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MANIFESTATION OF FUZZY TOPOLOGY IN OTHER FUZZY MATHEMATICAL STRUCTURES

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Abstract. The aim of the paper is to show manifestation of fuzzy topology in theories of three mathematical structures based on fuzzy sets that were created in the last decade of the previous century and the beginning of XXI. Namely, these are the theories of fuzzy rough sets, fuzzy morphology and fuzzy concept lattices. We present some known and new results, illustrating the manifestation of fuzzy topology in these theories.

Keywords: fuzzy topology, fuzzy rough sets, fuzzy mathematical morphology, fuzzy (pre)concept lattices.

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1. Introduction

Just three years after the appearance of the celebrated Lotfi A. Zadeh paper *Fuzzy sets* that actually opened a new era in science and technology, the first paper on fuzzy topology was published. It was C.L. Chang's, a student of Zadeh, work *Fuzzy topological spaces* [20] and, as far as we know, it was the first work considering issues of theoretical mathematics in the context of fuzzy sets. Probably it was also one of the very first works where the expression "fuzzy set" was used. Soon after Chang's article, interest of many researchers focused on this topic and "fuzzy topology" became a fashionable. In the first period of "Fuzzy Topology" that we conditionally restrict by the first 20 years, there were published, according to our records, more than 250 papers on this subject. Among them there are such outstanding works that left an important mark on the further development of Fuzzy Topology as papers by J.A, Goguen [40], B. Hutton [53], [54] et al., R. Lowen [71], [72], [73] et al., U. Höhle [45], [46], et al., S.E. Rodabaugh [92], [93] et al., A.K. Katsaras [57] et al., T.E. Gantner R.C. Steinlage and R.H. Warren [36], Pu Pao Ming and Liu Yingming [86], [87], U. Cerruti [17], C. de Mitri and E.Pascali [27], P. Eklund [30], T. Kubiak [65], etc.

Being in a focus of interest as a field of study for many researches as itself, fuzzy topology attracted attention also as a useful tool, method or as a context for many other directions of theoretic and applied "mathematics of fuzzy sets". In particular, fuzzy topological methods, constructions and results find their applications and were used in the theory of fuzzy dynamical systems, decision making in fuzzy environment, fuzzy cluster analysis, fuzzy control systems, etc. Just a brief enumeration of the works in which methods, constructions or tools inherent to fuzzy topology were used, would take several pages and this is far beyond the aim of this paper. Our goal is to illustrate how fuzzy topological concepts and ideas "manifest themselves" in theories of other fuzzy mathematical structures. Besides, we restrict here with the three fuzzy mathematical structures the study of which was initiated in 90-ies of the previous century and which now are in the focus of interest of many researchers, both in theoretic and applied areas of

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mathematics. Namely, we mean here Fuzzy Rough Sets, Fuzzy Morphology and Fuzzy Concept Lattices.

The paper consists of seven sections, including this, introductory, section and a short Conclusion section. In the first, preliminary, section basic auxiliary concepts used throughout the paper are recalled. The main goal of second section, Fuzzy Topology, is to acquaint the reader with concepts and terms related to fuzzy topology, which are used in sections 3,4 and 5.

Section 3 is devoted to fuzzy rough sets and focuses on their topological-type properties. The section consists of six subsections in which we give a brief introduction into the subject of rough and fuzzy rough sets, expose the basics of the theory of fuzzy rough sets in a form, convenient for our merits, present some information about extensional properties of L-fuzzy rough approximation operators, introduce the category of L-fuzzy rough approximation spaces and give it an L-fuzzy topological interpretation. Further, we introduce the measure of L-rough approximation of an L-fuzzy set that forms the basis for the category of L-valued L-rough approximation spaces; this category is interpreted as a special category of L-valued fuzzy (di)topological spaces.

In Section 4 we are addressing the problem of topological aspects of fuzzy mathematical morphology. In the first three short subsections we briefly touch the subject of (fuzzy) mathematical morphology, basic operators of (fuzzy) mathematical morphology and abstract algebraic approach to the subject of fuzzy morphology. In the fourth subsection we deal with *fuzzy relational morphology.* Just on this approach to fuzzy morphology we base further exposition in this paper. We introduce basic operations of mathematical morphology - erosion and dilation in the context of fuzzy relational morphology and comment their properties by means of topological terminology. Further two categories defined by erosion-dilation pairs are introduced and studied as categories of L-fuzzy (di)pretopological spaces. The last subsection deals with the "second level" operators of fuzzy morphology: openings and closings. As different from erosion and dilation, these operators are idempotent and this enables to use them for defining L-fuzzy supra (di)topologies.

The subject of Section 5 are fuzzy concept and fuzzy preconcept lattices. We start with a brief introduction into (fuzzy) concept analysis and, in particualr, highlight some deficiency of fuzzy concept lattices that hampers their use for practical problems. Just this was the reason to replace them by fuzzy graded preconcept lattices. We propose two methods allowing to determine the grade showing the distinction of a fuzzy preconcept from "being a real fuzzy concept" and study the corresponding graded preconcept lattices. The first method of gradation is based on the evaluation of mutual "contentment" of the fuzzy sets of objects and attributes. The lattice of fuzzy preconcepts endowed with thus defined grade evaluation is called by a $\mathcal{D}$-graded preconcept lattice. The alternative approach is based on the measure of distinction of a fuzzy preconcept from its conceptual hull and conceptual kernell . The value obtained in this way is called the measure of conceptuality and the preconcept lattice endowed with this gradation is called by the $\mathcal{M}$-graded lattice. We give topology-related comments on each of these approaches of gradation.

In the last, Conclusion, section the work presented in this paper is briefly summarized and some perspectives for future work are sketched out. Besides we indicate some other mathematical structures where manifestation of fuzzy topology could be of interest. A list of references completes the paper; this list is inevitably very extended, since it reflects necessary information from four, actually different fields: Fuzzy Topology, Fuzzy Rough Sets, Fuzzy Morphology and Fuzzy Conceptual Lattices.

2. Preliminaries

2.1. **Lattices.** We recall here some well known concepts from the theory of lattices that will be used in the paper, see e.g. [7], [38], [80], [24] for the details.

Given a set L, a binary relation $\leq$ on L is called a partial order if it is (1) reflexive, that is $a \leq a$; (2) anti-symmetric, that is $a \leq b$ and $b \leq a$ implies that $a = b$ and (3) transitive, that is $a \leq b$ and $b \leq c$ implies $a \leq c$ for all $a, b, c \in L$.

Given $a, b \in L$, element $c = a \vee b \in L$ is called the upper bound of a and b if $a \leq c, b \leq c$ and $c \leq d \in L$ whenever $a \leq d$ and $b \leq d$ for every $d \in \mathbb{L}$. The lower bound $a \wedge b$ is defined in the dual way. A partially ordered set is called a lattice, if any two its elements have upper and lower bounds.

Let $M \subseteq L$. Element $c = \bigvee M \in L$ is called the upper bound of M if (1) $k \leq c$ for every $k \in M$ and (2) if $d \in \mathbb{L}$ and $k \leq d$ for every $k \in M$, then $c \leq d$. The lower bound $d = \bigwedge M \in L$ of $M \subseteq L$ is defined in a dual way. A lattice is called complete, if for every non-empty $M \subseteq L$ there exists the upper and the lower bounds. In particular, the upper bound of L is its top element 1_L and the lower bound of L is its bottom element 0_L.

A complete lattice L is called infinitely distributive or join-distributive if $a \wedge \left(\bigvee_{i \in I} b_i\right) = \bigvee_{i \in I}(a \wedge b_i)$ for every $a \in L$ and every $\{b_i \mid i \in I\} \subseteq L$. A complete lattice L is called co-infinitely distributive or meet-distributive, if $a \vee \left(\bigwedge_{i \in I} b_i\right) = \bigwedge_{i \in I}(a \vee b_i)$ for every $a \in L$ and every $\{b_i \mid i \in I\} \subseteq L$. A complete lattice is called bi-infinitely distributive if it is infinitely and co-infinitely distributive. It is known, that every completely distributive lattice is bi-infinitely distributive. Every join-distributive MV-algebra, is also meet-distributive.

2.2. **Quantales and residuated lattices.** The notion of a quantale first appears in Rosenthal's paper [99]. Let L be a complete lattice and $* : L \times L \to L$ be a binary associative monotone operation. Then the tuple $(L, \leq, \wedge, \vee, *)$ is called *a quantale* if $*$ distributes over arbitrary joins:

$$a * \left(\bigvee_{i \in I} b_i\right) = \bigvee_{i \in I}(a * b_i), \left(\bigvee_{i \in I} b_i\right) * a = \bigvee_{i \in I}(b_i * a) \quad \forall a \in L, \ \{b_i | i \in I\} \subseteq L.$$

Operation $*$ will be referred to as the product in L. A quantale is called integral if the top element 1_L acts as the unit, that is $1_L * a = a * 1_L = a$ for every $a \in L$; in this case 0_L acts as the zero: $a * 0_L = 0_L$. A quantale is called commutative, if the product is commutative. In what follows saying *quantale* we mean *a commutative integral quantale.* A typical example of a quantale is the unit interval endowed with a lower semicontinuous t-norm, see e.g. [62], [101]. In particular, each complete infinitely distributive lattice can be viewed as a quantale by taking $* = \wedge$.

In a quantale a further binary operation $\mapsto : L \times L \to L$, the residuum, can be introduced as associated with operation $*$ of the quantale $(\mathbb{L}, \leq, \wedge, \vee, *)$ via the Galois connection, that is

$$a * b \leq c \Longleftrightarrow a \leq b \mapsto c \ \text{ for all } \ a, b, c \in L.$$

Explicitly the residuum can be defined by $a \mapsto b = \bigvee\{l \in L \mid l * a \leq b\}$.

A quantale $(L, \leq, \wedge, \vee, *)$ provided with the derived operation $\mapsto$, that is the tuple $(L, \leq, \wedge, \vee, *, \mapsto)$, is known also as a residuated lattice [80].

In the following proposition we collect well-known properties of the residuum:

Proposition 2.1. (see e.g. [47], [48])

(1) $(\bigvee_i a_i) \mapsto b = \bigwedge_i (a_i \mapsto b)$ *for all* $\{a_i \mid i \in I\} \subseteq L$, *for all* $b \in L$;
(2) $a \mapsto (\bigwedge_i b_i) = \bigwedge_i (a \mapsto b_i)$ *for all* $a \in L$, *for all* $\{b_i \mid i \in I\} \subseteq L$;
(3) $1_L \mapsto a = a$ *for all* $a \in L$;
(4) $a \mapsto b = 1_L$ *whenever* $a \leq b$
(5) $a * (a \mapsto b) \leq b$ *for all* $a, b \in L$;
(6) $(a \mapsto b) * (b \mapsto c) \leq a \mapsto c$ *for all* $a, b, c \in L$;
(7) $a \mapsto b \leq (a * c \mapsto b * c)$ *for all* $a, b, c \in L$.
(8) $a * b \leq a \wedge b$ *for any* $a, b \in L$.
(10) $(a * b) \mapsto c = a \mapsto (b \mapsto c)$ *for any* $a, b, c \in L$.

2.3. **L-fuzzy sets.** The concept of a fuzzy set was introduced by Zadeh [123] and then extended to a more general concept of an L-fuzzy set by J.A. Goguen [39] where L is a quantale, in particular, a complete infinitely complete lattice. We assume that the reader is well acquainted with terminology related to L-fuzzy sets. Just to clarify the details we recall here the following.

Given a set X, its L-fuzzy subset is a mapping $A : X \to L$. The lattice and the quantale structure of L is extended point-wise to the set L^X of all L-fuzzy subsets of X. Specifically, the union and intersection of a family of L-fuzzy sets $\{A_i | i \in I\} \subseteq L$ are defined by their join $\bigvee_{i\in I} A_i(x)$ and meet $\bigwedge_{i\in I} A_i(x)$ respectively. The product of an L-fuzzy set A by an element $a \in L$ is defined by $(a * A)(x) = a * A(x)$

2.4. **L-fuzzy relations.** An L-fuzzy relation between two sets X and Y is an L-fuzzy subset of the product $X \times Y$, that is a mapping $R : X \times Y \to L$, see, e.g. [113], [124].

An L-fuzzy relation R from X to Y is called left connected if $\bigwedge_{y\in Y}\bigvee_{x\in X} R(x, y) = 1$. If for every $y \in Y$ there exists $x \in X$ such that $R(x, y) = 1_L$, then R is called *strongly left connected.* An L-fuzzy relation R from X to $Y$is called right connected if $\bigwedge_{x\in X}\bigvee_{y\in Y} R(x, y) = 1$. If for every $x \in X$ there exists $y \in Y$ such that $R(x, y) = 1_L$, then R is called strongly right connected. An L-fuzzy relation $R : X \times X \to L$ is called reflexive, if $R(x, x) = 1$ for every $x \in X$. Obviously, an reflexive L-fuzzy relation on a set X is both strogly right and strongly left connected.

Let L be a quantale. An L-fuzzy relation $R : X \times X \to L$ is called transitive if $R(x, y) * R(y, z) \leq R(x, z)$ for all $x, y, z \in X$. A reflexive transitive L-fuzzy relation is called L-fuzzy preoder relation. A reflexive transitive symmetric L-fuzzy relation is called an L-fuzzy equivalence.

A set X endowed with a reflexive transitive L-fuzzy relation is called an L-fuzzy preodered set. Given two L-fuzzy preodered sets (X, R_X) and (Y, R_Y) a mapping $f : X \to Y$ is called isotone if $R_X(x, x') \leq R_Y(f(x), f(x'))$ for all $x, x' \in X$. Let L-**Preod** be the category, whose objects are L-fuzzy preodered sets and morphisms are isotone mappings $f : (X, R_X) \to (Y, R_Y)$.

2.5. **Measure of inclusion of L-fuzzy sets.** Many results in this work are based on the measure inclusion between fuzzy sets. We present here a brief introduction into this field.

In order to fuzzify the inclusion relation $A \subseteq B$ "a fuzzy set A is a subset of a fuzzy set B", we have to interpret it as a certain fuzzy inclusion $\hookrightarrow$ based on "if - then" rule, that is on some implication $\Rightarrow L \times L \to L$. In the result we come to the formula $A \hookrightarrow B = \inf_{x\in X}(A(x) \Rightarrow B(x))$. As far as we know, for the first time this approach was applied in [105], where it was based on the Kleene-Dienes implication $\Rightarrow$. Later the fuzzified relation of inclusion between fuzzy sets was studied and used by many authors, see e.g. [16], [23], [35], [2], [58], [125], et.al. In most of the papers the implication $\Rightarrow$ was defined by means of residuum $\mapsto$ of the underlying quantale $(L, \wedge, \vee, *)$. This implication behaves in this situation "much better" than Kleene-Dienes or some other implication on $(L, \wedge, \vee, *)$. In our paper we stick to the residuum based measure of inclusion specified in the following definition:

Definition 2.1. *By setting $A \hookrightarrow B = \bigwedge_{x\in X}(A(x) \mapsto B(x))$ for all $A, B \in L^X$, we obtain a mapping $\hookrightarrow : L^X \times L^X \to L$. Equivalently, $\hookrightarrow$ can be defined by $A \hookrightarrow B = \inf(A \mapsto B)$, where the infimum of the L-fuzzy set $A \mapsto B \in L^X$ is taken in the lattice L. We call $A \hookrightarrow B$ by the (L-valued) measure of inclusion of the L-fuzzy set A into the L-fuzzy set B.*

Definition 2.2. *Given two L-fuzzy sets $A, B \in L^X$ we define the measure of their equality by $A \cong B = (A \hookrightarrow B) \wedge (B \hookrightarrow A)$.*

In this work we need the properties of relation $A \hookrightarrow B$ collected in the following proposition:

Proposition 2.2. (see e.g. [41], [42].)
Mapping $\hookrightarrow : L^X \times L^X \to L$ satisfies the following properties:

(1) $(\bigvee_i A_i) \hookrightarrow B = \bigwedge_i (A_i \hookrightarrow B)$ *for all* $\{A_i \mid i \in I\} \subseteq L^X$ *and for all* $B \in L^X$;
(2) $A \hookrightarrow (\bigwedge_i B_i) = \bigwedge_i(A \hookrightarrow B_i)$ *for all* $A \in L^X$, *and for all* $\{B_i \mid i \in I\} \subseteq L^X$;
(3) $A \hookrightarrow B = 1_L$ *whenever* $A \leq B$;

(4) $1_X \hookrightarrow A = \bigwedge_x A(x)$ *for all* $A \in L^X$;
(5) $(A \hookrightarrow B) \leq (A * C \hookrightarrow B * C)$ *for all* $A, B, C \in L^X$;
(6) $(A \hookrightarrow B) * (B \hookrightarrow C) \leq (A \hookrightarrow C)$ *for all* $A, B, C \in L^X$;
(7) $(\bigwedge_i A_i) \hookrightarrow (\bigwedge_i B_i) \geq \bigwedge_i (A_i \hookrightarrow B_i)$ *for all* $\{A_i : i \in I\}$, $\{B_i : i \in I\} \subseteq L^X$.
(8) $(\bigvee_i A_i) \hookrightarrow (\bigvee_i B_i) \geq \bigwedge_i (A_i \hookrightarrow B_i)$ *for all* $\{A_i : i \in I\}$, $\{B_i : i \in I\} \subseteq L^X$.

