TWMS J. Pure Appl. Math. V.12, N.2, 2021, pp.193-198

EXTENDED KUNG-TRAUB-TYPE METHOD FOR SOLVING EQUATIONS

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Abstract. We are motivated by a Kung-Traub-type method for solving equations on the real line. In particular, we extend this method for Banach space valued operators. The radius of convergence is also obtained as well as error bounds on the distances involved and a uniqueness result. Our convergence analysis avoids Taylor expansions and the computation of higher order than one derivatives.

Keywords: Kung-Traub method, local convergence, Banach space.

AMS Subject Classification: 65H10, 65J15, 65G99, 49M17.

1. Introduction

In this paper, we are motivated by the Kung-Traub-type method [16] defined for each $n = 0, 1, 2, \dots$ by

$$\begin{aligned} z_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f'(x_n)f(z_n)}{f[z_n, x_n]^2}, \end{aligned}$$

where $f : \Omega \subset \mathbb{R} \longrightarrow \mathbb{R}$ is a continuously differentiable operator, Ω is a nonempty, convex and open set $x_0 \in \Omega$ is an initial point. This method generates a sequence approximating a solution x^* of equation $f(x) = 0$.

The convergence of this method is shown using Taylor expansions and hypotheses on the derivative of order at least three. However, there are simple numerical examples, where this method cannot apply. Hence, the applicability of the method is limited. Let us consider an example, let us define function F on $X = [-\frac{1}{2}, \frac{3}{2}]$ by

$$F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Choose $x^* = 1$. We have that

$$\begin{aligned} F'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \; F'(1) = 3, \\ F''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x, \\ F'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22. \end{aligned}$$

Then, obviously function F does not have bounded third derivative in X. We shall make this paper useful in solving not only equations on the real line but also equations of the form

$$F(x) = 0, \tag{1}$$

where $F : \Omega \subseteq \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ is a Fréchet-differentiable operator, $\mathcal{B}_1, \mathcal{B}_2$ are Banach spaces and Ω is a nonempty open and convex set. The iterative method corresponding to the Kung-Traub method in this setting is defined by

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Manuscript received November 2018.

$$z_k = x_k - F'(x_k)^{-1}F(x_k),$$
$$x_{k+1} = x_k - F'(x_k)^{-1}F(x_k) - F[z_k, x_k]^{-1}F'(x_k)F[z_k, x_k]^{-1}F(z_k), \tag{2}$$

where $F[.,.] : \Omega \times \Omega \longrightarrow \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$, the space of bounded linear operators from $\mathcal{B}_1$ into $\mathcal{B}_2$. Clearly, if $F = f$ and $\Omega = \mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}$, then method (2) reduces to Kung-Traub method. Related methods can be found in [1-16].

In this paper, we present the local convergence analysis of method (2). We shall find the radius of convergence, computable error bounds on the distances $\|x_n - x^*\|$ and we shall establish the uniqueness of the solution x^* inside a certain ball based on some Lipschitz constants. The computation of radius of convergence is important in the study of iterative methods, since it provides the degree of difficulty in determining initial points x_0.

The rest of the paper is structured as follows. Section 2 contains the local convergence, whereas in Section 3, the numerical examples appear.

