$$

$$\left.\left. - \mu_{1k}^2e^{\mu_{1k}(t-\tau)}\right)d\tau\right] + \frac{1}{1+\lambda_k^2}F_k(t;u,a) \ \ (k = 1,2,...). \tag{29}$$

To determine the first component of the classical solution to the problem (1)-(3), (8), (9) we substitute the expressions $u_k(t)$ $(k = 0,1,...)$ into (23) and obtain

$$u(x,t) = \varphi_0 + t\left(\psi_0 + \int_0^T p(t)u_0(t)dt\right) + \int_0^t (t-\tau)F_0(\tau;u,a)d\tau +$$

$$+\sum_{k=1}^{\infty}\left\{\frac{1}{\gamma_k}\left[\left(\mu_{2k}e^{\mu_{1k}t} - \mu_{1k}e^{\mu_{2k}t}\right)\varphi_k + \left(e^{\mu_{2k}t} - e^{\mu_{1k}t}\right)\left(\psi_k + \int_0^T p(t)u_k(t)dt\right) + \right.\right.$$

$$\left.\left. + \frac{1}{1+\lambda_k^2}\int_0^t F_k(\tau;u,a)\left(e^{\mu_{2k}(t-\tau)} - e^{\mu_{1k}(t-\tau)}\right)d\tau\right]\right\}\cos\lambda_k x. \tag{30}$$

It follows from (9) and (23) that

$$a(t) = [h(t)]^{-1}\left\{h''(t) - f(0,t) + \sum_{k=1}^{\infty}\left(\lambda_k^2u_k''(t) + \alpha\lambda_k^2u_k'(t) + \beta\lambda_k^2u_k(t)\right)\right\}. \tag{31}$$

By (24) and (29) we have:

$$\lambda_k^2u_k''(t) + \alpha\lambda_k^2u_k'(t) + \beta\lambda_k^2u_k(t) = F_k(\tau;u,a) - u_k''(t) =$$

$$= -\frac{1}{\gamma_k}\left[\mu_{1k}\mu_{2k}\left(\mu_{1k}e^{\mu_{1k}t} - \mu_{2k}e^{\mu_{2k}t}\right)\varphi_k + \left(\mu_{2k}^2e^{\mu_{2k}t} - \mu_{1k}^2e^{\mu_{1k}t}\right)\left(\psi_k + \int_0^T p(t)u_k(t)dt\right) + \right.$$

$$\left. + \frac{1}{1+\lambda_k^2}\int_0^t F_k(\tau;u,a)\left(\mu_{2k}^2e^{\mu_{2k}(t-\tau)} - \mu_{1k}^2e^{\mu_{1k}(t-\tau)}\right)d\tau\right] +$$

$$+ \frac{\lambda_k^2}{1+\lambda_k^2}F_k(t;u,a) \ \ (0 \le t \le T;\ k = 1,2,...). \tag{32}$$

By substituting expression (32) into (31), we obtain the equation for the second component of the solution to problem (1) - (3), (8), (9):

$$a(t) = [h(t)]^{-1}\{h''(t) - f(0,t) +$$

$$-\sum_{k=1}^{\infty}\left[\frac{1}{\gamma_k}\left[\mu_{1k}\mu_{2k}\left(\mu_{1k}e^{\mu_{1k}t}-\mu_{2k}e^{\mu_{2k}t}\right)\varphi_k+\left(\mu_{2k}^2e^{\mu_{2k}t}-\mu_{1k}^2e^{\mu_{1k}t}\right)\left(\psi_k+\int_0^T p(t)u_k(t)dt\right)+\right.\right.$$

$$\left.\left.+\frac{1}{1+\lambda_k^2}\int_0^t F_k(\tau;u,a)\left(\mu_{2k}^2e^{\mu_{2k}(t-\tau)}-\ \mu_{1k}^2e^{\mu_{1k}(t-\tau)}\right)d\tau\right]-\frac{\lambda_k^2}{1+\lambda_k^2}F_k(t;u,a)\right]\Bigg\}. \qquad (33)$$

Thus, the solution of problem (1) - (3), (8), (9) was reduced to the solution of system (30), (33) with respect to the unknown functions $u(x,t)$ and $a(t)$.