3. Fuzzy topologies

The aim of this section is to give a brief introduction into the history of Fuzzy Topology and to clarify terminology related to Fuzzy Topology used in this paper. More detailed survey about the first period of the development of Fuzzy Topology can be found in [106] and [107].

3.1. **Chang-Goguen fuzzy topologies.**

The first definition of a fuzzy topology was introduced by C.L. Chang [20], a student of L.A. Zadeh, in a paper published in 1968, that is just 3 years after the celebrated work of Zadeh [123]. The basics for the theory of fuzzy topological spaces were also laid in this paper. Probably, it was the first work where fuzzy sets appeared in a purely theoretical mathematical context. In 1973 J.A. Goguen [40], also a student of L.A. Zadeh, introduced the notion of an L-fuzzy topology where L is a quantale, specifically a complete infinitely distributive lattice, and proved some important theorems about L-fuzzy topological spaces. In particular, he established the L-fuzzy version of the fundamental Tychonoff theorem about the compactness of products of compact L-fuzzy topological spaces. Here we present the definition of Chang-Goguen L-fuzzy topological space.

Definition 3.1. *Let L be a complete infinitely distributive lattice and X be a set. An L-fuzzy topology on X is a family $\tau \subseteq L^X$, such that*

(1_t) $0_X, 1_X \in \tau$ *were* 0_X *and* 1_X *are constant fuzzy sets with values* 0_L *and* 1_L *respectively;*
(2_t) *If* $U_1, U_2 \in \tau$, *then* $U_1 \wedge U_2 \in \tau$;
(3_t) *If* $\{U_i \mid i \in I\} \subseteq \tau$ *then* $\bigvee_{i \in I} U_i \in \tau$.

The pair (X, τ) is called a (Chang-Goguen) L-fuzzy topological space.

An important kind of L-fuzzy topologies are so called Alexandroff (cf. e.g. [1], [21], [63]), or principal, fuzzy topologies obtained by replacing axiom (2_t) by a stronger axiom

(2_t^A) If $\{U_i \mid i \in I\} \subseteq \tau$, then $\bigwedge_{i \in I} U_i \in \tau$.

Definition 3.2. *Given two L-fuzzy topological space $(X, \tau_X), (Y, \tau_Y)$ a mapping $f : (X, \tau_X) \to (Y, \tau_Y)$ is called continuous if $f^{-1}(V) \in \tau_X$ for every $V \in \tau_Y$.*

In 70-ties and 80-ties of the previous century the theory of Chang-Goguen fuzzy topological spaces was probably the most fast developing theory in “Fuzzy Mathematics”. In particular, important results in this period were obtained by Pu Paoming and Liu Yingming [86] [87], B. Hutton [53], [54] A. Katsaras [57], R. Lowen [71], [72], [73], S.E. Rodabaugh [92], [93], Kubiak [65] et.al. Specifically, R. Lowen,[71], [73] suggested to replace axiom (1_t) in Chang-Goguen definition by a stronger axiom

(1_t^s) $a_X \in \tau$ for all $a \in L$ were a_X is a_X constant fuzzy set with value a.

Fuzzy topologies satisfying this axiom are called stratified, see e.g. [86]. The benefits of this assumption is that the resulting theory behaves more similar to classical general topology. For example, as different from general Chang-Goguen fuzzy topology, constant functions between stratified L-fuzzy topological spaces are always continuous, the projections in the product of such spaces are always open, etc. On the other hand, the request to include axiom (1_t^s) in the very definition of fuzzy topology would essentially restricts the field of research. For example, this approach excludes the possibility to view classical topologies as a special kind of fuzzy topologies.

Speaking about Chang-Goguen L-fuzzy topologies we have to mention also a series of works by S.E. Rodabaugh, [94], [97] et al. in which variable-base categories of L-fuzzy topologies were defined. As different from Chang-Goguen approach where the lattice L is arbitrary, but fixed, variable-base approach assumes L-fuzzy topologies with different lattices L in the same context. We shall not present here the definition of variable-base fuzzy topologies since such fuzzy topologies will not appear in this work.

In case when the lattice L is equipped with an order reversing involution, that is a mapping $^c : L \to L$ such that $\alpha \leq \beta, \Longrightarrow \alpha^c \geq \beta^c$ and $(\alpha^c)^c = \alpha$ for all $\alpha, \beta \in L$[1], then, having a family of open L-fuzzy sets one can define the family of closed L-fuzzy sets. Namely, to declare a fuzzy set $A \in L^X$ as closed if and only if its complement is open, that is $A^c \in \tau$. However generally L need not have an involution. Moreover, even if L has a naturally defined order-reversing involution, closed L-fuzzy sets may be defined independently of open L-fuzzy sets. Specifically, this is a typical situation in our work. This problem necessitates to introduce, parallel to the concept of an L-fuzzy topology, the concept of the L-fuzzy cotopology. This is done in the next definition:

Definition 3.3. *Let L be a lattice and X be a set. An L-fuzzy cotopology on a set X is a family $\sigma \in L^X$ such that*

(1_{ct}) $0_X, 1_X \in \sigma$;
(2_{ct}) *If* $A_1, A_2 \in \sigma$, *then* $A_1 \vee A_2 \in \sigma$;
(3_{ct}) *If* $\{A_i \mid i \in I\} \subseteq \sigma$ *then* $\bigwedge_{i\in I} A_i \subseteq \sigma$.

The Alexandroff and stratified versions for L-fuzzy cotopologies are defined in an obvious way.

Definition 3.4. *Given two L-fuzzy cotopological spaces $(X, \sigma_X), (Y, \sigma_Y)$, a mapping $f : (X, \sigma_X) \to (Y, \sigma_Y)$ is called continuous if $f^{-1}(B) \in \sigma_X$ for every $B \in \sigma_Y$.*

Finally the situation when both an L-fuzzy topology and an L-fuzzy cotopology are independently defined on a set X, leads us to the concept of a L-fuzzy ditopology, first distinguished by L.M. Brown, see e.g. [12] et al.:

Definition 3.5. *An L-fuzzy ditopology on a set X ia pair (τ, σ) were τ is an L-fuzzy topology and σ is an L-fuzzy cotopology. The triple (X, τ, σ) is called an L-fuzzy ditopological space.*

The continuity of mappings of L-fuzzy ditopological spaces is defined in the obvious way. We denote the category of L-fuzzy ditopological spaces and their continuous mappings by L-**FDiTop**.

3.2. Fuzzifying topologies.

In 1980 U. Höhle [46] presented an alternative viewpoint on the subject of Fuzzy Topology. Namely, in Chang-Goguen approach the sets are fuzzy, but the topology is actually crisp: every fuzzy set is either open or not. U. Höhle suggests an approach according to which the sets are crisp, while the topology is fuzzy: for each set the degree to which it is open is a certain number in [0,1]. Later Migsheng Ying [120], [121] independently rediscovered this approach. He came to this definition making an analysis of topological axioms by means of fuzzy logic tools and called the obtained structure by a fuzzifying topology. Just this term is used now when dealing with such "topologies". We reproduce here the L-version of a fuzzifying topology.

Definition 3.6. *Let L be a lattice and X be a set. An L-fuzzifying topology on a set X is a mapping $\mathcal{T} : 2^X \to L$ such that*

(1_{fft}) $\mathcal{T}(\emptyset) = \mathcal{T}(X) = 1$;
(2_{fft}) $\mathcal{T}(U_1 \cap U_2) \geq \mathcal{T}(U_1 \wedge \mathcal{T}(U_2)$ *for all* $U_1, U_2 \in 2^X$;
(3_{fft}) $\mathcal{T}(\bigcup_{i\in I} U_i \geq \bigwedge_{i\in I} \mathcal{T}(U_i)$ *for each family* $\{U_i \mid i \in I\} \subseteq 2^X$.

In an obvious way the concepts of an L-fuzzifying cotopology and an L-fuzzifying ditopology can be defined.

[1]Such lattices are known also as De Morgan algebras

3.3. **L-valued fuzzy topologies.**
The idea to combine the Chang-Goguen and Höhle approaches was realized (independently) by T. Kubiak in [64] and A.Šostak in [105]; it lead to the concept which now usually is called a manyvalued (or, specifically, an L-valued) or graded fuzzy topology.

Definition 3.7. [64], [105], [106]. *Let L be a complete infinitely distributive lattice. An L-valued fuzzy topology on a set X is a mapping $\mathcal{T} : L^X \to L$ satisfying the following conditions:*

(1_{vt}) $\mathcal{T}(0_X) = \mathcal{T}(1_X) = 1_L$;
(2_{vt}) $\mathcal{T}(U_1 \wedge U_2) \geq \mathcal{T}(U_1) \wedge \mathcal{T}(U_2)$ *for all* $U_1, U_2 \in L^X$;
(3_{vt}) $\mathcal{T}(\bigvee_{i\in I} U_i) \geq \bigwedge_{i\in I} \mathcal{T}(U_i)$ *for any family* $\{U_i \mid i \in I\} \subseteq L^X$.

L-valued fuzzy topology $\mathcal{T}$ is called Alexandroff if it satisfies a stroger version of axiom (2_{vt}):

(2^A_{vt}) $\mathcal{T}(\bigwedge_{i\in I} U_i) \geq \bigwedge_{i\in I} \mathcal{T}(U_i)$ *for any family* $\{U_i \mid i \in I\} \subseteq L^X$;

We call $\mathcal{T}(U)$ the degree of openness of an L-fuzzy set U. The pair $(X, \mathcal{T})$ is called an L-valued fuzzy topological space.

Definition 3.8. *(see e.g.* [64], [106]*) Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be L-valued fuzzy topological spaces. A mapping $f : X \to Y$ is called continuous if $\mathcal{T}_X(f^{-1}(V)) \geq \mathcal{T}_Y(V)$ for every $V \in L^Y$.*

Remark 3.1. In [66] a more general concept of an (L, M)-fuzzy topology was introduced as a mapping $\mathcal{T} : L^X \to M$ satisfying conditions analogous to the ones in the previous definition. In this case it is possible to distinguish the properties that lattice L must satisfy from the properties required for M in the proof of each concrete result for the analysis of some situations. However here, in order to make exposition homogeneous, we will not consider this case of fuzzy topology.

In case when we can apply the order reversing involution $^c : L \to L$, the degree of closedness of an L-fuzzy set A can be defined as the degree of openness of .its complement, that is as $\mathcal{T}(A^c)$. However, in case when we have to avoid the use of the involution (and this is just the case that we encounter in this work) we define the L-valued fuzzy cotopology on a set X as follows:

Definition 3.9. *Let L be a complete infinitely distributive lattice and X be a set. An L-valued fuzzy cotopology on a set X is a mapping $\mathcal{S} : L^X \to L$ satisfying the following conditions:*

(1_{vct}) $\mathcal{S}(0_X) = \mathcal{S}(1_X) = 1_L$;
(2_{vct}) $\mathcal{S}(A_1 \vee A_2) \geq \mathcal{S}(A_1) \wedge \mathcal{S}(A_2)$ *for all* $A_1, A_2 \in L^X$;
(3_{vct}) $\mathcal{S}(\bigwedge_{i\in I} A_i) \geq \bigwedge_{i\in I} \mathcal{S}(A_i)$ *for any family* $\{A_i \mid i \in I\} \subseteq L^X$.

The Alexandroff version of a L-valued fuzzy cotopology is defined in an obvious way.

Definition 3.10. *Given two L-valued fuzzy cotopological spaces $(X, \mathcal{S}_X)$ and $(Y, \mathcal{S}_Y)$ A mapping $f : X \to Y$ is called continuous if $\mathcal{S}_X(f^{-1}(B)) \geq \mathcal{S}_Y(B)$ for every $B \in L^Y$.*

In what follows, we will meet also the following special kinds of L-valued fuzzy topologies and cotopologies in case L is a quantale:

Definition 3.11. (see e.g. [50]) *An L-valued fuzzy cotopology $\mathcal{S} : L^X \to L^X$ is called stratified if*

(4^s_{vct}) $\mathcal{S}(A * a_X) \geq \mathcal{S}(A) \forall A \in L^X, \forall a \in L$.

An L-valued fuzzy topology $\mathcal{T} : L^X \to L^X$ is called costratified if

(4^{cs}_{vt}) $\mathcal{T}(a_X \mapsto U) \geq \mathcal{T}(U) \forall U \in L^X, \forall a \in L$.

Remark 3.2.

- Notice that axiom (4^s_{vct}) implies that $\mathcal{S}(a_X) = 1_L$ for every $a \in L$, and axiom (4^{cs}_{vt}) implies that that $\mathcal{T}(a_X) = 1_L$ for every $a \in L$.
- Mutual correspondence of concepts stratified and costratified and their roles in the definitions of L-valued topology and L-fuzzy cotopology are justified by the rules of fuzzy logic., see e.g. [50].

Definition 3.12. [13] *An L-valued fuzzy ditopology on a set X is a pair $(\mathcal{T}, \mathcal{S})$ where $\mathcal{T}$ is an L-valued fuzzy topology and $\mathcal{S}$ is an L-valued fuzzy cotopology on X. The corresponding triple $(X, \mathcal{T}, \mathcal{S})$ is called an L-valued fuzzy ditopological space. An L-valued fuzzy ditopological space is called stratified if its cotopology is stratified and its topology costratified.*

Continuity for mappings of L-valued fuzzy ditopological spaces is defined in an obvious way.

The category of L-valued fuzzy ditopological spaces and their continuous mappings will be denoted L-**VFDiTop** and its full subcategory of stratified spaces will be denoted L-**VFDiTop**^{s}

At present there is a vast literature where L-valued fuzzy topologies, cotopologies and ditopologies and related structures are studied, see e.g. [50], [51], [19], etc.

3.4. **Point-free approach to fuzzy topology and L-valued fuzzy topology.**

L-fuzzy topology on a set X is a family $\tau \subseteq L^X$ satisfying certain axioms. The structure of L^X is determined by the lattice structure of the lattice L. Therefore one can revise the definition of an L-fuzzy topology by replacing the lattice L^X with some abstract lattice $\mathcal{L}$ (in particular, $\mathcal{L} = L^X$) and redefining all concepts by replacing families of L-fuzzy subsets of X with "crisp" subsets of the lattice $\mathcal{L}$. In other words, instead of viewing A as an object of L^X, we can interpret it as an element from $\mathcal{L}$. In this case a fuzzy topology on a lattice $\mathcal{L}$ is defined as a family $\tau \subseteq L^X$ satisfying axioms analogous to the axioms of the Definition 3.1. This approach, ignoring the points of the set X, but just basing on the lattice structure of the lattice $\mathcal{L}(= L^X)$, is called point-free. B. Hutton was the "the founder" of this approach, see [53], [54]; a similar approach based on the so called fuzzy molecular lattices was created by G.J. Wang [114] (the first Chinese version of this work is dated by 1979). Later this approach has been developed in papers by S.E.Rodabaugh [95], [96], see also work done by T. Kubiak, J. Guttierez, J. Kortelainen, U. H́ohle, P. Eklund et al. Of course, all concepts considered in Section 3.1 can be reformulated in the point-free context.

Similary, one can easily come to the point-free version of an L-valued fuzzy topology. Indeed, an L-valued fuzzy topology on X is a mapping $\mathcal{T} : L^X \to L$ satisfying some axioms. By ignoring the points $x \in X$ but just considering L^X as some abstract lattice $\mathcal{L}$ and defining a point-free L-valued fuzzy topology on $\mathcal{L}$ as a mapping $\mathcal{T} : \mathcal{L} \to L$ satsfying axioms, analogous to the axioms from Definition 3.7 (in this case $\mathcal{L}$ and L do not need to be related). Without problem, all definitions and results from Section 3.3 can be reformulated for the point-free case.

In some investigations, in partucar in the results that will be discussed in this work it is necessary to consider generalizations of L-fuzzy topologies and L-valued fuzzy topologies in which one of the axioms is weaken or omitted. Such generalizations are introduced in the next two subsections.