2. Local convergence

Let $\varphi_0 : \mathbb{R}_+ \cup \{0\} \longrightarrow \mathbb{R}$ be a continuous and nondecreasing function with $\varphi_0(0) = 0$. Suppose that equation
$$\varphi_0(t) = 1 \tag{3}$$
has at least one positive root. Denote by ρ_0 the smallest such root. Let $\varphi : [0, \rho_0) \longrightarrow \mathbb{R}$ be a continuous and nondecreasing function with $\varphi(0) = 0$. Define functions ψ_1 and μ_1 on $[0, \rho_0)$ by
$$\psi_1(t) = \frac{\int_0^1 \varphi((1-\theta)t)d\theta}{1 - \varphi_0(t)}$$
and
$$\mu_1(t) = \psi_1(t) - 1.$$
We have that $\mu_1(0) = -1$ and $\mu_1(t) \longrightarrow +\infty$ as $t \longrightarrow \rho_0^-$. It then follows from the intermediate value theorem that function μ_1 has zeros in the interval $(0, \rho_0)$. Denote by ρ_1 the smallest such zero. Let $\varphi_1 : [0, \rho_1) \times [0, \rho_1) \longrightarrow \mathbb{R}$ be a continuous and nondecreasing function in both variables such that $\varphi_1(0, 0) = 0$. Define function $\bar{\varphi}_1(t) := \varphi_1(\psi_1(t)t, t) - 1$. Suppose that
$$\varphi_1(\psi_1(\rho_1)\rho_1, \rho_1) < 1. \tag{4}$$
Denote by $\bar{\rho}_1$ the smallest zero of function $\bar{\varphi}_1$ on $[0, \rho_1)$. Moreover, let $\varphi_2 : [0, \bar{\rho}_1) \longrightarrow \mathbb{R}$ and $\varphi_3 : [0, \bar{\rho}_1) \longrightarrow \mathbb{R}$ be continuous and nondecreasing functions. Define functions ψ_2 and μ_2 on $[0, \bar{\rho}_1)$ by
$$\psi_2(t) = (1 + \varphi_2(t)\varphi_3(\psi_1(t)t))\psi_1(t)$$
and
$$\mu_2(t) = \psi_2(t) - 1.$$
We get that $\mu_2(0) = -1 < 0$ and $\mu_2(t) \longrightarrow +\infty$ as $t \longrightarrow \bar{\rho}_1^-$. Denote by ρ the smallest zero of function μ_2 on the interval $(0, \bar{\rho}_1)$. Then, we have that for each $t \in [0, \rho)$
$$0 \leq \psi_1(t) < 1, \tag{5}$$
$$0 \leq \varphi_1(\psi_1(t)t, t) < 1 \tag{6}$$
and
$$0 \leq \varphi_2(t) < 1. \tag{7}$$
Let $U(u, \tau), \bar{U}(u, \tau)$ stand for the open and closed balls in Ω, respectively with center $u \in \mathcal{B}_1$ and of radius $\tau > 0$. The local convergence of method (2) is based on the conditions (C):

(c1) $F : \Omega \subset \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ is a continuously Fréchet-differentiable operator and there exists a divided difference of order one $F[.,.] : \Omega \times \Omega \longrightarrow \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$.

(c2) (4) holds for ρ_0 given in (1) and ρ_1 satisfying $\mu_1(\rho_1) = 0$.

(c3) There exist $x^* \in \Omega$ and a continuous and nondecreasing function $\varphi_0 : \mathbb{R}_+ \longrightarrow \mathbb{R}$ with $\varphi_0(0) = 0$ such that $F(x^*) = 0$ and for each $x \in \Omega, F'(x^*)^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$ and

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le \varphi_0(\|x - x^*\|).$$

Set $\Omega_0 = \Omega \cap \bar{U}(x^*, \rho_0)$.

(c4) There exist continuous and nondecreasing functions $\varphi : [0, \rho_0) \longrightarrow \mathbb{R}$, $\varphi_1 : [0, \rho_0)^2 \longrightarrow \mathbb{R}$ with $\varphi(0) = \varphi(0, 0) = 0$ such that for each $x, y \in \Omega_0$

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \le \varphi(\|x - y\|)$$

and

$$\|F'(x^*)^{-1}(F[x, y] - F'(x^*))\| \le \varphi_1(\|x - x^*\|, \|y - x^*\|).$$

(c5) There exist continuous and nondecreasing functions $\varphi_2 : [0, \bar{\rho}_1) \longrightarrow \mathbb{R}, \varphi_3 : [0, \bar{\rho}_1) \longrightarrow \mathbb{R}$ such that for each $x \in \Omega_0$

$$\|F'(x^*)^{-1}F'(x)\| \le \varphi_2(\|x - x^*\|)$$

and

$$\|F'(x^*)^{-1}F[x, y]\| \le \varphi_3(\|x - x^*\|),$$

where $\bar{\rho}_1$ satisfies $\bar{\varphi}_1(\bar{\rho}_1) = 0$.

(c6) $\bar{U}(x_0, \rho) \subseteq \Omega$, where ρ satisfies $\mu_2(\rho) = 0$.

(c7) There exists $\bar{\rho} \ge \rho$ such that

$$\varphi_0(\bar{\rho}) < 1. \tag{8}$$

Next, we show the local convergence analysis of method (2) using the conditions (C) and the preceding notation.