Lemma 3.1. *If $\{u(x,t),a(t)\}$ is any solution to problem (1) - (3), (8), (9), then the functions*

$$u_k(t)=m_k\int_0^1 u(x,t)\cos\lambda_k x dx,\quad k=0,1,2,\ldots,$$

satisfy the system (26), (27) in $C[0,T]$.

It follows from Lemma 2.2. that

Corollary 3.1. *Let system (30)-(33) have a unique solution. Then problem (1) - (3), (8), (9) cannot have more than one solution, i.e. if the problem (1) - (3), (8), (9) has a solution, then it is unique.*

With the purpose to study the problem (1) - (3), (8), (9), we consider the following functional spaces.

Denote by $B_{2,T}^3$ [8] a set of all functions of the form

$$u(x,t)=\sum_{k=0}^{\infty}u_k(t)\cos\lambda_k x,\ \ \lambda_k=k\pi,$$

considered in the region D_T, where each of the function $u_k(t)$ $(k=0,1,2,...)$ is continuous over an interval $[0,T]$ and satisfies the following condition:

$$J(u)\equiv\|u_0(t)\|_{C[0,T]}+\left\{\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_k(t)\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}<+\infty.$$

The norm in this set is defined by

$$\|u(x,t)\|_{B_{2,T}^3}=J(u).$$

It is known that $B_{2,T}^3$ is Banach space.

Obviously, $E_T^3=B_{2,T}^3\times C[0,T]$ with the norm

$$\|z(x,t)\|_{E_{,T}^3}=\|u(x,t)\|_{B_{2,T}^3}+\|a(t)\|_{C[0,T]}$$

is also Banach space.

Now, consider the operator

$$\Phi(u,a)=\{\Phi_1(u,a),\Phi_2(u,a)\},$$

in the space E_T^3, where

$$\Phi_1(u,a)=\tilde{u}(x,t)\equiv\sum_{k=0}^{\infty}\tilde{u}_k(t)\cos\lambda_k x,\quad \Phi_2(u,a)=\tilde{a}(t)$$

and the functions $\tilde{u}_0(t)$, $\tilde{u}_k(t)$, $k=1,2,...$, and $\tilde{a}(t)$ are equal to the right-hand sides of (26), (27), and (33), respectively.

It is easy to see that

$$\mu_{ik}<0,\quad e^{\mu_{ik}t}<1,\quad e^{\mu_{ik}(t-\tau)}<1,\quad (i=1,2;\ 0\le t\le T;\ 0\le\tau\le t),$$

$$|\mu_{ik}|\le\lambda_k\left(\frac{\alpha\lambda_k}{2(1+\lambda_k^2)}+\sqrt{\frac{\alpha^2\lambda_k^2}{4(1+\lambda_k^2)^2}-\frac{\beta}{1+\lambda_k^2}}\right)\le\frac{\alpha\lambda_k^2}{1+\lambda_k^2}\le\alpha\quad(i=1,2),$$

$$|\mu_{1k}\mu_{2k}|\le\frac{\beta\lambda_k^2}{1+\lambda_k^2}\le\beta,\ \frac{1}{\gamma_k}=\frac{1}{2\sqrt{\frac{\lambda_k^2}{1+\lambda_k^2}\left(\frac{\alpha^2\lambda_k^2}{4(1+\lambda_k^2)^2}-\beta\right)}}\le\frac{1}{2\sqrt{\frac{1}{2}\left(\frac{\alpha^2}{8}-\beta\right)}}\equiv\gamma_0.$$