3.5. **L-fuzzy supratopology and L-valued fuzzy supratopology.**
Starting to discuss generalizations of L-fuzzy and L-valued fuzzy topologies needed for our merits, we prefer to begin with the cotopological approach. Recall the following classic definition; we formulate in a form, covenient for our merits:

Definition 3.13. (see, e.g. [7], [79]) *A mapping $c : L^X \to L^X$ is called a closure operator on the set L^X, if it satisfies the following conditions:*

($1_{\rm cl}$) $A \le \mathrm{cl}(A)$*, that is operator* $\mathrm{cl} : L^X \to L^X$ *is extensional;*
($2_{\rm cl}$) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ *that is operator* $\mathrm{cl} : L^X \to L^X$ *is idempotent.*
($3_{\rm cl}$) $A_1 \le A_2 \Longrightarrow \mathrm{cl}_X(A_1) \le \mathrm{cl}_X(A_2)$ *that is operator* $\mathrm{cl}_X : L^X \to L^X$ *is isotone.*

Closure operator $\mathrm{cl} : L^X \to L^X$ is called Kuratowski or topological closure operator if it satisfies axioms

($1_{\rm cl}$) $A \le \mathrm{cl}(A)$;
($2_{\rm cl}$) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$;

($3^{\mathrm{ad}}_{\mathrm{cl}}$) $\mathrm{cl}(A_1 \vee A_2) = \mathrm{cl}(A_1) \vee \mathrm{cl}(A_2)$[2];
(4_{cl}) $\mathrm{cl}(0_X) = 0_X$;

It is well known, that given a Kuratowski closure operator, by setting $\sigma_{\mathrm{cl}} = \{A \in L^X \mid \mathrm{cl}(A) = A\}$ an L-fuzzy cotopology on X is defined. Conversely, if we have an L-fuzzy cotopology σ on a set X, by setting $\mathrm{cl}_\sigma(A) = \bigwedge\{B \mid B \geq A, B \in \sigma\}$ we obtain a Kuratowski closure operator.

We undertake the same procedure for a general closure operator. Namely, let $\mathrm{cl} : L^X \to L^X$ be a closure operator and let $\sigma_{\mathrm{cl}} = \{A \in L^X \mid \mathrm{cl}(A) = A\}$. We show that σ_{cl} is closed under arbitrary meets. Indeed, $\mathrm{cl}\left(\bigwedge_{i\in I} A_i\right) \leq \mathrm{cl}(A_i) = A_i$ for every $i \in I$ and hence $\mathrm{cl}\left(\bigwedge_{i\in I} A_i\right) \leq \bigwedge_{i\in I} A_i$. On the other hand, since $\mathrm{cl}\left(\bigwedge_{i\in I} A_i\right) \geq \bigwedge_{i\in I} A_i$ by extensionality of operator cl, we get the equality $\mathrm{cl}\left(\bigwedge_{i\in I} A_i\right) = \bigwedge_{i\in I} A_i$. Hence $\bigwedge\{A_i \mid i \in I\} \in \sigma_{\mathrm{cl}}$ whenever $A_i \in \sigma_{\mathrm{cl}}$ for every $i \in I$.

In the literature one can find the following definition which first in crisp case appeared in Levin's theory of generalized closed sets [69]:

Definition 3.14. (cf. e.g. [69], [31], [76]) *A family $\sigma \in L^X$ is called an L-fuzzy supra cotopology, if it is invariant under taking arbitrary non-empty meets and contains 0_X.*

Thus a closure operator generates an L-fuzzy supra cotopology up to the condition $0_X \in \sigma_{\mathrm{cl}}$. This condition will be satisfied as soon as $\mathrm{cl}(0_X) = 0_X$.

Dualizing the above considerations we come to the concepts of interior operator, Kuratowski interior operator, and L-fuzzy supratopology:

Definition 3.15. (cf e.g. [31], [76]) *A family $\tau \in L^X$ is called an L-fuzzy supra topology, if it is invariant under taking arbitrary non-empty joins and contains 1_X.*

The triple (X, τ, σ) where τ is an L-fuzzy supra topology and σ is an L-fuzzy supra cotopology on a set X is called an L-fuzzy supra ditopological space. Continuity for mappings of L-fuzzy supra topological, L-fuzzy supra cotopological and L-fuzzy supra ditopological spaces is defined in an obvious way, patterned after definitions 3.2, 3.4. The category of L-fuzzy supra ditopological spaces and their continuous mappings is denoted L-**FSDiTOP**.

In the sequel we will need also the L-valued versions of these concepts:

Definition 3.16. *Let L be a lattice and X be a set. A mapping $\mathcal{S} : L^X \to L$ is called an L-valued fuzzy supra cotopology on X if*
(1_{vcl}) $\mathcal{S}(\bigwedge_{i\in I} A_i) \geq \bigwedge_{i\in I} \mathcal{S}(A_i)$ *for every* $\{A_i \mid i \in I\} \subseteq L^X$;
(2_{vcl}) $\mathcal{S}(0_X) = 1_L$.
A mapping $\mathcal{T} : L^X \to L$ is called an L-valued fuzzy supra topology on X if
(1_{vint}) $\mathcal{T}(\bigvee_{i\in I} A_i) \geq \bigwedge_i \mathcal{T}(A_i)$ *for every* $\{A_i \mid i \in I\} \subseteq L^X$;
(2_{vint}) $\mathcal{T}(1_X) = 1_L$.
The triple $(X, \mathcal{T}, \mathcal{S})$ is called an L-valued fuzzy supra ditopological space.

The continuity of L-valued fuzzy supra ditopological spaces is defined patterned after definion of continuity of L-valued fuzzy ditopological spaces. Let L-**VFSDiTop** denote the category of L-valued fuzzy supra ditopological spaces and their continuous mappings.

3.6. L-fuzzy pretopology and L-fuzzy precotopology.

In our work we need also an alternative generalization of the concept of an L-fuzzy topology. Again, we start with the analysis of the Kuratowski system of axioms of an L-fuzzy topology. While in the previous subsection we "sacrifised" the additivity axiom and replaced it by a weaker, isotonisity axiom, here we omit the second, idempotence, axiom and retain the additivity axiom ($3^{\mathrm{ad}}_{\mathrm{cl}}$) in full. In the result we come to the following definition:

[2] Note that (3^{ad}_{cl}) $\Longrightarrow$ (3_{cl})

Definition 3.17. *An L-fuzzy preclosure operator on a set X is a mapping* $\mathrm{cl} : L^X \to L^X$ *satisfying the following axioms:*

(1_{cl}) $A \leq \mathrm{cl}(A)$ *for all* $A \in L^X$*;*
($3^{\mathrm{ad}}_{\mathrm{cl}}$) $\mathrm{cl}(A_1 \vee A_2) = \mathrm{cl}(A_1) \vee \mathrm{cl}(A_2)$ *for all* $A_1, A_2 \in L^X$*;*
(4_{cl}) $\mathrm{cl}(0_X) = 0_X$.

Given a preclosure operator, let $\sigma_{\mathrm{cl}} = \{A \in L^X \mid \mathrm{cl}(A) = A\}$. It is easy to notice that σ_{cl} is an L-fuzzy cotopology. Indeed,

(1) $1_X \in \sigma_{\mathrm{cl}}$ by axiom (4_{cl});

(2) If $A_1, A_2 \in \sigma_{\mathrm{cl}}$, then, by axiom ($3^{\mathrm{ad}}_{\mathrm{cl}}$) $A_1 \vee A_2 = \mathrm{cl}(A_1) \vee \mathrm{cl}(A_2) = \mathrm{cl}(A_1 \vee A_2)$ and hence $A_1 \vee A_2 \in \sigma_{\mathrm{cl}}$;

(3) Let $A_i \in \sigma_{\mathrm{cl}}$ for all $i \in I$. Then $\bigwedge_i A_i = \bigwedge_i \mathrm{cl}(A_i) \geq \mathrm{cl}(\bigwedge_i A_i)$ by isotonisity of cl that follows from ($3^{\mathrm{ad}}_{\mathrm{cl}}$). The opposite inequality follows from the extensionality of the preclosure operator.

Fuzzy cotopology σ_{cl} is called induced by the fuzzy preclosure operator $\mathrm{cl} : L^X \to L^X$.

Dualizing the above considerations, we come to the concept of an L-fuzzy preinterior operator as a mapping $\mathrm{int} : L^X \to L^X$ satisfying the following axioms:

(1_{int}) $A \geq \mathrm{int}(A)$;
($3^{\mathrm{ml}}_{\mathrm{int}}$) $\mathrm{int}(A_1 \wedge A_2) = \mathrm{int}(A_1) \wedge \mathrm{int}(A_2)$;
(4_{int}) $\mathrm{int}(1_X) = 1_X$

and the induced L-fuzzy topology $\tau_{\mathrm{int}} = \{A \in L^X \mid \mathrm{int}(A) = A\}$.

It is important to emphasize, that as different from the case of L-fuzzy supra topology and L-fuzzy supra cotopology, the structure of an L-fuzzy pretopology and precotopology (as a family of L-fuzzy sets) is not apriory clear because of the absence of idempotency of preclosure and preinterior operators. In particuar, it is not excluded that the L-fuzzy cotopology induced by a *nontrivial* preclosure operator is just the one point family $\{1_X\}$. Similarly for a fuzzy topology: it can reduce to the one point family $\{0_X\}$. Therefore, when dealing with L-fuzzy pretopology and L-fuzzy precotopology we will remain on the level defined by preclosure and preinterior operators and not deal with their realization as families of L-fuzzy sets. In its turn it forces to define continuity of mappings between L-fuzzy preclosure and L-fuzzy preinterior spaces. We do it in the next definition; it is justified by the characterization of continuity of mappings of topological spaces in terms of closure and interior operators, well known in General Topology, see e.g. [33].

Definition 3.18.
- *Let $(X, \mathrm{cl}_X), (Y, \mathrm{cl}_Y)$ be L-fuzzy precotopological spaces. A mapping $f : X \to Y$ is called continuous if $f(\mathrm{cl}_X(A)) \leq \mathrm{cl}_Y(f(A))$ for all $A \in L^X$.*
- *Let $(X, \mathrm{int}_X), (Y, \mathrm{int}_Y)$ be L-fuzzy pretopological spaces. A mapping $f : X \to Y$ is called continuous if $f^{-1}(\mathrm{int}(B)) \leq \mathrm{int}(f^{-1}(B))$ for all $B \in L^Y$.*
- *Let $(X, \mathrm{cl}_X, \mathrm{int}_X), (Y, \mathrm{cl}_Y, \mathrm{int}_Y)$ be L-fuzzy preditopological spaces. A mapping $f : X \to Y$ is called continuous if it is continous as a mapping $f : (X, \mathrm{cl}_X) \to (Y, \mathrm{cl}_Y)$ and as a mapping $f : (X, \mathrm{int}_X) \to (Y, \mathrm{int}_Y)$*

Let L-**FPDiTop** denote the category of L-fuzzy preditopological spaces and their continuous mappings.

Remark 3.3. We do not introduce here the L-valued versions of L-fuzzy closure and L-fuzzy interior operators, because they will not appear in this paper. Besides, their definition would necessitate some auxiliary construction that will allow to work on the level of operators.

4. Fuzzy approximation systems and fuzzy rough sets

4.1. **Introduction.** The concept of a rough subset of a set equipped with an equivalence relation was introduced by Pawlak [83] in 1982 as a tool for dealing with imperfect knowledge. For reader's convenience we recall here the "classic" Pawlak's definition of a rough set.

Let X be a set endowed with an equivalence relation $E \subseteq X \times X$ and let $A \subseteq X$. Then a rough set determined by A can be defined as the pair $(A^{\blacktriangledown}, A^{\blacktriangle})$, where $A^{\blacktriangledown} = \{x \in A \mid [x]_E \cap A \neq \emptyset\}$ and $A^{\blacktriangle} = \{x \in A \mid [x]_E \subseteq A\}$; here $[x]_E$ denotes the E-equivalence class of the element $x \in A$.

Rough sets approach has found important applications in real-world problems. In particular it is of fundamental importance in artificial intelligence and cognitive sciences, especially in data analysis and knowledge discovery. The main advantage of rough set theory in data analysis is that it does not need any preliminary or additional information about data like probability distributions needed in statistics and a grade of membership or the value of possibility needed in fuzzy set theory. One can find many papers devoted to theoretical aspects of fuzzy rough sets and their applications in different fields, see e.g. the recent Skowron's survey [103].

Soon after Pawlak's original work, some mathematicians showed interest in the fuzzy version for rough sets, that is to extend the concept of a rough set to the context of fuzzy sets; the paper by Dubois and Prade [28] was the first work in this direction. Fuzzy rough sets could combine the advantages of the both approaches for study problems in which incomplete, imprecise or vague information is used. At present there is a vast literature where fuzzy rough sets are investigated and applied. In particular, important contribution to the theory of fuzzy rough sets is done by Kortelainen [63], Yao [118],[119], Radzikowska and Kerre [90] Qin and Pei [89], Ciucci [22], Tiwari and Srivastava [112], etc. just to mention a few of the numerous works dealing with theoretical and applied problems of fuzzy rough sets.

Several authors who were interested in the theory of rough (in particular, L-fuzzy rough) sets, noticed its undoubted relation with (fuzzy) topology. Probably, the papers by A. Wiweger [116] and Skowron [102] were the first ones in which such relations were revealed. Later many authors showed interest in the topological aspects of fuzzy rough sets. In particular, these problems were touched in [63], [34], [68], [56], [77], [112], [41], etc.

The aim of this section is to outline basics of the theory of L-fuzzy rough sets and to reveal some aspects of topological base underlying this theory. The exposition is based on inerpreting categories related to L-fuzzy rough sets as categories of L-fuzzy ditopological and L-valued ditopological spaces.

4.2. **L-fuzzy rough approximation of an L-fuzzy set.**
For the exposition of the basic concepts of L-fuzzy rough set theory we start, as also many researches in this field do, with an L-fuzzy preodered set (X, R), where L is a fixed quantale (Subsection 2.4).[3]

Definition 4.1. (see, e.g. [117], [56], [108]) *The upper L-fuzzy rough approximation operator $u_R : L^X \to L^X$ is defined by*

$$u_R(A)(x) = \bigvee_y (R(y,x)) * A(y)) \ \ \forall A \in L^X, \ \forall x \in X.$$

The lower L-fuzzy rough approximation operator $l_R : L^X \to L^X$ is defined by

$$l_R(A)(x) = \bigwedge_y (R(x,y) \mapsto A(y)) \ \ \forall A \in L^X \ \forall x \in X.$$

The pair $(u_R(A), l_R(A))$ can be interpreted as the L-fuzzy rough set induced by an L-fuzzy subset of an L-fuzzy preodered set (X, R). Note that in case $L = \{0, 1\}$ is a two-point element, $A \subset X$ and $R : X \times X \to \{0, 1\}$ is an equivalence ralation, then $(u_R(A), l_R(A))$ is just Pawlak's rough set $(A^{\blacktriangledown}, A^{\blacktriangle})$.

The analogues of the following two theorems under different assumptions on the lattice L or on the L-fuzzy relation R can be found in the works of different people. We formulate them as it is presented in [56], [32]

[3]Some authors request R to be an L-fuzzy equivalence on the set X, while others do not request reflexivity or transitivity. For our merits the most appropriate context is provided by L-fuzzy preoders

Theorem 4.1. *The upper L-fuzzy rough approximation operator satisfies the following properties:*

(1_u) $u_R(a_X) = a_X$ *for every* $a \in L$;
(2_u) $A \leq u_R(A)$ $\forall A \in L^X$;
(3_u) $u_R(\bigvee_i A_i) = \bigvee_i u_R(A_i)$ $\forall \{A_i \mid i \in I\} \subseteq L^X$;
(4_u) $u_R(u_R(A)) = u_R(A)$ $\forall A \in L^X$.

Theorem 4.2. *The lower L-fuzzy rough approximation operator satisfies the following properties:*

(1_l) $l_R(a_X) = a_X$;
(2_l) $A \geq l_R(A)$ $\forall A \in L^X$;
(3_l) $l_R(\bigwedge_i A_i) = \bigwedge_i l_R(A_i)$ $\forall \{A_i \mid i \in I\} \subseteq L^X$;
(4_l) $l_R(l_R(A)) = l_R(A)$ $\forall A \in L^X$.

Dealing with L-fuzzy subsets of an L-fuzzy preodered set, an important property is extensionality.

Definition 4.2. (see, e.g. [47]) *An L-fuzzy subset A of an L-fuzzy preodered set (X, R) is called extensional, if $A(x) * R(x, y) \leq A(y)$ for all $x, y \in X$.*

There are many works dealing with the problem of extensionality, especially in case when R is an L-fuzzy equivalence relation. Our interest focuses on the extensionality of upper and lower rough approximations.

4.2.1. *Extensional properties of upper and lower fuzzy rough approximation operators.*

Definition 4.3. *Extensional hull of an L-fuzzy set A is the smallest ($\leq$) extensional L-fuzzy set $\tilde{A}$ containing the given L-fuzzy set A (that is $A \leq \tilde{A}$).*

One can find the proof of the following proposition in the papers [47], [48], [61].