Theorem 2.1. *Suppose that the conditions (C) hold. Then, sequence $\{x_n\}$ generated for $x_0 \in U(x^*, \rho) - \{x^*\}$ by method (2) is well defined in $U(x^*, \rho)$, remains in $U(x^*, \rho)$ for each $n = 0, 1, 2, \ldots$ and converges to x^*. Moreover, the following error estimates hold*

$$\|z_k - x^*\| \le \psi_1(\|x_k - x^*\|)\|x_k - x^*\| \le \|x_k - x^*\| < \rho \tag{9}$$

and

$$\|x_{k+1} - x^*\| \le \psi_2(\|x_k - x^*\|)\|x_k - x^*\| \le \|x_k - x^*\|, \tag{10}$$

where functions ψ_1 and ψ_2 are defined previously. Furthermore, the limit point x^* is the only solution of equation $F(x) = 0$ in $\Omega_1 = \Omega \cap \bar{U}(x^*, \rho)$.

Proof. We shall show that sequence $\{x_n\}$ is well defined in $U(x^*, \rho)$ and converges to x^* so that estimates (9) and (10) hold. Let $x \in U(x^*, \rho)$. Using the definition of ρ and (c3) we have that

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le \varphi_0(\|x - x^*\|) \le \varphi_0(\rho_0) \le \varphi_0(\rho) < 1. \tag{11}$$

It follows from (11) and the Banach perturbation lemma [2] that $F'(x)^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$ and

$$\|F'(x)^{-1}F'(x^*)\| \le \frac{1}{1 - \varphi_0(\|x - x^*\|)}. \tag{12}$$

In particular, estimate (12) hold for $x = x_0$, since $x_0 \in U(x^*, \rho) - \{x^*\}$. We can write by the first substep of method (2) for $k = 0$ and (c3)

$$\begin{aligned} z_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) \\ &= [F'(x_0)^{-1}F'(x^*)] \\ &\quad \times [\int_0^1 F'(x^*)^{-1}F'(x^* + \theta(x_0 - x^*)) - F'(x_0)](x_0 - x^*)d\theta]. \end{aligned} \tag{13}$$

By (c4), (5), (12), (13) and the definition of ρ, we get in turn that

$$\begin{aligned}
\|z_0 - x^*\| &\leq \|F'(x_0)^{-1}F'(x^*)\| \\
&\quad \times \|\int_0^1 F'(x^*)^{-1}[F'(x^* + \theta(x_0 - x^*)) - F'(x_0)](x_0 - x^*)d\theta\| \\
&\leq \frac{\int_0^1 \varphi((1-\theta)\|x_0 - x^*\|)d\theta\|x_0 - x^*\|}{1 - \varphi_0(\|x_0 - x^*\|)} \\
&\leq \psi_1(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < \rho, \qquad (14)
\end{aligned}$$

which shows (9) for $k = 0$ and $z_1 \in U(x^*, \rho)$. We shall show that x_1 is well defined by proving the invertability of $F[z_0, x_0]$. Indeed, using (4), (6), (c4) and (14), we obtain in turn that

$$\begin{aligned}
\|F'(x^*)^{-1}([F[z_0, x_0] - F'(x^*))\| &\leq \varphi_1(\|z_0 - x^*\|, \|x_0 - x^*\|) \\
&\leq \varphi_1(\psi_1(\|x_0 - x^*\|)\|x_0 - x^*\|, \|x_0 - x^*\|) \\
&\leq \varphi_1(\psi_1(\rho)\rho, \rho) < 1, \qquad (15)
\end{aligned}$$

so $F[z_0, x_0]^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$ and

$$\|F[z_0, x_0]^{-1}F'(x^*)\| \leq \frac{1}{1 - \varphi_1(\psi_1(\|x_0 - x^*\|)\|x_0 - x^*\|, \|x_0 - x^*\|)}. \qquad (16)$$

By (c3) and (c5), we get the estimates

$$\|F'(x^*)^{-1}F'(x_0)\| \leq \varphi_2(\|x_0 - x^*\|) \qquad (17)$$

and

$$(18)$$

$$\begin{aligned}
&\|F'(x^*)^{-1}F(z_0)\| \\
= &\|F'(x^*)^{-1}(F(z_0) - F(x^*))\| \\
= &\|\int_0^1 F'(x^*)^{-1}F'(x^* + \theta(z_0 - x^*))(z_0 - x^*)d\theta\| \\
\leq &\varphi_3(\|z_0 - x^*\|)\|z_0 - x^*\| \\
\leq &\varphi_3(\psi_1(\|x_0 - x^*\|)\|x_0 - x^*\|)\psi_1(\|x_0 - x^*\|)\|x_0 - x^*\|. \qquad (19)
\end{aligned}$$