Taking into account these relations, by means of simple transformations we find:

$$\|\tilde u_0(t)\|_{C[0,T]}\le|\varphi_0|+T|\psi_0|+T^2\|p(t)\|_{C[0,T]}\|u_0(t)\|_{C[0,T]}+$$
$$+T\sqrt{T}\left(\int_0^T|f_0(\tau)|^2d\tau\right)^{\frac12}+T^2\|a(t)\|_{C[0,T]}\|u_0(t)\|_{C[0,T]},\tag{34}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^3\|\tilde u_k(t)\|_{C[0,T]})^2\right)^{\frac12}\le\sqrt5\alpha\gamma_0\left(\sum_{k=1}^{\infty}(\lambda_k^3|\varphi_k|)^2\right)^{\frac12}+$$
$$+\sqrt5\gamma_0\left(\sum_{k=1}^{\infty}(\lambda_k^3|\psi_k|)^2\right)^{\frac12}+\gamma_0\sqrt{5T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k|f_k(\tau)|)^2d\tau\right)^{\frac12}+$$
$$+\sqrt5T\gamma_0(\|p(t)\|_{C[0,T]}+\|a(t)\|_{C[0,T]})\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k(t)\|_{C[0,T]})^2\right)^{\frac12},\tag{35}$$

$$\|\tilde a(t)\|_{C[0,T]}\le\left\|[h(t)]^{-1}\right\|_{C[0,T]}\times$$
$$\times\left\{\left\|h''(t)-f(0,t)\right\|_{C[0,T]}+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac12}\left[2\alpha\beta\gamma_0\left(\sum_{k=1}^{\infty}(\lambda_k^3|\varphi_k|)^2\right)^{\frac12}+\right.\right.$$
$$+2\alpha^2\gamma_0\left(\sum_{k=1}^{\infty}(\lambda_k^3|\psi_k|)^2\right)^{\frac12}+2\alpha^2\gamma_0T\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k(t)\|_{C[0,T]})^2\right)^{\frac12}+$$
$$+2\alpha^2\gamma_0\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k|f_k(\tau)|)^2d\tau\right)^{\frac12}+2\alpha^2\gamma_0T\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k(t)\|_{C[0,T]})^2\right)^{\frac12}+$$
$$\left.\left.+\left(\sum_{k=1}^{\infty}(\lambda_k\|f_k(t)\|_{C[0,T]})^2\right)^{\frac12}+\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k(t)\|_{C[0,T]})^2\right)^{\frac12}\right]\right\}.\tag{36}$$

Suppose that the data for problem (1)-(3), (8), (9) satisfy the assumptions:

A) $\varphi(x)\in C^2[0,1],\ \varphi'''(x)\in L_2(0,1),\ \varphi'(0)=\varphi'(1)=0;$
B) $\psi(x)\in C^2[0,1],\ \psi'''(x)\in L_2(0,1),\ \psi'(0)=\psi'(1)=0;$
C) $f(x,t),f_x(x,t),f_x(x,t)\in C(D_T),\ f_{xxx}(x,t)\in L_2(D_T),f_x(0,t)=f_x(1,t),\ 0\le t\le T;$
D) $p(t)\in C[0,T],\ h(t)\in C^2[0,T],\ h(t)\ne0,\ 0\le t\le T;$
E) $\alpha>0,\ \beta>0,\ \frac{\alpha^2}{8}-\beta>0.$

Then from (34)-(36) we correspondingly find

$$\|\tilde{u}_0(t)\|_{C[0,T]} \leq \|\varphi(x)\|_{L_2(0,1)} + T\|\psi(x)\|_{L_2(0,1)} + T\sqrt{T}\|f(x,t)\|_{L_2(D_T)} + \\ +T^2\|p(t)\|_{C[0,T]}\|u_0(t)\|_{C[0,T]} + T^2\|a(t)\|_{C[0,T]}\|u_0(t)\|_{C[0,T]}, \quad (37)$$

$$\left\{\sum_{k=1}^{\infty}(\lambda_k^3\|\tilde{u}_k(t)\|_{C[0,T]})^2\right\}^{\frac{1}{2}} \leq \sqrt{5}\alpha\gamma_0\|\varphi'''(x)\|_{L_2(0,1)} + \sqrt{5}\gamma_0\|\psi'''(x)\|_{L_2(0,1)} + \\ +\gamma_0\sqrt{5T}\|f_x(x,t)\|_{L_2(D_T)} + \sqrt{5}T\gamma_0\|p(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} + \\ +\sqrt{5}T\gamma_0\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}, \quad (38)$$