Proposition 4.1. *The extensional hull of an L-fuzzy set A is equal to its upper L-fuzzy rough approximation: $\tilde{A} = u_R(A)$.*

Definition 4.4. *Extensional kernel of an L-fuzzy set A is the largest ($\geq$) extensional L-fuzzy set A^o contained in A.*

Theorem 4.3. ([32]) *For every L-fuzzy set A the extensional kernel of A is equal to its lower L-fuzzy rough approximation: $A^o = l_R(A)$.*

The proof of this theorem as given in [32], uses Proposition 4.2, which is of its own interest. Note that in case $L = [0, 1]$ and $* = \wedge$, this statement can be found in [90, Proposition 9].

Proposition 4.2. *For every L-fuzzy set A, it holds $u_R(l_R(A)) = l_R(A)$ and $l_R(u_R(A)) = u_R(A)$.*

4.3. **Category of L-fuzzy rough approximation spaces.**

Definition 4.5. *Let (X, R) be an L-fuzzy preodered set where L is a fixed quantale. The tuple (X, R, u_R, l_R) is called an L-fuzzy rough approximation space.*

Before defining the morphisms for the catgory L-**FRAS** of L-fuzzy rough approximation spaces, we analyse ralations of L-fuzzy rough approximation spaces from topological point of view.

Notice that properties of the upper L-fuzzy rough approximation operator collected in Theorem 4.1 characterize u_R as Kuratowski closure operator, see Definition 3.13. Hence, by setting $\sigma_{u_R} = \{A \in L^X \mid u_R(A) = A\}$ we obtain an L-fuzzy cotopology on the set X. Besides, from property (1_u) we can conclude that σ_{u_R} is stratified and referring to Proposition 4.2 one can show that σ_{u_R} is Alexandroff (see Subsection 3.1).

In its turn, the properties of the lower L-fuzzy rough approximation operator collected in Theorem 4.2 characterize l_R as Kuratowski interiour operator. Hence, by setting $\tau_{l_R} = \{A \in L^X \mid l_R(A) = A\}$, we obtain an L-fuzzy topology on the set X. Besides, from property (1_l) we can conclude that τ_{l_R} is stratified and refering to Proposition 4.2 one can show that τ_{l_R} is Alexandroff. The obtained results can be combined now as follows:

Theorem 4.4. *Let (X, R) be an L-fuzzy predered set and (X, R, u_R, l_R) be the corresponding L-fuzzy rough approximation space. Then $(X, \tau_{l_R}, \sigma_{u_R})$ is a stratified Alexandroff L-fuzzy ditopological space.*

Thus L-fuzzy rough approximation spaces can be viewed as a kind of L-fuzzy ditopological spaces, and this observation indicates the way how morphisms in the category L-**FRAS** must be defined. We call these morphisms continuous mappings and introduce them in terms of interior and closure operators, see Definition 3.18:

Definition 4.6. *Given L-fuzzy rough approximation spaces (X, R, u_{R_X}, l_{R_X}) and (Y, R, u_{R_Y}, l_{R_Y}), a mapping $f : X \to Y$ is called continuous, if*

(1) *$f(u_{R_X}(A)) \leq u_{R_Y}(f(A))$ for every $A \in L^X$, and*
(2) *$f^{-1}(l_{R_Y}(B)) \leq l_{R_X}(f^{-1}(B))$ for every $B \in L^Y$.*

Definition 4.7. *L-**FRAS** is a category, whose objects are L-fuzzy rough approximation spaces and whose morphisms are continuous mappings between such spaces.*

From the above considerations it follows that the category L-**FRAS** is a full subcategory of the category L-**FDiTop** of L-fuzzy ditopologicas spaces and their continuous mappings.

The "corectness" of the previous definition is justified also by the next easily provable theorem and its corollary:

Theorem 4.5. *Let (X, R_X) and (Y, R_Y) be L-fuzzy predered sets and let $f : X \to Y$ be an isotone mapping. Then f is continuous as a mapping of the corresponding L-fuzzy rough approximation spaces, that is as a mapping $f : (X, R_X, u_{R_X}, l_{R_X}) \to (Y, R, u_{R_Y}, l_{R_Y})$.*

Corollary 4.1. *By assigning to an L-fuzzy predered set (X, R) an L-fuzzy rough approximation space (X, R, u_R, l_R) and interpreting isotone mappings $f : (X, R_X) \to (Y, R_Y)$ as mappings $f : (X, R_X, u_{R_X}, l_{R_X}) \to (Y, R_Y, u_{R_Y}, l_{R_Y})$, we get an embedding functor from the category L-**Preod** of L-fuzzy predered sets into the category L-**FRAS** of L-fuzzy rough approximation spaces $\Phi : L$-**Preod** $\longrightarrow L$-**FRAS**.*

It is clear from the definitions, that the category L-**FRAS** of L-fuzzy rough approximation spaces is a full subcategory of the category L-**FDiTop** of L-fuzzy ditopological spaces, therefore we get also the following

Corollary 4.2. *By assigning to an L-fuzzy predered set (X, R) an L-fuzzy ditopological space $(X, R, \tau_{l_R}, \sigma_{u_R})$ and interpreting isotone mappings $f : (X, R_X) \to (Y, R_Y)$ as mappings of the corresponding L-fuzzy ditopological spaces $f : (X, R_X, \tau_{l_{R_X}}, \sigma_{u_{R_X}}) \to (Y, R, \tau_{l_{R_Y}}, \sigma_{u_{R_Y}})$, we get an embedding functor from the category L-**Preod** into the category L-**FDiTop** of L-fuzzy ditopological spaces $\Phi' : L$-**Preod** $\longrightarrow L$-**FDiTop**.*

4.4. Measure of L-fuzzy rough approximation of an L-fuzzy set.

The first attempt to measure the degree of roughness of a fuzzy set, or to state it in another way, to measure, "how much rough," is a given L-fuzzy subset of an L-fuzzy predered set (X, R), was undertaken as far was we know, in [41], [42]:

Definition 4.8. *Let (X, R) be an L-fuzzy predered set and $(X, R_X, u_{R_X}, l_{R_X})$ the corresponding L-fuzzy rough approximation space. Given an L-fuzzy set $A \in L^X$ we define the measure $\mathcal{U}(A)$ of its upper L-fuzzy rough approximation by $\mathcal{U}_R(A) = u_R(A) \hookrightarrow A$ and the measure $\mathcal{L}(A)$ of its lower L-fuzzy rough approximation by $\mathcal{L}_R(A) = A \hookrightarrow l_R(A)$.*

Theorem 4.6. [41], [42] *If R is also symmetric, that is an L-fuzzy equivalence, then $\mathcal{U}_R(A) = \mathcal{L}_R(A)$ for every L-fuzzy set A.*

In this case we define $\mathcal{R}A(A) = \mathcal{U}_R(A) = \mathcal{L}_R(A)$ and call it *the measure of rough approximation* of an L-fuzzy set A.

Changing $A \in L^X$, we come to the L-valued operators of upper and lower L-valued fuzzy rough approximation $\mathcal{U}_R : L^X \to L$ and $\mathcal{L}_R : L^X \to L$ and the operator of L-valued fuzzy rough approximation $\mathcal{R}A_R : L^X \to L$ if R is symmetric.

In the next two theorems we collect the main properties of the operators $\mathcal{U}_R : L^X \to L$, $\mathcal{L}_R : L^X \to L$ and $\mathcal{R}A_R : L^X \to L$ established in [41], [42]:

Theorem 4.7. $(1_\mathcal{U})$ *$\mathcal{U}_R(a_X) = 1_L$ for every $a \in L$;*
$(2_\mathcal{U})$ *$\mathcal{U}_R(u_R(A)) = 1_L$ for every $A \in L^X$;*
$(3_\mathcal{U})$ *$\mathcal{U}_R(\bigvee_i A_i) \geq \bigwedge_i \mathcal{U}_R(A_i)$ for every family of L-fuzzy sets $\{A_i \mid i \in I\} \subseteq L^X$;*
$(4_\mathcal{U})$ *$\mathcal{U}_R(\bigwedge_i A_i) \geq \bigwedge_i \mathcal{U}_R(A_i)$ for every family of L-fuzzy sets $\{A_i \mid i \in I\} \subseteq L^X$;*
$(5^s_\mathcal{U})$ *$\mathcal{U}_R(a_X * A) \geq \mathcal{U}_R(A)$ for all $A \in L^X$ and all constant L-fuzzy sets a_X;*

Theorem 4.8. $(1_\mathcal{L})$ *$\mathcal{L}_R(a_X) = 1_L$ for every $a \in L$;*
$(2_\mathcal{L})$ *$\mathcal{L}_R(l_R(A)) = 1_L$ for every $A \in L^X$;*
$(3_\mathcal{L})$ *$\mathcal{L}_R(\bigwedge_i A_i) \geq \bigwedge_i \mathcal{L}_R(A_i)$ for every family of L-fuzzy sets $\{A_i \mid i \in I\} \subseteq L^X$;*
$(4_\mathcal{L})$ *$\mathcal{L}_R(\bigvee_i A_i) \geq \bigwedge_i \mathcal{L}_R(A_i)$ for every family of L-fuzzy sets $\{A_i \mid i \in I\} \subseteq L^X$;*
$(5^{cs}_\mathcal{L})$ *$\mathcal{L}_R(a_X \mapsto A) \geq \mathcal{L}_R(A)$ for all $A \in L^X$ and all constant L-fuzzy sets a_X.*

Corollary 4.3. $(1_{\mathcal{R}A})$ *$\mathcal{R}A_R(a_X) = 1_L$ for every $a \in L$;*
$(2_{\mathcal{R}A})$ *$\mathcal{R}A_R(u_R(A)) = \mathcal{R}A_R(l_R(A)) = 1_L$ for every $A \in L^X$;*
$(3_{\mathcal{R}A})$ *$\mathcal{R}A_R(\bigvee_i A_i) \geq \bigwedge_i \mathcal{R}A_R(A_i)$ for every family $\{A_i \mid i \in I\} \subseteq L^X$;*
$(4_{\mathcal{R}A})$ *$\mathcal{R}A_R(\bigwedge_i A_i) \geq \bigwedge_i \mathcal{R}A_R(A_i)$ for every family $\{A_i \mid i \in I\} \subseteq L^X$.*
$(5^s_{\mathcal{R}A})$ *$\mathcal{R}A_R(a_X * A) \geq \mathcal{R}A_R(A)$ for all $A \in L^X$ and all constant L-fuzzy sets a_X;*
$(6^{cs}_{\mathcal{R}A})$ *$\mathcal{R}A_R(a_X \mapsto A) \geq \mathcal{L}_R(A)$ for all $A \in L^X$ and all constant L-fuzzy sets a_X.*

Remark 4.1. Referring to subsection 4.2.1 we know that $u_R(A)$ is the extensional hull of the L-fuzzy set A and $l_R(A)$ is the extensional kernel of A. This observation allows to interpret $\mathcal{U}_R(A)$ as the *measure of upper extensionality* and $\mathcal{L}_R(A)$ as the *measure of lower extensionality* of the L-fuzzy set A. In case the L-fuzzy preoder R is symmetric, $\mathcal{R}A_R(A)$ can be interpreted as the *measure of extensionality* of the L-fuzzy set A.

Remark 4.2. Our operators $\mathcal{U}_R$ and $\mathcal{L}$ are related to the operators $\triangle\ (R)$ and $\nabla(R)$ introduced by Fang [34] in case $L = [0, 1]$. However, as different from our approach originating from L-fuzzy rough approximation spaces, Fang bases his approach on fuzzified sets of the family of upper sets in (X, R) and relates these operators to some specialization orders.

4.5. **The category of L-valued fuzzy rough approximation spaces.**
Let (X, R) be an L-fuzzy preodered set and operators $\mathcal{U}_R : L^X \to L$ and $\mathcal{L}_R : L^X \to L$ be defined as in Definition 4.8.

Definition 4.9. *The quadruple $(X, R, \mathcal{U}_R, \mathcal{L}_R)$ is called an L-valued fuzzy approximation space induced on the L-fuzzy preodered set (X, R).*

Before defining the morphisms in the catgory L-**VFRAS** of L-valued fuzzy rough approximation spaces, we analyse properties of L-valued fuzzy rough approximation spaces from topological point of view. Note that the properties of the upper and lower L-valued fuzzy rough approximation operaors $\mathcal{U}_R$ and $\mathcal{L}_R$, collected in theorems 4.7 and 4.8, correspond respectively to the properties of the (stratified) L-valued fuzzy cotopology (definitions 3.9, 3.11) and to the properties of (costratified) L-valued fuzzy topology (definition 3.7), 3.11) Thus, the triple $(X, R, \mathcal{U}_R, \mathcal{L}_R)$ can be viewed as a special kind of a stratified L-valued fuzzy ditopological space. This observation suggests the following definition for morphisms in the category L-**VFRAS**:

Definition 4.10. *Let $(X, R_X, \mathcal{U}_{R_X}, \mathcal{L}_{R_X})$ and $(Y, R_Y, \mathcal{U}_{R_Y}, \mathcal{L}_{R_Y})$ be L-valued fuzzy rough approximation spaces. A mapping $f : X \to Y$ is called continuous if*

$(1_{\mathcal{U}})$ $\mathcal{U}_{R_X}(f^{-1}(B)) \geq \mathcal{U}_Y(B)\ \forall B \in L^Y$;

$(2_{\mathcal{L}})$ $\mathcal{L}_{R_X}(f^{-1}(B)) \geq \mathcal{L}_Y(B)\ \forall B \in L^Y$.

*Category L-**VFRAS** is defined as the category of L-valued fuzzy rough approximation spaces as objects and their continuous mappings as morphisms.*

The "correctness" of this definition is justified also by the following theorem and its corollary:

Theorem 4.9. [41], [42] *Let (X, R_X) and (Y, R_Y) be L-fuzzy preodered sets and $(X, R_X, \mathcal{U}_{R_X}, \mathcal{L}_{R_X})$ and $(Y, R_Y, \mathcal{U}_{R_Y}, \mathcal{L}_{R_Y})$ be induced L-valued fuzzy rough approximation spaces. If $f : (X, R_X) \to (Y, R_Y)$ be an isotone mapping. Then*

- $\mathcal{U}_{R_X}(f^{-1}(B)) \geq \mathcal{U}_{R_Y}(B)$ *for every* $B \in L^Y$;
- $\mathcal{L}_{R_X}(f^{-1}(B)) \geq \mathcal{L}_{R_Y}(B)$ *for every* $B \in L^Y$.

Corollary 4.4. [41], [42] *By assigning to an L-fuzzy preodered set (X, R_X) an L-valued fuzzy rough approximation space $(X, R_X, \mathcal{U}_{R_X}, \mathcal{L}_{R_X})$ and interpreting an isotone mapping $f : (X, R_X) \to (Y, R_Y)$ as a mapping $(X, R_X, \mathcal{U}_{R_X}, \mathcal{L}_{R_X}) \to (Y, R_Y, \mathcal{U}_{R_Y}, \mathcal{L}_{R_Y})$, we define a covariant functor $\Psi : L$-**Preod** $\longrightarrow L$-**VFRAS**.*

It is clear from the definitions, that the category L-**VFRAS** of L-valued fuzzy rough approximation spaces is a full subcategory of the category L-**VFDiTop**^{s} of stratified L-valued fuzzy ditopological spaces. Hence we get the following

Corollary 4.5. *By assigning to a L-fuzzy preodered set (X, R) an L-valued fuzzy ditopological space $(X, R, \mathcal{L}_R, \mathcal{U}_R)$ and interpreting isotone mappings $f : (X, R_X) \to (Y, R_Y)$ as mappings of the corresponding L-valued fuzzy ditopological spaces $f : (X, R_X, \mathcal{L}_{R_X}, \mathcal{U}_{R_X}) \to (Y, R_Y, \mathcal{L}_{R_Y}, \mathcal{U}_{R_Y})$, we get an embedding functor from the category L-**Preod** into the category L-**VFDiTop**^{s} $\Psi' : L$-**Preod** $\longrightarrow L$-**VFDiTop**$^{\mathbf{s}}$.*

4.6. **Examples of M-valued measures for rough approximation of L-fuzzy sets.**
In this subsection we show how operators $\mathcal{U}$ and $\mathcal{L}$ in case when $L = [0, 1]$ is endowed with the three basic t-norms: Łukasiewicz, Product and Minimum, see, e.g. [62], [**?**].

4.6.1. *The case of Łukasiewicz t-norm.* Let $*_L$ be the Łukasiewicz t-norm on the unit interval $L = [0, 1]$, that is $\alpha *_L \beta = \min(\alpha + \beta - 1, 1)$. Then, given an L-fuzzy relation R on a set X and $A \in L^X$, we have: $\mathcal{U}_R(A) = \bigwedge_x \bigwedge_{x'} (2 - A(x) + A(x') - R(x, x'))$ and $\mathcal{L}_R(A) = \bigwedge_x \bigwedge_{x'} (2 - A(x) + A(x') - R(x', x))$. In particular, if $R : X \times X \to [0, 1]$ is discrete relation, we have $\mathcal{R}A_R(A) = 1$ for every $A \in L^X$.