Then, we get by the second substep of method (2), (7), (15)–(19) that

$$\begin{aligned}
\|x_1 - x^*\| &\leq \|z_0 - x^*\| + \|F[z_0, x_0]^{-1}F'(x^*)\|^2 \\
&\quad \times \|F'(x^*)^{-1}F'(x_0)\|\|F'(x^*)^{-1}F(z_0)\| \\
&\leq \psi_1(\|x_0 - x^*\|)\|x_0 - x^*\| \\
&\quad + \frac{\varphi_2(\|x_0 - x^*\|)\varphi_3(\|z_0 - x^*\|)\|z_0 - x^*\|}{(1 - \varphi_1(\psi_1(\|x_0 - x^*\|)\|x_0 x -^* \|, \|x_0 - x^*\|))^2} \\
&\leq \psi_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < \rho, \qquad (20)
\end{aligned}$$

which shows (10) for $n = 0$ and $x_1 \in U(x^*, \rho)$. The induction for (9) and (10) is completed in an analogous way, if we replace x_0, z_0, x_1 by x_m, z_m, x_{m+1} in the preceding estimates. Then, by the estimate

$$\|x_{m+1} - x^*\| \leq c\|x_m - x^*\| < \rho, \qquad (21)$$

where $c = \psi_2(\|x_0 - x^*\|) \in [0, 1)$, we deduce that $\lim_{m \longrightarrow +\infty} x_m = x^*$ and $x_{m+1} \in U(x^*, \rho)$. Then, for the uniqueness part, we let $y^* \in \Omega_1$ such that $F(y^*) = 0$. Define linear operator

$Q = \int_0^1 F'(x^* + \theta(y^* - x^*))d\theta$. Using (c3) and (8) we get in turn that

$$\|F'(x^*)^{-1}(Q - F'(x^*))\| \leq \int_0^1 \varphi_0(\theta\|y^* - x^*\|)d\theta \leq \varphi_0(\bar{\rho}) < 1, \tag{22}$$

so $Q^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$. We can write that

$$0 = F(y^*) - F(x^*) = Q(y^* - x^*), \tag{23}$$

so $x^* = y^*$.

$\square$

3. Numerical examples

In this Section the divided difference is given by $F[x, y] = \int_0^1 F'(y + \theta(x - y))d\theta$.

Example 1. Returning back to the example in the introduction, we have for $\varphi_0(t) = \varphi(t) = 147t, \varphi_1(s, t) = \frac{147}{2}(s + t), \varphi_2(t) = \varphi_3(t) = 2$. Using the definition of ρ we obtain

$$\rho = 0.0011.$$

Example 2. Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}^3, \Omega = \bar{U}(0, 1), x^* = (0, 0, 0)^T$. Define function F on Ω for $w = (x, y, z)^T$ by

$$F(w) = (e^x - 1, \frac{e - 1}{2}y^2 + y, z)^T.$$

The Fréchet-derivative is defined by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e - 1)y + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, we have $\varphi_0(t) = (e - 1)t, \varphi(t) = e^{\frac{1}{e-1}}t$, $\varphi_1(s, t) = \frac{1}{e-1}(s + t)$, $\varphi_2(t) = \varphi_3(t) = e^{\frac{1}{e-1}}$. Using the definition of ρ we obtain

$$\rho = 0.1825$$

Example 3. Let $\mathcal{X} = \mathcal{Y} = C[0, 1]$, be the space of continuous functions on $[0, 1])$ equipped with the max-norm. Let $\Omega = \bar{U}(0, 1)$. Define F on Ω by

$$F(\varphi)(x) = \varphi(x) - 10\int_0^1 x\theta\varphi(\theta)^3 d\theta.$$

We have that

$$[F'(\varphi(\xi))](x) = \xi(x) - 30\int_0^1 x\theta\varphi(\theta)^2 d\theta, \text{ for each } \xi \in D.$$

Then, we get that $x^* = 0$, $\varphi_0(t) = 15t$, $\varphi(t) = 30t$, $\varphi_1(s, t) = \frac{15}{2}(s + t)$, $\varphi_2(t) = \varphi_3(t) = 30$. We obtain

$$\rho = 7.3910e - 05.$$

4. Conclusion

The Kung-Traub method has been used to solve nonlinear equations on the real line. The convergence order is shown to be four under the assumption that the fifth order derivative (not on the method) exists limiting its applicability. But we have extended the applicability of the method in a Banach space setting using hypotheses only on the derivative and divided difference of order one that actually appear in the method.

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