$$\|\tilde{a}(t)\|_{C[0,T]} \leq \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t)-f(0,t)\right\|_{C[0,T]} + \left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\times\right. \\ \times\left[2\alpha\beta\gamma_0\|\varphi'''(x)\|_{L_2(0,1)} + 2\alpha^2\gamma_0\|\psi'''(x)\|_{L_2(0,1)} + 2\alpha^2\gamma_0\sqrt{T}\|f_x(x,t)\|_{L_2(D_T)} + \right. \\ +2\alpha^2\gamma_0T(\|p(t)\|_{C[0,T]} + \|a(t)\|_{C[0,T]})\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} + \\ \left.\left. + \left\|\|f_x(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)} + \|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^3\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]\right\}. \quad (39)$$

It follows from (32) and (33) that

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}} \leq A_1(T) + B_1(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}} + C_1(T)\|u(x,t)\|_{B^3_{2,T}}, \quad (40)$$

where

$$A_1(T) = \|\varphi(x)\|_{L_2(0,1)} + T\|\psi(x)\|_{L_2(0,1)} + T\sqrt{T}\|f(x,t)\|_{L_2(D_T)} + \\ +\sqrt{5}\alpha\gamma_0\|\varphi'''(x)\|_{L_2(0,1)} + \sqrt{5}\gamma_0\|\psi'''(x)\|_{L_2(0,1)} + \gamma_0\sqrt{5T}\|f_x(x,t)\|_{L_2(D_T)},$$

$$B_1(T) = T^2 + \gamma_0\sqrt{5}T,$$

$$C_1(T) = T(T+\sqrt{5}\gamma_0)\|p(t)\|_{C[0,T]}.$$

Further from (34), we may also write

$$\|\tilde{a}(t)\|_{C[0,T]} \leq A_2(T) + B_2(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B^3_{2,T}} + C_2(T)\|u(x,t)\|_{B^3_{2,T}}, \quad (41)$$

where

$$A_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t)-f(0,t))\right\|_{C[0,T]} + \left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\times\right. \\ \times\left[2\alpha\beta\gamma_0\|\varphi'''(x)\|_{L_2(0,1)} + 2\alpha^2\gamma_0\|\psi'''(x)\|_{L_2(0,1)} + \right. \\ \left.\left. +2\alpha^2\gamma_0\sqrt{T}\|f_x(x,t)\|_{L_2(D_T)} + \left\|\|f_x(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)}\right]\right\},$$

$$B_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^{-2})\right)^{\frac{1}{2}}(2\alpha^2\gamma_0T+1),$$

$$C_2(T) = \left\| [h(t)]^{-1} \right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} (\lambda_k^{-2}) \right)^{\frac{1}{2}} T \left\| p(t) \right\|_{C[0,T]}.$$

From the inequalities (39) and (40), we conclude that

$$\| \tilde{u}(x,t) \|_{B^3_{2,T}} + \| \tilde{a}(t) \|_{C[0,T]} \le A(T) + B(T) \| a(t) \|_{C[0,T]} \| u(x,t) \|_{B^3_{2,T}} + C(T) \| u(x,t) \|_{B^3_{2,T}}, \quad (42)$$

where

$$A(T) = A_1(T) + A_2(T), \quad B(T) = B_1(T) + B_2(T), \quad C(T) = C_1(T) + C_2(T).$$

Thus, we can prove the following theorem:

Theorem 3.1. *Assume that statements A-E and the condition*

$$(B(T)(A(T)+2) + C(T))(A(T)+2) < 1, \quad (43)$$

hold, then the problem (1)-(3), (8), (9) has a unique solution in the ball $K = K_R(\|z\|_{E^3_T} \le R \le A(T)+2)$ *of the space* E^3_T.

Remark 3.1. *Inequality (43) is satisfied for sufficiently small values of* T.