4.6.2. *The case of the minimum t-norm.* Let $* = \wedge$ be the minimum t-norm on the unit interval $L = [0, 1]$. Then given an L-fuzzy preoder R on a set X and $A \in L^X$ we have:
$\mathcal{U}_R(A) = \inf_{x,x'}(A(x') \wedge R(x, x') \mapsto A(x))$ and $\mathcal{L}_R(A) = \inf_{x,x'}(A(x') \wedge R(x', x) \mapsto A(x))$.
In particular, $\mathcal{R}A_R(A) = 1$ for every $A \in L^X$ in case the relation R is symmetric.

4.6.3. *The case of the product t-norm.* Let $* = \cdot$ be the product t-norm on the unit interval [0,1]. Then, for an L-fuzzy preoder relation we have: $\mathcal{U}(A) = \inf_{x,x'}(A(x') \cdot R(x, x') \mapsto A(x))$ and $\mathcal{L}(A) = \inf_{x,x'}(A(x') \cdot R(x', x) \mapsto A(x))$. In particular in case R is symmetric: $\mathcal{R}A(A) = \inf_{x,x'}(A(x') \cdot R(x, x') \mapsto A(x))$.

5. Fuzzy Mathematical morphology

5.1. Mathematical morphology: historical comments and basic concepts.

Mathematical morphology has its origins in geological problems centered in the processes of erosion and dilation. The founders of mathematical morphology are engineers G. Matheron [75] and J. Serra [100]. The idea of the classical mathematical morphology can be explained as the process of modifying a subset A (image) of the n-dimensional Euclidean space $\mathbb{R}^n$ by cutting out pieces of the apriory chosen S from A (in case of erosion $\mathcal{E}$) or gluing them down to A (in case of dilation $\mathcal{D}$). The set S, having "good" shape and intuitively, small if compared with A, is called the structuring element. The process of cutting and gluing is based on the transfer of S by means of the linear structure of the underlying space $\mathbb{R}^n$. The successive application of dilation and erosion leads to the operation $\mathcal{O}$ called opening and the successive application of erosion and dilation leads to the operation of closing $\mathcal{C}$. Opening and closing give two alternative kinds of "smoothing" the given image A. In case of the original problem studied by J. Serra, these operations allowed to smooth out the influence of migrating calcite, which "spoiled" the global picture of iron deposits. Now operations of mathematical morphology are widely and effectively used in image processing and image analysis for improving the image or highlighting its special areas. This is very important in the research of different theoretical and applied areas of science, for example, in medical diagnostics, biology, geology, etc., [100],[104], [82], [11] et al.

5.2. **Fuzzy mathematical morphology in Euclidean spaces.**
The above described operators of mathematical morphology are quite satisfactory when dealing with processing of crisp binary images. However they are not applicable in case of a greyscale or colour images. The desire to solve this problem aroused interest in creating a fuzzy version of mathematical morphology. As different from the classic approach, now the image A can be represented by a fuzzy subset of the space $\mathbb{R}^n$ and the structuring element S can remain crisp or assumed to be fuzzy as well. The linear structure of the underlying space $\mathbb{R}^n$ still remains as the base for this approach. Among the first works on fuzzy mathematical morphology were papers by B. De Baets and co-authors [25], [26]; operators of fuzzy morphology in these papers were based on Minkowski fuzzy addition in Euclidean spaces. Later many papers on fuzzy mathematical morphology in Euclidean spaces were published. The authors of these works proposed different conjunctions and implicators to be used in the definitions of operators of fuzzy mathematical morphology. In the detailed survey by M. Nachtegael and E.E. Kerre [81] different approaches to fuzzy mathematical morphology in $\mathbb{R}^n$ are described and compared.

5.3. **Abstract fuzzy mathematical morphology.**
Since the last decade of the previous century the interest of some researchers was drawn to the problem of a more general view on fuzzy mathematical morphology. Namely, they were interested to extend the subject of fuzzy morphology to the context that would avoiding the use of the linear structure of the space $\mathbb{R}^n$ but at the same time will preserve "the essence" of classical mathematical morphology. And as one of the main principles for the abstract extension of fuzzy mathematical morphology was taken the property of adjunctness of the erosion-dilation pair $(\mathcal{E}, \mathcal{D})$, which is fundamental in "classical" mathematical morphology. This idea was first discussed in [44] and further developed in a series of papers by I. Bloch, Ronse and Heijsman [9], [10] [8] et al. In the general form it looks as folows.

Given two partially ordered sets $(\mathbb{L}_1, \leq_1)$ and $(\mathbb{L}_2, \leq_1)$, a pair of mappings $\mathcal{E} : \mathbb{L}_1 \to \mathbb{L}_2$ and $\mathcal{D} : \mathbb{L}_2 \to \mathbb{L}_1$ is said to form an adjunction $(\mathcal{E}, \mathcal{D})$ if $\beta \leq_2 \mathcal{E}(\alpha) \Longleftrightarrow \mathcal{D}(\beta) \leq_1 \alpha$ for all $\alpha \in \mathbb{L}_1, \beta \in \mathbb{L}_2$. In this case $\mathcal{E} : \mathbb{L}_1 \to \mathbb{L}_2$ is treated as an abstract erosion and $\mathcal{D} : \mathbb{L}_2 \to \mathbb{L}_1$ is treated as an abstract dilation. Respectively, openning and closing are defined by $\mathcal{O} = \mathcal{D} \circ \mathcal{E} : \mathbb{L}_1 \to \mathbb{L}_1$ and $\mathcal{C} = \mathcal{E} \circ \mathcal{D} : \mathbb{L}_2 \to \mathbb{L}_2$.

5.4. **Relational fuzzy mathematical morphology.**
In [74] the general abstract approach to fuzzy mathematical morphology was applied in the special case when $\mathbb{L}_1 = L^X$ and $\mathbb{L}_2 = L^Y$ where L is a complete lattice, X, Y are sets and $\mathcal{E}$ and $\mathcal{D}$ are induced by an L-fuzzy relation $R : X \times Y \to L$. In the sequel of this paper we stick to this approach to the subject of fuzzy mathematical morphology.

5.4.1. *Image and preimages induced by L-fuzzy relations.*

We start with recalling well-known operators of image and preimage induced by L-fuzzy relations, see e.g.[96]:

Definition 5.1. *The upper image of an L-fuzzy set $A \in L^X$ under an L-fuzzy relation $R : X \times Y \to L$ is an L-fuzzy set $R^{\to}(A) \in L^Y$ defined by $R^{\to}(A)(y) = \bigvee_x R(x,y) * A(x)$.*
The upper preimage of L-fuzzy set $B \in L^Y$ under L-fuzzy relation $R : X \times Y \to L$ is an L-fuzzy set $R^{\leftarrow}(B) \in L^X$ defined by $R^{\leftarrow}(B)(x) = \bigvee_y R(x,y) * B(y)$.

Remark 5.1. If the fuzzy relation R represents an ordinary function $f : X \to Y$, then the above definitions reduce respectively, to the definitions of a forward and backward L-powerset operators $f^{\to}$ and $f^{\leftarrow}$, as they are defined by S.E. Rodabaugh [97]. Specifically $f^{\to}(A)$ is the image $f(A)$ of $A \in L^X$ and $f^{\leftarrow}(B)$ is the preimage $f^{-1}(B)$ of $B \in L^Y$. The properties of these operators from category theory point of view were studied by S.E. Rodabaugh [97].

Changing L-fuzzy sets $A \in L^X$ and L-fuzzy sets $B \in L^Y$, we get the image and preimage operators $R^{\to} : L^X \to L^Y$ and $R^{\leftarrow} : L^Y \to L^X$.

The image and preimage operators considered above were obtained by applying the idea of Zadeh extension principle [123] and rely on the interpretation of operation $*$ as a logical conjunction. On the other hand, further we consider alternative definitions of image and preimage operators relying on the interpretation of IF-THEN rule as a logical implication.

Definition 5.2. *The lower image of an L-fuzzy set $A \in L^X$ is an L-fuzzy set $R^{\Rightarrow}(A) \in L^Y$ defined by $R^{\Rightarrow}(A)(y) = \bigwedge_{x\in X} R(x,y) \mapsto A(x)$.*
The lower preimage of an L-fuzzy set $B \in L^Y$ is the L-fuzzy set $R^{\Leftarrow}(B) \in L^X$ defined by $R^{\Leftarrow}(B)(x) = \bigwedge_{y\in Y} R(x,y) \mapsto B(y)$.

Remark 5.2. In case when L-fuzzy relation R is realized by an ordinary function $f : X \to Y$, then $R^{\Rightarrow}(A)$ consists of all $y \in Y$ such that $y = f(x)$ for some $x \in A$ and $y \neq f(x)$ if $x \notin A$. In particular, this property holds if f is injective. In case when L-fuzzy relation R is realized by an ordinary function f, then $R^{\leftarrow}(B) = R^{\Leftarrow}(B) = f^{-1}(B)$, that is the preimage of B under function f. Strong right connectedness in this case just means that the function f is surjective.

Proposition 5.1. (see, e.g. [32]) *If L-fuzzy relation is strongly left connected, then $R^{\Rightarrow}(A) \leq R^{\to}(A)$ for every $A \in L^X$. In case $R : X \times Y \to L$ is strongly right connected, then $R^{\leftarrow} \geq R^{\Leftarrow}$.*

5.4.2. *Fuzzy relational erosion and fuzzy relational dilation.*

Definition 5.3. [74] *Given $A \in L^X$, its erosion $\mathcal{E}_R(A) \in L^Y$ is defined by*

$$\mathcal{E}_R(A)(y) = \bigwedge_{x\in X} (R(x,y) \mapsto A(x)).$$

Considering erosion for all $A \in L^X$, we get the operator of erosion $\mathcal{E}_R : L^X \to L^Y$.

Proposition 5.2.

($1_\mathcal{E}$) *$\mathcal{E}_R(1_X) = 1_Y$ and if R is left connected (2.4), then $\mathcal{E}_R(a_X) = a_Y$ for every $a_X \in L^X$, and in particular $\mathcal{E}_R(0_X) = 0_Y$*

($2_\mathcal{E}$) *For a family of L-fuzzy sets $\{A_i \mid i \in I\} \subseteq L^X$, we have $\mathcal{E}_R\left(\bigwedge_{i\in I} A_i\right) = \bigwedge_{i\in I} \mathcal{E}_R(A_i)$.*

($3_\mathcal{E}$) *If fuzzy relation R is strongly left connected, then $\mathcal{E}_R(A) \leq R^{\to}(A)$ for every $A \in L^X$.*

Corollary 5.1. *Operator $\mathcal{E} : L^X \to L^Y$ is isotone: $A_1 \leq A_2 \in L^X \Longrightarrow \mathcal{E}_R(A_1) \leq \mathcal{E}_R(A_2)$.*

Definition 5.4. [74] *Given $B \in L^Y$, its dilation $\mathcal{D}_R(B) \in L^X$ is defined by*

$$\mathcal{D}_R(B)(x) = \bigvee_{y\in Y} R(x,y) * B(y).$$

Considering dilation for all $B \in L^Y$, we get the operator of dilation $\mathcal{D}_R : L^Y \to L^X$.

Proposition 5.3. (1) *$\mathcal{D}_R(0_Y) = 0_X$ and if R is right connected, then $\mathcal{D}_R(b_Y) = b_X$ for every $b_Y \in L^Y$ and in particular $\mathcal{D}_R(1_Y) = 1_X$*

(2) *Given a family of L-fuzzy sets $\{B_i \mid i \in I\} \subseteq L^Y$, we have* $\mathcal{D}_R\left(\bigvee_{i\in I} B_i\right) = \bigvee_{i\in I}\mathcal{D}_R(B_i)$.
(3) $\mathcal{D}_R(B) = R^{\leftarrow}(B)$

Corollary 5.2. *Operator* $\mathcal{D} : L^X \to L^Y$ *is isotone that is* $B_1 \le B_2 \in L^Y \Longrightarrow \mathcal{D}_R(B_1) \le \mathcal{D}_R(B_2)$.

5.4.3. *Adjunction* $(\mathcal{E}_R, \mathcal{D}_R)$.
Let id_X and id_Y denote the identity mappings $\mathrm{id}_X : X \to X$ and $\mathrm{id}_Y : Y \to Y$.

Theorem 5.1. (cf [74]) $\mathcal{D}_R \circ \mathcal{E}_R \le \mathrm{id}_X$ *and* $\mathcal{E}_R \circ \mathcal{D}_R \ge \mathrm{id}_Y$.

Proof. Given $A \in L^X$, and $x \in X$ we have $\mathcal{D}_R(\mathcal{E}_R(A))(x) = \bigvee_{y\in Y} R(x,y) * \mathcal{E}_R(A)(y) = \bigvee_{y\in Y} R(x,y) * (R(x,y)) \mapsto A(x)) \le A(x)$.
Further, given $B \in L^Y$ and $y \in Y$, we have $\mathcal{E}_R(\mathcal{D}_R(B))(y) = \bigwedge_{x\in X}(R(x,y) \mapsto \mathcal{D}_R(B)(x)) = \bigwedge_{x\in X} (R(x,y) \mapsto (R(x,y) * B(y))) \ge B(y)$. □
Referring to a well-known fact, see e.g. [38, Theorem 0.3.6], we get

Corollary 5.3. *The pair* $(\mathcal{E}_R, \mathcal{D}_R)$ *is an adjunction.*

5.5. **Categories $(\mathcal{E}, \mathcal{D}^{-1})$ and $(\mathcal{E}^{-1}, \mathcal{D})$ of L-fuzzy pre-ditopological spaces.**
Let $R : X \times Y \to L$ be an L-fuzzy relation interpreted as *a relation from X to Y* and let relation $R^{-1} : Y \times X \to L$ be defined by $R^{-1}(y,x) = R(x,y)$ and interpreted as *a relation from Y to X*.

The properties of erosion operator $\mathcal{E}_R$ collected in Proposition 5.2 remind the properties of a (stratified Alexandroff) L-fuzzy preinterior operator defined on the L^X but taking values in L^Y. On the other hand, the properties of the operator of dilation $\mathcal{D}_R$ collected in Proposition 5.3 and reformulated for $\mathcal{D}_{R^{-1}}$ remind the properties of a (stratified Alexandroff) L-fuzzy preclosure operator defined on L^X and taking values in L^Y. It is clear that $\mathcal{E}_R(A) = R^{\Rightarrow}(A)$ and $\mathcal{D}_{R^{-1}}(A) = R^{\rightarrow}(A)$ for every $A \in L^X$. Now, referring to Proposition 5.1 we know that if R is strongly right connected, then $R^{\Rightarrow}(A) \le R^{\rightarrow}(A)$. These observations lead to the interpretation of $\mathcal{E}_R(A)$ as the L-fuzzy preinterior of $A \in L^X$ transferred to L^Y and $\mathcal{D}_{R^{-1}}(A)$ as the L-fuzzy preclosure of A transferred to L^Y. In the result we interpret the pair $(\mathcal{E}_R, \mathcal{D}_{R^{-1}})$ as a (stratified Alexandroff) L-fuzzy pre-ditopology transformed from L^X to L^Y.

Definition 5.5. *An $(\mathcal{E}, \mathcal{D}^{-1})$-fuzzy pre-ditopological space or an $(\mathcal{E}, \mathcal{D}^{-1})$-space for short a is a tuple $(X, Y, L, R, \mathcal{E}_R, \mathcal{D}_{R^{-1}})$ where X, Y are sets, L is a fixed quantale, $R : X \times Y \to L$ is an L-fuzzy relation and $\mathcal{E}_R, \mathcal{D}_{R^{-1}} : L^X \to L^Y$ are erosion and dilation operators.*

To view $(\mathcal{E}, \mathcal{D}^{-1})$-spaces as a category we must specify its morphisms. We do it in the next definition justified by the topological interpretation of the pairs $(\mathcal{E}_R, \mathcal{D}_{R^{-1}})$ given above:

Definition 5.6. *A continuous transformation from an $(\mathcal{E}, \mathcal{D}^{-1})$-space $(X_1, Y_1, L, R_1, \mathcal{E}_{R_1}, \mathcal{D}_{R_1^{-1}})$ to an $(\mathcal{E}, \mathcal{D}^{-1})$-space $(X_2, Y_2, L, R_2, \mathcal{E}_{R_2}, \mathcal{D}_{R_2^{-1}})$, is a pair of mappings (φ, ψ) where $\varphi : X_1 \to X_2$, $\psi : Y_1 \to Y_2$ satsfying the following conditions:*

(1_{fmcon}) $\mathcal{E}_{R_2}(\varphi(A)) \le \psi\left(\mathcal{E}_{R_1}(A)\right)$ $\forall A \in L^{X_1}$;
(2_{fmcon}) $\mathcal{D}_{R_2^{-1}}(\varphi(A)) \le \psi\left(\mathcal{D}_{R_1^{-1}}(A)\right)$ $\forall A \in L^{X_1}$.

Given three $(\mathcal{E}, \mathcal{D}^{-1})$-spaces and continuous transformations
$(\varphi, \psi) : (X_1, Y_1, L, R_1, \mathcal{E}_{R_1}, \mathcal{D}_{R_1^{-1}}) \to (X_2, Y_2, L, R_2, \mathcal{E}_{R_2}, \mathcal{D}_{R_2^{-1}})$,
$(\varphi', \psi') : (X_2, Y_2, L, R_2, \mathcal{E}_{R_2}, \mathcal{D}_{R_2^{-1}}) \to (X_3, Y_3, L, R_3, \mathcal{E}_{R_3}, \mathcal{D}_{R_3^{-1}})$ we define their composition as $(\varphi' \circ \varphi, \psi' \circ \psi) : (X_1, Y_1, L, R_1, \mathcal{E}_{R_1}, \mathcal{D}_{R_1^{-1}}) \to (X_3, Y_3, L, R_3, \mathcal{E}_{R_3}, \mathcal{D}_{R_3^{-1}})$.

Proposition 5.4. *$(\mathcal{E}, \mathcal{D}^{-1})$-spaces and their continuous transformations with composition form the category $(\mathcal{E}, \mathcal{D}^{-1})$-**FPDiTop**.*

In a similar way, defining erosion on the base of relation R^{-1} and dilation on the base of relation R we come to the following

Definition 5.7. *An $(\mathcal{E}^{-1},\mathcal{D})$-fuzzy pre-ditopological space or an $(\mathcal{E}^{-1},\mathcal{D})$-space for short, is a tuple $(X, Y, L, R, \mathcal{E}_{R^{-1}}, \mathcal{D}_R))$ where X, Y are sets, L is a fixed quantale, $R : X \times Y \to L$ is an L-fuzzy relation and $\mathcal{E}_{R^{-1}}, \mathcal{D}_R : L^Y \to L^X$ are erosion and dilation operators.*

Thus we come to the category $(\mathcal{E}^{-1}\mathcal{D})$-**FPDiTop** whose objects are $(\mathcal{E}^{-1},\mathcal{D})$-spaces and whose morphisms are continuous transformations defined patterned after the definition of continuous transformations of $(\mathcal{E}\mathcal{D}^{-1})$-spaces.

5.6. **Fuzzy relational morphology: Closing and openning.**
Let $R : X \times Y \to L$ be an L-fuzzy relation from X to Y and let $\mathcal{E}_R : L^X \to L^Y, \mathcal{D}_R : L^Y \to L^X$ be the corresponding erosion and dilation. Basing on erosion and dilation the derived operators, opening and closing, can be defined as follows:

Definition 5.8. (cf e.g. [9], [10]) *An opening is an operator $\mathcal{O}_R : L^X \to L^X$ defined by $\mathcal{O}_R(A) = \mathcal{D}_R(\mathcal{E}_R(A))$ for every $A \in L^X$. A closing is an operator $\mathcal{C}_R : L^Y \to L^Y$ defined by $\mathcal{C}_R(B) = \mathcal{E}_R(\mathcal{D}_R(B))$ for every $B \in L^Y$.*

The fundamental for us is the following

Theorem 5.2. *Operator $\mathcal{O}_R : L^X \to L^X$ is an interior operator on the set L^X. Operator $\mathcal{C}_R : L^Y \to L^Y$ is a closure operator on the set L^Y, see Definition 3.13.*

Proof. We prove the statement for the closing operator $\mathcal{C}_R$. The case of operator $\mathcal{O}_R$ can be proved in a similar way.

From Theorem 5.1 we know that $\mathcal{C}_R = \mathcal{E}_R \circ \mathcal{D}_R \geq id_{L^Y}$ where $id_{L^Y} : L^Y \to L^Y$ is the identity mapping and hence operators $\mathcal{C}_R : L^Y \to L^Y$ is extensional. Now to complete the proof we have to show the idempotence of $\mathcal{C}_R$. We do it as follows.

From the inequality $\mathcal{E}_R \circ \mathcal{D}_R \geq id_{L^Y}$ and isotonisity of operators $\mathcal{E}_R$ and $\mathcal{D}_R$ we have $\mathcal{E}_R \circ \mathcal{D}_R \circ \mathcal{E}_R \geq \mathcal{E}_R$ and further $\mathcal{E}_R \circ \mathcal{D}_R \circ \mathcal{E}_R \circ \mathcal{D}_R \geq \mathcal{E}_R \circ \mathcal{D}_R$, that is $\mathcal{C}_R \circ \mathcal{C}_R \geq \mathcal{C}_R$.

On the other hand, from the same theorem 5.1 we have $\mathcal{D}_R \circ \mathcal{E}_R \leq id_{L^X}$. Again, by the isotonisity of operators $\mathcal{E}_R$ and $\mathcal{D}_R$, we have $\mathcal{E}_R \circ \mathcal{D}_R \circ \mathcal{E}_R \leq \mathcal{E}_R$ and further $\mathcal{E}_R \circ \mathcal{D}_R \circ \mathcal{E}_R \circ \mathcal{D}_R \leq \mathcal{E}_R \circ \mathcal{D}_R$, that is $\mathcal{C}_R \circ \mathcal{C}_R \leq \mathcal{C}_R$. □

Let now $\sigma_{\mathcal{C}_R} = \{B \in L^Y \mid \mathcal{C}_R(B) = B\}$. Then referring to Subsection 3.5, we know that the family $\sigma_{\mathcal{C}_R}$ is closed under taking arbitrary meets. Hence in case $0_X \in \mathcal{C}(0_X)$ the family $\sigma_{\mathcal{C}_R}$ is a supratopology on L^Y. And the last condition holds if $\mathcal{C}_R(0_X) = 0_X$, that, in its turn, is guaranteed in case R is left connected, see propositions 5.2, 5.3. Thus we get

Corollary 5.4. *If $R : X \times Y \to L$ is a left connected L-fuzzy relation, then σ_R is an L-fuzzy supra cotopology on Y.*

In a similar way knowing that operator $\mathcal{O}_R : L^X \to L^X$ is an interior operator on the set L^X we get the following

Corollary 5.5. *If $R : X \times Y \to L$ is a right connected L-fuzzy relation, then $\tau_{\mathcal{O}_R} = \{A \in L^X \mid \mathcal{O}_R(A) = A\}$ is an L-fuzzy supra topology on X.*

Assume now that $R : X \times X \to L$ is an L-fuzzy relation on a set X. Then from the above we have:

Theorem 5.3. *If R is reflexive, then $(\tau_{\mathcal{O}_R}, \sigma_{\mathcal{C}_R})$ is an L-fuzzy supra ditopology on X and the triple $(X, \tau_{\mathcal{O}_R}, \sigma_{\mathcal{C}_R})$ is an L-fuzzy supra ditopological space. We call it $(\mathcal{O}_R, \mathcal{C}_R)$-ditopological space.*

Let now (X, R_X), (Y, R_Y) be two sets endowed with L-fuzzy relations and let $(X, \tau_{\mathcal{O}_{R_X}}, \sigma_{\mathcal{C}_{R_X}})$ and $(Y, \tau_{\mathcal{O}_{R_Y}}, \sigma_{\mathcal{C}_{R_Y}})$ be the corresponding L-fuzzy supra ditopological spaces. We call a mapping $f : (X, \tau_{\mathcal{O}_{R_X}}, \sigma_{\mathcal{C}_{R_X}}) \to (Y, \tau_{\mathcal{O}_{R_Y}}, \sigma_{\mathcal{C}_{R_Y}})$ continuous if $f^{-1}(V) \in \tau_{\mathcal{O}_{R_X}}$ for every $V \in \tau_{\mathcal{O}_{R_Y}}$ and $f^{-1}(B) \in \sigma_{\mathcal{C}_{R_X}}$ for every $V \in \sigma_{\mathcal{C}_{R_Y}}$. Obviously, composition of continuous mappings is continuous and the identity mapping is continuous. Thus we come to the category $(\mathcal{O}\mathcal{C})$**FSDiTop** having L-fuzzy supra ditopological spaces $(X, \mathcal{O}_R, \mathcal{C}_R)$ as objects and their continuous mappings as morphisms. From the definition and the above observation we conclude that $(\mathcal{OC})$**FSDiTop** is a full subcategory of the category L-**FSDiTop** of L-fuzzy supra ditopological spaces.

6. Fuzzy concept lattices and fuzzy topology

6.1. **Introduction.** Formal concept analysis or just concept analysis for short was developed mainly in eighties of the previous century by R. Wille and B. Ganter [115]. [37]. The concept analysis starts with the notion of a (formal) context that is a triple (X, Y, R) where X and Y are sets and $R \subseteq X \times Y$ is a relation. The elements of X are interpreted as some abstract objects, the elements of Y are interpreted as some abstract properties or attributes, and the entry $(x, y) \in R$ means that object x has attribute y. The aim of the concept analysis is to reveal all pairs (A, B) of sets $A \subseteq X$ and $B \subseteq Y$ (called concepts) such that every object $x \in A$ has all properties $y \in B$ and every property $y \in B$ holds for all objects $x \in A$. The set of all such pairs in the given context (X, Y, R) endowed with a certain partial order makes a lattice, called a concept lattice - the principal object of research in concept analysis.

In the first decade of the XXI century different fuzzy counterparts of the formal concept were introduced. In the fuzzy case a context is a tuple (X, Y, L, R) where X and Y are non-empty sets, L is a lattice, and $R : X \times Y \to L$ is an L-fuzzy relation. Fuzzy concepts in this fuzzy context are pairs (A, B), where A and B are L-fuzzy subsets of the sets X and Y respectively, which are interrelated in a certain way (see Definition 6.3). The most important work in this field was carried out by R. Belohlavek [3], [4], [5], [6], see also [15], [8], [67], [14], [85], etc.

Concept analysis and concept lattices, crisp as well as fuzzy, aroused interest among theorists in mathematics and among practicing researchers. The theoretical interest in concept lattices can be explained, in particular, by the fact that they form interesting non-trivial internal connections with other mathematical structures. For example, every complete lattice can be obtained as a concept lattice for some formal context [37]. There are interesting relations between fuzzy concept analysis, fuzzy rough sets and fuzzy morphology, see e.g.[67],[8].

Since its inception, crisp concept analysis has found important applications in the study of "real-world" problems. Starting with illustrative examples of application of crisp lattices given in [115], there appeared many serious works in which concept lattices were used in research of medical-related problems [52], [59], [70], problems related to biology [91], [43], social type problems [78], and in other applied sciences; see also [37]. On the other hand, we found only a few works, where fuzzy concept analysis is used in the research of any practical problems. In our opinion, the problem to use fuzzy concept lattices in practice is that the request of the *precise correspondence* between the fuzzy set A of objects and the fuzzy set B of attributes in "real-world" situations is (almost?) inpracticable. In this case one sooner has to deal with the weaker request *that the correspondence between A and B must hold up to a certain degree.* In order to provide a theoretical basis for the research in this situation, in our paper [111] we first replaced the notion of a fuzzy concept by a much weaker notion of a fuzzy preconcept, and then proposed technique, allowing to evaluate "how far a fuzzy preconcept is from the nearest fuzzy concept" thus coming to the notion of a *graded preconcept lattice.* In this section, following mainly [111], we expose basics of the theory of graded fuzzy peconcept lattices with special attention to the manifistation of topological ideas in this theory.

6.2. **Preconcepts and preconcept lattices.**
Let L be a quantale, let X, Y be sets and $R : X \times Y \to L$ be an L-fuzzy relation. Following terminology accepted in (fuzzy) concept analysis, see e.g. [115], [3], [5],[6], we refer to the tuple (X, Y, L, R) as a fuzzy context.

Definition 6.1. *Given a fuzzy context (X, Y, L, R), a pair $P = (A, B) \in L^X \times L^Y$ is called a fuzzy preconcept*[4].

[4]The notion of a fuzzy preconcept is not related with the notion of a preconcept as it is defined in [37, p. 59]

On the set $L^X \times L^Y$ of all fuzzy preconcepts we introduce a partial order $\preceq$ as follows. Given $P_1 = (A_1, B_1)$ and $P_2 = (A_2, B_2)$, we set $P_1 \preceq P_2$ if and only if $A_1 \leq A_2$ and $B_1 \geq B_2$. Let $(\mathbb{P}, \preceq)$ be the set $L^X \times L^Y$ endowed with this partial order. Further, given a family of fuzzy concepts $\{P_i = (A_i, B_i) : i \in I\} \subseteq L^X \times L^Y$, we define its join (supremum) by $\veebar_{i\in I} P_i = (\bigvee_{i\in I} A_i, \bigwedge_{i\in I} B_i)$ and its meet (infimum) as $\barwedge_{i\in I} P_i = (\bigwedge_{i\in I} A_i, \bigvee_{i\in I} B_i)$.

Theorem 6.1. ([111]) $(\mathbb{P}, \preceq, \barwedge, \veebar)$*is a complete lattice. Besides, if* L *is a bi-infinitely distributive lattice, then* $(\mathbb{P}, \preceq, \barwedge, \veebar)$ *is also a bi-infinitely distributive lattice.*

6.3. **Operators $R^\uparrow$ and $R^\downarrow$ on L-powersets.**
Let X and Y be sets and let $R : X \times Y \to L$ be a fuzzy relation, where L is a fixed *quantale*. Given a fuzzy context (X, Y, L, R), we define operators $R^\uparrow : L^X \to L^Y$ and $R^\downarrow : L^Y \to L^X$ as follows:

Definition 6.2. (see e.g. [5], [4]) *Given* $A \in L^X$*, we define* $A^\uparrow \in L^Y$ *by setting*

$$A^\uparrow(y) = \bigwedge\nolimits_{x\in X}(A(x) \mapsto R(x, y)) \ \forall y \in Y.$$

Given $B \in L^Y$ *we define* $B^\downarrow \in L^X$ *by setting*

$$B^\downarrow(x) = \bigwedge\nolimits_{y\in Y}(B(y) \mapsto R(x, y)) \ \forall x \in X.$$

In the crisp case, that is when $A \subseteq X, B \subseteq Y$ and $R : X \times Y \to \{0, 1\}$, this definition is obviously equivalent to the original definition of operators $A \longrightarrow A'$ and $B \longrightarrow B'$ in [115]. By changing A over L^X and B over L^Y, we get operators $R^\uparrow : L^X \to L^Y$ and $R^\downarrow : L^Y \to L^X$ respectively. From the properties of the residuum one can easily justify the following

Proposition 6.1. *Operators* $R^\uparrow : L^X \to L^Y$ *and* $R^\downarrow : L^Y \to L^X$ *are non-increasing:*

$$A_1 \leq A_2 \Rightarrow A_1^\uparrow \geq A_2^\uparrow; \quad B_1 \leq B_2 \Rightarrow B_1^\downarrow \geq B_2^\downarrow.$$

In the sequel we write $A^{\uparrow\downarrow}$ instead of $(A^\uparrow)^\downarrow$ and $B^{\downarrow\uparrow}$ instead of $(B^\downarrow)^\uparrow$. We write also $R^{\uparrow\downarrow}$ for the composition $R^\downarrow \circ R^\uparrow : L^X \to L^X$ and $R^{\downarrow\uparrow}$ for the composition $R^\uparrow \circ R^\downarrow : L^Y \to L^Y$.

Proposition 6.2. (cf. [115], [6]) $A^{\uparrow\downarrow} \geq A$ *for every* $A \in L^X$ *and* $B^{\downarrow\uparrow} \geq B$ *for every* $B \in L^Y$.

Proposition 6.3. (cf. [115], [6]) $A^\uparrow = A^{\uparrow\downarrow\uparrow}$ *for every* $A \in L^X$ *and* $B^\downarrow = B^{\downarrow\uparrow\downarrow}$ *for every* $B \in L^Y$.

Example 6.1. Let a fuzzy context (X, Y, L, R) be given and let $A \subseteq X$. Then for every $y \in Y$ $A^\uparrow(y) = \bigwedge_{x\in X} A(x) \mapsto R(x, y) = \bigwedge_{x\in A} R(x, y)$. In the same way we prove that if $B \subseteq Y$, then $B^\downarrow(x) = \bigwedge_{y\in B} R(x, y)$. Hence, even in case when $A \subseteq X$, $B \subseteq Y$ the pair (A, B) can be a concept (either crisp or fuzzy!) only in case when R is also crisp, that is $R : X \times Y \to \{0, 1\}$. This shows already the limitation for the use of concept lattices in the case of a fuzzy context and gives an additional evidence in favour of the graded approach to fuzzy preconcept lattices.