Proof. In the space E^3_T, consider the operator equation

$$z = \Phi z, \quad (44)$$

where $z = \{u, a\}$, and the components $\Phi_i(u,a)$ $(i = 1,2)$, of operator $\Phi(u,a)$ defined by the right sides of (30) and (33), respectively and the following inequalities hold:

$$\|\Phi z\|_{E^3_T} \le A(T) + B(T) \|a(t)\|_{C[0,T]} \|u(x,t)\|_{B^3_{2,T}} + C(T) \|u(x,t)\|_{B^3_{2,T}} \le$$
$$\le A(T) + B(T) R^2 + C(T) R = A(T) + (B(T)(A(T)+2) + C(T))(A(T)+2), \quad (45)$$
$$\|\Phi z_1 - \Phi z_2\|_{E^3_T} \le B(T) R (\|u_1(x,t) - u_2(x,t)\|_{B^3_{2,T}} + \|a_1(t) - a_2(t)\|_{C[0,T]}) +$$
$$+ C(T) \|u_1(x,t) - u_2(x,t)\|_{B^3_{2,T}}. \quad (46)$$

Then it follows from (43), (45), and (46) that the operator Φ acts in the ball $K = K_R$, and satisfy the conditions of the contraction mapping principle. Therefore, the operator Φ has a unique fixed point $\{z\} = \{u, a\}$ in the ball $K = K_R$ which is a solution of equation (44); i.e. the pair $\{u, a\}$ is the unique solution of the systems (30) and (33) in $K = K_R$.

Then the function $u(x,t)$ as an element of space $B^3_{2,T}$ is continuous and has continuous derivatives $u_x(x,t)$ and $u_{xx}(x,t)$ in D_T.

Now, from (28) it is obvious that $u'_k(t)$ $(k = 1,2,...)$ is continuous in $[0,T]$ and from the same relation we get:

$$\left(\sum_{k=1}^{\infty} (\lambda_k^3 \|u'_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \le 2\sqrt{3}\beta\gamma_0 \|\varphi'''(x)\|_{L_2(0,1)} + 2\sqrt{3}\alpha \|\psi'''(x)\|_{L_2(0,1)} +$$

$$+ 2\alpha\sqrt{3T} \|f_{xxx}(x,t)\|_{L_2(D_T)} + 2\alpha\sqrt{3}T(\|p(t)\|_{C[0,T]} + \|a(t)\|_{C[0,T]}) \left(\sum_{k=1}^{\infty} (\lambda_k^3 \|u_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}}.$$

Hence, it follows that $u_t(x,t)$, $u_{tx}(x,t)$, and $u_{txx}(x,t)$ are continuous in D_T.

Next, from (24) it follows that $u''_k(t)$ $(k = 1,2,...)$ are continuous in $[0,T]$ and consequently we have: Equation (24) gives

$$\left(\sum_{k=1}^{\infty} (\lambda_k \|u''_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \le 2\alpha \left(\sum_{k=1}^{\infty} (\lambda_k^3 \|u'_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} +$$

$$+2\beta\left(\sum_{k=1}^{\infty}(\lambda_k^3\left\|u_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}+2\left\|\left\|f_x(x,t)+a(t)u_x(x,t)\right\|_{C[0,T]}\right\|_{L(0,1)}.$$

From the last relation it is obvious that $u_t(x,t)$, $u_{tx}(x,t)$, and $u_{txx}(x,t)$ are continuous in D_T.

It is easy to verify that equation (1) and conditions (2), (3), (8), (9) satisfy in the usual sense. So, $\{u(x,t),a(t)\}$, is a solution of (1)-(3), (8), (9), and by Lemma 2 it is unique in the ball $K=K_R$. □

In summary, from Theorem 2.1 and Theorem 3.1, straightforward implies the unique solvability of the original problem (1)-(5).

Theorem 3.2. *Suppose that all assumptions of Theorem 3.1, and the conditions*

$$\int_0^1\varphi(x)dx=0,\ \int_0^1\psi(x)dx=0,\ \int_0^1 f(x,t)dx=0,\ t\in[0,T],\ p(t)\le 0,\ t\in[0,T],$$

$$h(0)=\varphi(0),\ h'(0)=\int_0^T p(t)h(t)dt+\psi(0),$$

$$\left(\|p(t)\|_{C[0,T]}+\frac{1}{2}(A(T)+2)\right)T<1,$$

hold. Then problem (1)-(5) has a unique classical solution in the ball $K=K_R(\|z\|_{E_T^3}\le A(T)+2)$ *of the space* E_T^3.

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