Continuing the previous example we calculate $A^{\uparrow\downarrow}$ and $B^{\downarrow\uparrow}$ in case of crisp sets A and B:

$$A^{\uparrow\downarrow}(x) = \bigwedge\nolimits_{y\in Y}\left(\bigwedge\nolimits_{x'\in A} R(x', y) \mapsto R(x, y)\right), B^{\downarrow\uparrow}(y) = \bigwedge\nolimits_{x\in X}\left(\bigwedge\nolimits_{y'\in B} R(x, y') \mapsto R(x, y)\right). \square$$

Proposition 6.4. (cf e.g. [115], [6]) *Given a family* $\{A_i \mid i \in I\} \subseteq L^X$*, we have* $\left(\bigvee_{i\in I} A_i\right)^\uparrow = \bigwedge_{i\in I} A_i^\uparrow$*. Given a family* $\{B_i \mid i \in I\} \subseteq L^Y$*, we have* $\left(\bigvee_{i\in I} B_i\right)^\downarrow = \bigwedge_{i\in I} B_i^\downarrow$.

6.3.1. *Some topology-related comments.*
Given a fuzzy set $A \in L^X$ let $c_X(A) = A^{\uparrow\downarrow}$. Note first that

(1) $A \subseteq c_X(A)$ (by Proposition 6.2), that is operator $c_X : L^X \to L^X$ is extensional,
(2) $A_1 \leq A_2 \Longrightarrow c_X(A_1) \leq c_X(A_2)$ that is operator $c_X : L^X \to L^X$ is isotone, and
(3) $c_X(c_X(A) = (A^{\uparrow\downarrow})^{\uparrow\downarrow} = A^{\uparrow\downarrow} = c_X(A)$, that is operator $c_X : L^X \to L^X$ is idempotent.

Thus $c_X : L^X \to L^X$ is *a closure operator* (Definition 3.13). We call $c_X(A)$ the *closure of the fuzzy set A in the fuzzy context (X,Y,L,R),* and say that A *closed in the fuzzy context (X,Y,L,R),* if $c_X(A) = A$. Let $\mathcal{S}_X$ be the family of all closed fuzzy subsets of L^X in the fuzzy context (X,Y,L,R). According to Subsection 3.5, this family is closed under arbitrary meets.

Thus to conclude that the family $\mathcal{S} = \{A^{\uparrow\downarrow} \mid A \in L^X\}$ is an L-fuzzy supra cotopology, (Definition 3.14), we have to find out whether $0_X^{\uparrow\downarrow} = 0_X$. We calculate $0_X^{\uparrow\downarrow}$ as follows: $0_X^{\uparrow}(y) = \bigwedge_{x\in X}((0_X(x) \mapsto R(x,y)) = 1_Y$ further $0_X^{\uparrow\downarrow}(x) = \bigwedge_{y\in Y} R(x,y)$. So to get the desired $0_X^{\uparrow\downarrow} = 0_X$ we have to request that for every $x \in X$ it holds $\bigwedge_{y\in Y} R(x,y) = 0_L$. This, obviously is not true in general, but it holds in some important cases. In particular, it is fulfilled if for every object $x \in X$ there exists some property $y \in Y$ which is not satisfied by x and such situation seems to be quite natural in all "practical" cases.

In a similar way, we can consider the closure operator $c_Y : L^Y \to L^Y$ in the fuzzy context (X,Y,L,R) defined by $c_Y(B) = B^{\downarrow\uparrow}$ and consider the system $\mathcal{S}_Y \subseteq L^Y$ of all c_Y-closed fuzzy sets that constitutes an (almost) L-fuzzy supra cotopology on Y The difference here from the above case is that to be a "real" supra cotopology we need $0_Y^{\downarrow\uparrow}(y) = \bigwedge_{x\in X} R(x,y) = 0_Y$ and this is satisfied in case when for every property y one can find an object that does not have this property. □

6.4. **Concepts and concept lattices.**
Let, as before, L be a quantale and let (X,Y,L,R) be a fuzzy context. Referring to the definition of a (fuzzy) concept given in [115], [6], we reformulate it as follows:

Definition 6.3. *An L-fuzzy preconcept (A,B) is called an L-fuzzy concept if $A^{\uparrow} = B$ and $B^{\downarrow} = A$.*

Let $\mathbb{C} = \mathbb{C}(X,Y,R,L)$ be the subset of $\mathbb{P} = \mathbb{P}(X,Y,R)$ consisting of all fuzzy concepts (A,B) and let $\preceq$ be the partial order on $\mathbb{C}$ induced by the partial order $\preceq$ from the lattice $(\mathbb{P},\preceq)$. Then $(\mathbb{C},\preceq)$ is a partially ordered subset of the lattice $(\mathbb{P},\preceq)$. Besides the definition of the order $\preceq$ on $\mathbb{C}$ can be simplified. Indeed, notice first , that if (A_1,B_1), (A_2,B_2) are fuzzy concepts, then $A_1 \leq A_2$, if and only if $B_1 \geq B_2$. Now partial order $\preceq$ can be redefined as follows:

$$(A_1,B_1) \preceq (A_2,B_2) \Longleftrightarrow A_1 \leq A_2 \Longleftrightarrow B_1 \geq B_2.$$

However, although $(\mathbb{C},\preceq)$ is a partially ordered subset of the lattice $(\mathbb{P},\preceq,\overline{\wedge},\underline{\vee})$, it is not its sublattice. The problem is that the join ($\underline{\vee}$) of even two concepts need not be a concept, as well as, the meet ($\overline{\wedge}$) of two concepts need not be a concept. The correct definition of meets ($\curlywedge$) and joins ($\curlyvee$) for fuzzy concepts is given in the next theorem:

Theorem 6.2. cf [115] for the crisp case, [6] for the fuzzy case) . *Let $\mathcal{C} = \{C_i = (A_i,B_i)\} \subseteq \mathbb{C}(X,Y,R,L,\preceq)$ be a family of fuzzy concepts. Then*

(1) $\curlywedge_{i\in I} C_i = \left(\bigwedge_{i\in I} A_i, \left(\bigvee_{i\in I} B_i\right)^{\downarrow\uparrow}\right)$ *is its infimum in the partially ordered set* $(\mathbb{C},\preceq)$.

(2) $\curlyvee_{i\in I} C_i = \left(\left(\bigvee_{i\in I} A_i\right)^{\uparrow\downarrow}, \bigwedge_{i\in I} B_i\right)$ *is its supremum in the partially ordered set* $(\mathbb{C},\preceq)$.

Taking into account that in a fuzzy concept (A_i,B_i) it holds $A_i^{\uparrow} = B_i$ and $B_i^{\downarrow} = A_i$ and applying Proposition 6.4, we get the following corollary from the previous Lemma 6.1:

Corollary 6.1. *Let $\mathcal{C} = \{C_i = (A_i,B_i)\} \subseteq \mathbb{C}$ be a family of fuzzy concepts. Then*

(1) $\curlywedge_{i\in I} C_i = \left(\bigwedge_{i\in I} A_i, \left(\bigwedge_{i\in I} A_i\right)^{\uparrow}\right)$ *is its infimum in the lattice* $(\mathbb{C},\preceq)$.

(2) $\curlyvee_{i\in I} C_i = \left(\left(\bigwedge_{i\in I} B_i\right)^{\downarrow}, \bigwedge_{i\in I} B_i\right)$ *is its supremum in the lattice* $(\mathbb{C},\preceq)$.

6.5. **Conceptuality degree of a fuzzy preconcept and $\mathcal{D}$-graded preconcept lattices.**

6.5.1. *Degrees of object and attribute based contentments.*
In this paragraph we introduce the L-fuzzy relations describing the degree of contentment of a fuzzy set A of objects with a fuzzy set B of attributes (object based contaitment of the preconcept (A,B) for short) and the degree of contentment of a fuzzy set B of attributes with the fuzzy set A of objects (or attribute based contentment of the preconcept (A,B) for short). These L-fuzzy relations are used to define the gradation of the lattice of preconcepts $\mathbb{P}$
Let (X,Y,L,R) be a fuzzy context and $(A,B) \in \mathbb{P}(X,Y,L,R)$.

Definition 6.4. *The degree of object based contentment of the preconcept (A,B) is defined by $\mathcal{D}^{\uparrow}(A,B) =_{def} A^{\uparrow} \cong B$ (Definition 2.2). The degree of attribute based contentment of the preconcept (A,B) is defined by $\mathcal{D}^{\downarrow}(A,B) =_{def} A \cong B^{\downarrow}$. The degree of mutual contentment of the preconcept (A,B) is defined by $\mathcal{D}(A,B) = \mathcal{D}^{\uparrow}(A,B) \wedge \mathcal{D}^{\downarrow}(A,B)$*

Changing pairs $(A,B) \in \mathbb{P}$, we obtain mappings $\mathcal{D}^{\uparrow} : \mathbb{P} \to L, \mathcal{D}^{\downarrow} : \mathbb{P} \to L$ and $\mathcal{D} : \mathbb{P} \to L$.

6.5.2. *Examples.*
We illustrate the evaluation of the conceptuality degree in the fuzzy context (X,Y,L,R) in some simple situations.

Example 6.2. Let $A \subseteq X, B \subseteq Y$, let $(L, \leq, \wedge, \vee, *)$ be an arbitrary quantale, $\mapsto : L \to L$ its residuum, and $R : X \times Y \to L$ an arbitrary fuzzy relation. Then by an easy calculation we get $\mathcal{D}^{\uparrow}(A,B) = \mathcal{D}^{\downarrow}(A,B) = \mathcal{D}(A,B) = \bigwedge_{x \in A, y \in B} R(x,y)$,

Example 6.3. Let now $X_a \subseteq X$, $L = [0,1]$, $a \in (0,1)$ and $A : X \to L = [0,1]$ be defined by

$$A(x) = \begin{cases} a & \text{if } x \in X_a \\ 0, & \text{if } x \notin X_a \end{cases}$$

Then $\mathcal{D}^{\uparrow}(A,B) = \bigwedge_{y \in B, x \in X_a} (a \mapsto R(x,y))$; $\mathcal{D}^{\downarrow}(A,B) = \bigwedge_{x \in X_a} \Big((a \leftrightarrow \bigwedge_{y \in B} R(x,y) \Big)$, where $\leftrightarrow$ is the biresiduum in L and $\mathcal{D}(A,B) = \bigwedge_{x \in X_a} \Big((a \leftrightarrow \bigwedge_{y \in B} R(x,y) \Big) \wedge (a \mapsto \bigvee_{y \in Y} R(x,y))$. $\square$

We calculate $\mathcal{D}(A,B)$ in case of the three basic t-norms $*$ on $[0,1]$: $*_{\wedge} = \wedge$ - the minimum t-norm, $*_L$ - the Łukasiewicz t-norm and $*_P$ - the product t-norm, see e.g. [62].

(1) In case of the *Łukasiewicz t-norm*:
$\mathcal{D}^{\uparrow}(A,B) = \Big(\bigwedge_{x \in X_a, y \in Y} (1 - a + R(x,y) \Big) \wedge 1, \quad \mathcal{D}^{\downarrow}(A,B) = \bigwedge_{x \in A, y \in Y} (1 - |a - R(x,y)|)$,
$\mathcal{D}(A,B) = \mathcal{D}^{\downarrow}(A,B) = \bigwedge_{x \in A, y \in Y} (1 - |a - R(x,y)|)$.

(2) In case of *the product t-norm*
$$\mathcal{D}^{\uparrow}(A,B) = \begin{cases} 1 & \text{if } a \leq \bigvee_{y \in B, x \in X_a} R(x,y) \\ \frac{\bigvee_{y \in Y, x \in X_a} R(x,y)}{a} & \text{otherwise} \end{cases}$$
To describe $\mathcal{D}^{\downarrow}$ we denote $A_1 = \{x \in X_a \mid a < \bigwedge_{y \in Y} R(x,y)\}, \quad A_2 = \{x \in X_a \mid a > \bigwedge_{y \in Y} R(x,y)\}$ and have
$$\mathcal{D}^{\downarrow}(A,B) = \begin{cases} \Big(\bigwedge_{x \in A_1} \frac{a}{\bigwedge_{y \in Y} R(x,y)} \Big) \wedge \Big(\bigwedge_{x \in A_2} \frac{\bigwedge_{y \in Y} R(x,y)}{a} \Big) & \text{if } A_1 \cup A_2 \neq \emptyset \\ 1 & \text{otherwise} \end{cases}$$

(3) In case of the minimum t-norm:
$$\mathcal{D}^{\uparrow}(A,B) = \begin{cases} \bigvee_{y \in B, x \in X_a} R(x,y) & \text{if } a > \bigvee_{y \in Y, x \in X_a} R(x,y) \\ 1 & \text{otherwise} \end{cases}$$
$$\mathcal{D}^{\downarrow}(A,B) = \begin{cases} \bigwedge_{x \in A_2, y \in Y} R(x,y) & \text{if } A_2 \neq \emptyset; \\ 1 & \text{otherwise} \end{cases}$$

$\square$

Example 6.4. The case when A and B are 3-valued fuzzy sets.

Let X, Y be sets and $R : X \times Y \to [0,1]$ be a fuzzy relation. Let $0 < a < 1$ and let $X = X_0 \cup X_a \cup X_1$ where sets X_0, X_a, X_1 are disjoint. Further, let $0 < b < 1$ and let $Y =$

$Y_0 \cup Y_b \cup Y_1$ where the sets Y_0, Y_b, Y_1 are disjoint. Define a fuzzy set $A : X \to [0,1]$ and a fuzzy set $B : Y \to [0,1]$ by $A(x) = \begin{cases} 0 & \text{if } x \in X_0 \\ a & \text{if } x \in X_a \\ 1 & \text{if } x \in X_1 \end{cases}$ and $B(y) = \begin{cases} 0 & \text{if } y \in Y_0 \\ b & \text{if } y \in Y_b \\ 1 & \text{if } y \in Y_1 \end{cases}$

Then $B \hookrightarrow A^{\uparrow} = \wedge \left(\bigwedge_{y\in Y_b, x\in X_1} (b \mapsto R(x,y)\right) \wedge \left(\bigwedge_{y\in Y_b, x\in X_a} (b \mapsto (a \mapsto R(x,y))\right)$,

$A^{\uparrow} \hookrightarrow B = \left(\bigwedge_{y\in Y_1, x\in X_1} R(x,y)\right) \wedge \left(\bigwedge_{y\in Y_1, x\in X_a} (a \mapsto R(x,y))\right) \wedge \bigwedge_{y\in Y_b} \left(\bigwedge_{x\in X_1} R(x,y) \mapsto b\right) \wedge \left(\bigwedge_{y\in Y_b} \left(a \mapsto \left(\bigwedge_{x\in X_a} R(x,y) \mapsto b\right)\right)\right)$;

$B^{\downarrow} \hookrightarrow A = \left(\bigwedge_{x\in X_1, y\in Y_1} R(x,y)\right) \wedge \left(\bigwedge_{x\in X_1, y\in Y_b} (b \mapsto R(x,y)\right) \wedge \left(\bigwedge_{x\in X_a} \left(\bigwedge_{y\in Y_1} R(x,y)\right) \mapsto a\right) \wedge \left(\bigwedge_{x\in X_a} \left(b \mapsto \left(\bigwedge_{y\in Y_b} R(x,y) \mapsto a\right)\right)\right)$;

$A \hookrightarrow B^{\downarrow} = \left(\bigwedge_{y\in Y_1, x\in X_1} R(x,y)\right) \wedge \left(\bigwedge_{x\in X_1, y\in Y_b} (b \mapsto R(x,y))\right) \wedge \left(\bigwedge_{x\in X_a, y\in Y_1} (a \mapsto R(x,y)\right) \wedge \left(\bigwedge_{x\in X_a} (a \mapsto \left(b \mapsto \bigwedge_{y\in Y_b} R(x,y)\right)\right)$.

6.5.3. *$\mathcal{D}$-graded preconcept lattices and their supra ditopological structure.*
Let (X, Y, L, R) be an fuzzy context and $\mathcal{D}^{\uparrow} : \mathbb{P}(X, Y, L, R) \to L$, $\mathcal{D}^{\downarrow} : \mathbb{P}(X, Y, L, R) \to L$, $\mathcal{D} : \mathbb{P}(X, Y, L, R) \to L$ be the operators of contentment degrees. Then we can consider three graded lattices of fuzzy preconcepts: object-based graded $(\mathbb{P}, \mathcal{D}^{\uparrow})$, attribute-based graded $(\mathbb{P}, \mathcal{D}^{\downarrow})$ and graded $(\mathbb{P}, \mathcal{D})$.

Theorem 6.3. [111] *Let $\mathbb{P} = (\mathbb{P}, \preceq \veebar, \barwedge)$ be an L-fuzzy preconcept lattice. Given a family of fuzzy preconcepts $\mathcal{P} = \{P_i = (A_i, B_i) \mid i \in I\} \subseteq \mathbb{P}$ it holds $\mathcal{D}^{\uparrow}(\veebar_{i\in I} P_i) \geq \bigwedge_{i\in I} \mathcal{D}^{\uparrow}(P_i)$ and $\mathcal{D}^{\downarrow}(\barwedge_{i\in I} P_i) \geq \bigwedge_{i\in I} \mathcal{D}^{\downarrow}(P_i)$.*

Referring to Subsection 3.4 and Definition 3.15 we reformulate this theorem as follows:

Theorem 6.4. [111] *$\mathcal{D}^{\uparrow} : \mathbb{P} \to L$ is an L-valued point-free fuzzy supra topology on the lattice $(\mathbb{P}, \preceq, \veebar, \barwedge)$, $\mathcal{D}^{\downarrow} : \mathbb{P} \to L$ is an L-valued point-free fuzzy supra cotopology on the lattice $(\mathbb{P}, \preceq, \barwedge, \veebar)$ and $(\mathcal{D}^{\uparrow}, \mathcal{D}^{\downarrow})$ is an L-valued point-free supra ditopology on the lattice $(\mathbb{P}, \preceq, \barwedge, \veebar)$.*

6.6. **Measure of conceptuality of a fuzzy preconcept and $\mathcal{M}$-graded fuzzy preconcept lattices.**
In the previous subsection we estimated the "deviation" of a fuzzy preconcept (A, B) from being a "real" concept by analysing the "mutual" contentment of given fuzzy sets A and B in the fuzzy context (X, Y, R, L). We did not take care of the location of the pair (A, B) in the lattice $\mathbb{P}(X, Y, L, R)$ in respect of the fuzzy conceptual lattice $\mathbb{C}(X, Y, L, R)$ that in a certain sense "surrounds" this pair. Therefore we referred to that approach as an inner one. On the other hand, in this section we consider the "closest" fuzzy concepts to a given fuzzy preconcept (A, B) and estimate their distinction. In this sense the approach proposed here looks like an outer one. In order to realize this idea we introduce the concepts of a fuzzy conceptual kernel and a fuzzy conceptual hull of a fuzzy preconcept (A, B).

6.6.1. *Conceptional hull and conceptional kernel of a fuzzy preconcept.*
Let (X, Y, L, R) be a fixed fuzzy context, $(\mathbb{P}(X, Y, L, R) \preceq)$ be the corresponding fuzzy preconcept lattice and let $(\mathbb{C}(X, Y, L, R) \preceq)$ be its partial ordered subset of fuzzy concepts. Further, let $(A, B) \in \mathbb{P}$. A natural question arises: how far is this preconcept (A, B) from a "real" concept? To state this question more precisely, we are interested to find the largest (in the sense of preoder $\preceq$ on $\mathbb{P}$) fuzzy concept which is smaller or equal than (A, B) and to find the smallest fuzzy concept that is larger or equal than (A, B).

Definition 6.5. *A fuzzy concept $K(A, B) =_{def} (A^0, B^0) \in \mathbb{C}$ is called the conceptual kernel of a fuzzy preconcept (A, B) if*

(1) *$(A^0, B^0) \preceq (A, B)$ and*

(2) *for every* $(C,D) \in \mathbb{C}$ *such that* $(C,D) \preceq (A,B)$ *it holds* $(A^0,B^0) \succeq (C,D)$.

A fuzzy concept $H(A,B) =_{def} (\bar{A},\bar{B}) \in \mathbb{C}$ *is called the conceptual hull of a fuzzy preconcept* (A,B) *if*

(1) $(\bar{A},\bar{B}) \succeq (A,B)$ *and*

(2) *for every* $(C,D) \in \mathbb{C}$ *such that* $(C,D) \succeq (A,B)$ *it holds* $(\bar{A},\bar{B}) \preceq (C,D)$.

The answer to the question on the existence of conceptual hulls and kernels for fuzzy preconcepts is given in the next theorem.

Theorem 6.5. [111] *Let a fuzzy preconcept* (A,B) *be given. If there exists a fuzzy concept* $(C,D) \preceq (A,B)$ *then there exists also the kernel* $K(A,B)$. *If there exists a fuzzy concept* $(C,D) \succeq (A,B)$*, then there exists also the hull* $H(A,B)$.

Proof. To prove the first statement let $\mathcal{C} = \{(C_i,D_i) \mid i \in I\} \subseteq \mathbb{C}$ be the family of all fuzzy concepts such that $(C_i,D_i) \preceq (A,B)$ and assume that this family is not empty. Take now $\curlyvee_{i\in I}(C_i,D_i)$. According to Theorem 6.1 $\curlyvee_{i\in I}(C_i,D_i) \in \mathbb{C}$ and besides from the construction it is clear that $\curlyvee_{i\in I}(C_i,D_i) \preceq (A,B)$. Hence $\curlyvee_{i\in I}(C_i,D_i) = K(A,B)$.

To prove the second statement let $\mathcal{C} = \{(C_i,D_i) \mid i \in I\} \subseteq \mathbb{C}$ be the family of all fuzzy concepts such that $(C_i,D_i) \succeq (A,B)$ and assume that this family is not empty. Take now $\curlywedge_{i\in I}(C_i,D_i)$. According to Theorem 6.1 $\curlywedge_{i\in I}(C_i,D_i) \in \mathbb{C}$ and besides, obviously $\curlywedge_{i\in I}(C_i,D_i) \preceq (A,B)$. From the construction it is clear that $\curlywedge_{i\in I}(C_i,D_i) = H(A,B)$.

□

As different from the problem of existence, the problem of finding the conceptional kernel and hull for a fuzzy preconcept seems to be quite difficult. However we have a special case when the kernel and the hull for a fuzzy preconcept (A,B) can be easily found. Namely, let a fuzzy preconcept (A,B) be given. Reasoning on the fuzzy conceptual hull of a (A,B), we have to minimally enlarge (in the sense of the order $\preceq$) the pair (A,B) in order to get a fuzzy concept. This leads to the idea to take $A \vee B^{\downarrow}$ as the set of objects, thus minimally enlarging A $(\leq)$ in order to satisfy all attributes from B and to take $A^{\uparrow} \wedge B$ as the set of attributes minimally reducing B $(\leq)$ in order to keep in accordance with all objects from A. Now, if we are lucky and $(A \vee B^{\downarrow}, A^{\uparrow} \wedge B)$ is a fuzzy concept, then it is the hull $H(A,B)$ of the fuzzy preconcept (A,B). Reasoning in a dual way, the pair $(A \wedge B^{\downarrow}, A^{\uparrow} \vee B)$ can pretend to be the fuzzy conceptual kernel of a fuzzy preconcept (A,B).

We realize this idea in two cases. First, take $(0_X, 1_Y)$, that is the minimal element in $(\mathbb{P},\preceq)$. Then $1_Y^{\downarrow}(x) = \bigwedge_{y\in Y} R(x,y)$, $\quad 0_X^{\uparrow}(y) = 1_Y(y) \quad (1_X^{\uparrow}(y) = 1_Y(y)$ and hence in this situation $\big((A \vee B^{\downarrow}), (A^{\uparrow} \wedge B)\big) = \Big(\bigwedge_{y\in Y} R(\cdot,y), 1_Y\Big)$. Directly checking, we get $\Big(\bigwedge_{y\in Y} R(\cdot,y)\Big)^{\uparrow} = 1_Y$ and $1_Y^{\downarrow} = \Big(\bigwedge_{y\in Y} R(\cdot,y)\Big)$ and hence $H(1_X,0_Y) = \Big(\bigwedge_{y\in Y} R(\cdot,y), 1_Y\Big)$ is the fuzzy conceptional hull of the minimal fuzzy preconcept $(0_X,1_Y) \in \mathbb{P}$. Obviously, the conceptional kernel of the minimal preconcept $(0_X,1_Y)$ does not exist unless $(0_X,1_Y)$ is a fuzzy concept itself.

As the second case, we take $(1_X,0_Y)$, that is the maximal element in the preconcept lattice $(\mathbb{P},\preceq)$ and are looking for its fuzzy conceptional kernel $K(1_X,0_Y)$. Now we get $A = 1_X, B^{\downarrow}(x) = 1_X(x), B = 0_Y, A^{\uparrow}(y) = \bigwedge_{x\in X} R(x,y)$ and hence in this situation $\big((A \wedge B^{\downarrow}), (A^{\uparrow} \vee B)\big) = \big(1_X, \bigwedge_{x\in X} R(x,\cdot)\big)$. Directly cheking, we conclude that it is indeed a fuzzy concept and hence $K(1_X,0_Y) = \big(1_X, \bigwedge_{x\in X} R(x,\cdot)\big)$ is the conceptional kernel of the maximal fuzzy preconcept $(1_X,0_Y) \in \mathbb{P}$. Obviously, the conceptional hull of the maximal preconcept $(1_X,0_Y)$ does not exist unless $(1_X,0_Y)$ is a fuzzy concept itself. □

Now we can make further clarification in Theorem 6.1:

Theorem 6.6. [111] *Let* (X,Y,L,R) *be a fuzzy context and let* $\preceq$ *be the partial order on* $\mathbb{C}$ *induced from the lattice* $\mathbb{P}(X,Y,L,R,\preceq)$. *Then* $\mathbb{C}(X,Y,L,R),\preceq)$ *is a complete lattice. Its top and bottom elements are respectively* $\top_{\mathbb{C}} = \big(1_X(\cdot), \bigwedge_{x\in X} R(x,\cdot)\big)$ *and* $\bot_{\mathbb{C}} = \Big(\bigwedge_{y\in Y} R(\cdot,y), 1_Y(\cdot)\Big)$.

6.6.2. *Ditopological interpretation of conceptional kernels and hulls.*
Let $\mathbb{P}^1(X,Y,L,R) = \{(A,B) \in \mathbb{P}(X,Y,L,R,\preceq) \mid (A,B) \preceq \top_{\mathbb{C}}\}$.

Theorem 6.7. [111] *Hull operators $H : \mathbb{P}^1(X,Y,L,R) \to \mathbb{P}^1(X,Y,L,R)$ defined by $H : (A,B) \to H(A,B)$ for all $(A,B) \in \mathbb{P}^1(X,Y,L,R)$ is a closure operator.*

Indeed, H is obviously extentional, $((A,B) \preceq H(A,B))$, isotone $((A,B) \preceq (A',B') \Longrightarrow H(A,B) \preceq H(A',B'))$. Further, since the fuzzy preconcept $H(A,B)$ is fuzzy a concept, therefore H is also idempotent $H(H(A,B)) = H(A,B)$.

Hence, referring to Definition 3.14 this closure operator H generates a fuzzy co-topology on $\mathbb{P}^1(X,Y,L,R)$. Moreover, notice that the image of $\mathbb{P}^1(X,Y,L,R)$ under H is obviously $\mathbb{C}(X,Y,L,R)$. Thus we get the following theorem:

Theorem 6.8. *The family $\mathbb{C}(X,Y,L,R)$ of all fuzzy concepts in a fuzzy preconcept lattice $\mathbb{P}(X,Y,L,R)$ is a fuzzy supra cotopology induced by the fuzzy hull operator $H : \mathbb{P}^1(X,Y,L,R) \to \mathbb{P}^1(X,Y,L,R)$.*

Further, let Let $\mathbb{P}^0(X,Y,L,R) = \{(A,B) \in \mathbb{P}(X,Y,L,R,\preceq) \mid (A,B) \succeq \perp_{\mathbb{C}}$ and let operator $K : \mathbb{P}^0(X,Y,L,R) \to \mathbb{P}^0(X,Y,L,R)$ be defined by $K : (A,B) \to K(A,B)$ for All $(A,B) \in \mathbb{P}^0(X,Y,L,R)$

Theorem 6.9. [111] *Kernel operator $K : \mathbb{P}^0(X,Y,L,R) \to \mathbb{P}^0(X,Y,L,R)$ defined by $K : (A,B) \to K(A,B)$ for all $(A,B) \in \mathbb{P}^0(X,Y,L,R)$ is an interior operator.*

Indeed, K is obviously anti-extentional, $((A,B) \succeq K(A,B))$, isotone $((A,B) \preceq (A',B') \Longrightarrow K(A,B) \preceq K(A',B'))$ and since $K(A,B)$ is a concept, therefore K is also idempotent $(K(K(A,B)) = K(A,B))$.

Hence this operator generates a fuzzy supratopology on $\mathbb{P}^0(X,Y,L,R)$. Moreover, notice that the image of $\mathbb{P}^0(X,Y,L,R)$ under K is obviously $\mathbb{C}(X,Y,L,R)$. [111] Thus we get the following theorem:

Theorem 6.10. *The family $\mathbb{C}(X,Y,L,R)$ of all fuzzy concepts in a fuzzy preconcept lattice $\mathbb{P}(X,Y,L,R)$ is a fuzzy supra topology induced by the L-fuzzy kernel operator $K : \mathbb{P}^0(X,Y,L,R) \to \mathbb{P}^0(X,Y,L,R)$.*

6.6.3. *Measure of conceptuality of a fuzzy preconcept and $\mathcal{M}$-graded preconcept lattices.*
In this paragraph we introduce measures of lower and upper conceptual approximations of a fuzzy preconcept (A,B) which are defined as a certain measure of distinctions between (A,B) and its fuzzy conceptual kernel $K(A,B)$ and hull $H(A,B)$ respectively. We start with the following definition.

Definition 6.6. *Let $(C,D),(E,F) \in (\mathbb{P}(X,Y,L,R),\preceq)$ and $(C,D) \preceq (E,F)$. We define the measure of inclusion of a fuzzy preconcept (E,F) into fuzzy preconcept (C,D) by $(E,F) \sqsubseteq (C,D) = (E \hookrightarrow C) \wedge (F \hookleftarrow D)$ and the measure of covering of a fuzzy preconcept (E,F) by a fuzzy preconcept (C,D) as $(C,D) \sqsupseteq (E,F) = (C \hookleftarrow E) \wedge (D \hookrightarrow F)$.*

Definition 6.7. *Given a fuzzy preconcept (A,B) in a fuzzy preconcept lattice $(\mathbb{P},\preceq)$, its lower measure of conceptuality is defined by $\mathcal{M}_L(A,B) = (A,B) \sqsubseteq K(A,B)$ and its upper measure of conceptuality is defined by $\mathcal{M}_U(A,B) = (A,B) \sqsupseteq H(A,B)$. Finally the measure of conceptuality of (A,B) is defined by $\mathcal{M}(A,B) = \mathcal{M}_L(A,B) \wedge \mathcal{M}_U(A,B)$.*

Thus the lower measure of approximation of a fuzzy preconcept (A,B) is defined as the measure of its inclusion in its kernel $K(A,B)$ and the upper measure of approximation is defined as the measure how its conceptional hull $H(A,B)$ is covered by (A,B)

Let a fuzzy context (X,Y,L,R) be given and let $\mathbb{P} =_{def} (\mathbb{P}(X,Y,L,R),\preceq)$ be the corresponding fuzzy preconcept lattice.

Definition 6.8. *The triple $(\mathbb{P},\sqsubseteq,\sqsupseteq)$ is called $\mathcal{M}$-graded preconcept lattice of the fuzzy context (X,Y,L,R).*

7. Conclusions

In this paper, we tried to reveal topological ideas that can be viewed as a certain hidden background of theories of fuzzy rough sets, fuzzy morphology and fuzzy (pre)concept lattices. Concerning theory of fuzzy rough sets, its relation with fuzzy topology was the subject of many papers, see the corresponding references in the text. On the other hand relation of fuzzy rough sets to manyvalued fuzzy topologies is known much less, and so it can be of most interest for a potential reader. Concerning the interest in topological background in fuzzy mathematical morphology and fuzzy concept lattices - we do not know any work in this field except of our few our papers, see the corresponding references in the text. In this work we tried to expose most important information concerning “manifestation” of fuzzy topological structures in the theories of fuzzy rough sets, fuzzy mathematical morphology and fuzzy concept lattices known to us; also some new results are presented here. The work in this area can be continued in different directions. In particular, it seems challenging to study further interrelations between the categories of fuzzy rough sets, fuzzy mathematical morphology and fuzzy rough sets (considered in this work as well as others) on one side with corresponding categories of topological-type structures on the other. We guess that revealing such interrelation will contribute to better understanding of all these fields of mathematics.

Of course, there are also other fuzzy mathematical structures with an interesting topological background. One of such fields is the theory of fuzzy transforms. We do not touch this area here since its relations to fuzzy rough sets and fuzzy topology is studied in detail by I. Perfilieva and co-authors in [84]. We did not consider here also fuzzy topological aspects of fuzzy metric spaces. However it is worth to note that in most papers where fuzzy metrics or related fuzzy distance functions are involved, these functions induce a *crisp* and nor a *fuzzy* topology. Fuzzy topology induced by fuzzy metrics up to now got a much less attention. We think that the work in this direction needs to be done.

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