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FIXED POINT ON CONVEX b-METRIC SPACE VIA ADMISSIBLE MAPPINGS

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Abstract. In this manuscript, we define a convex admissible mapping. Using this notion, we consider specific contraction involving rational terms via convex admissible mapping. We investigate the necessary and sufficient requirement to guarantee a fixed point in the framework of convex b-metric spaces.

Keywords: convex structure, fixed point theorems, b-metric.

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1. Introduction and preliminaries

Fixed point notions appeared in the papers that provided certain solutions to the particular differential equations at the end. Banach [8] abstracted the first independent metric fixed point theory. Since then, the connection between the metric fixed point theory and applied mathematics has been advanced, see e.g. [1, 4]. The concept of b-metric can be considered the most valuable generalization of the metric put forward to date. The idea of b-metric appeared in [12], first, in 1974. This notion was also announced as a quasi-metric [9, 10, 11]. After the papers of Czerwik [15, 16] and Bakhtin [7], b-metric began to attract the attention of researchers [2, 5, 6, 3, 17, 19, 20, 14, 18, 21]. Roughly speaking, although b-metric axioms are very similar to the metric, the topology produced by b-metric has severe structural differences. For instance, b-metric is not need to be continuous.

On the other hand, metric spaces endowed with a convex structure is one of the interesting research topic, see, e.g. [23]. Very recently, in [13], the authors considered convex b-metric spaces and proved a certain fixed point theorem in this framework.

In this paper, we first consider to define admissible mapping for the set endowed with a convex structure. We get new type contractions by employing this notion to contractions involving rational terms. We prove the existence of a fixed point of such mappings in the context of convex b-metric spaces.

We start by recalling the following basic definition. Let U be a non empty set, a number $s \geq 0$ and $\mathsf{m} : \mathsf{U} \times \mathsf{U} \to [0, +\infty)$ withe the following axioms:

(m_1) $\mathsf{m}(v, o) = 0 \Leftrightarrow v = o$;

(m_2) $\mathsf{m}(v, o) = \mathsf{m}(o, v)$;

(m_3) $\mathsf{m}(v, o) \leq \mathsf{m}(v, u) + \mathsf{m}(u, o)$;

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(m_4) $\mathsf{m}(v,o) \le s[\mathsf{m}(v,u)+\mathsf{m}(u,o)]$;
where $v,o,u \in \mathsf{U}$.
We say that the function m is a metric on U if satisfies the axioms (m_1), (m_2), (m_3) and it is a b-metric on U if satisfies (m_1), (m_2), (m_4). Moreover, a non-empty set endowed with a metric (b-metric) is called a metric (respectively, b-metric) space.
Related to b-metric space we recall the following important result.

Lemma 1.1. [22] *If $\{v_n\}$ is a sequence in a b-metric space (U,b) with the property that there exist $\kappa \in [0,1/s)$ and $K>0$ such that*

$$\mathsf{b}(v_n, v_{n+1}) \le \kappa^n K,$$

for any $n \in \mathbb{N}$, then $\{v_n\}$ is a Cauchy sequence.

Let now (U,d) be a metric space and $\mathsf{J}=[0,1]$. A mapping $\mathsf{w}:\mathsf{U}\times\mathsf{U}\times\mathsf{J}\to\mathsf{U}$ is a convex structure on U if

$$\mathsf{d}(u,\mathsf{w}(v,o;\lambda)) \le \lambda\mathsf{d}(u,v)+(1-\lambda)\mathsf{d}(u,o), \tag{1}$$

for each $(v,o,\lambda)\in\mathsf{U}\times\mathsf{U}\times\mathsf{J}$ and $u\in\mathsf{U}$. Moreover, the set U together with a convex structure w is said to be a convex metric space. (see [23]).
Recently, in [13], the notion of b-convex metric space was introduced.

Definition 1.1. [13] *Let (U,b) be a b-metric space (with $s\ge1$), $\mathsf{w}:\mathsf{U}\times\mathsf{U}\times\mathsf{J}\to\mathsf{U}$ be a convex structure on U and $\mathsf{J}=[0,1]$. The triplet $(\mathsf{U},\mathsf{b},\mathsf{w})$ is called a convex b-metric space.*

Example 1.1. [13] *Letting $\mathsf{U}=\mathbb{R}^n$ and $\mathsf{b}:\mathsf{U}\times\mathsf{U}\to[0,+\infty)$, with $\mathsf{b}(v,o)=\sum_{j=1}^{n}(v_j-o_j)^2$, with $v=(v_1,v_2,...,v_n)$, $o=(o_1,o_2,...,o_n)\in\mathsf{U}$ we get that (U,b) is a b-metric space ($s=2$). Moreover, choosing the function $\mathsf{w}:\mathsf{U}\times\mathsf{U}\times[0,1]\to\mathsf{U}$ defined as*

$$\mathsf{w}(v,o,\lambda)=\lambda v+(1-\lambda)o,$$

for $v,o\in\mathsf{U}$, then $(\mathsf{U},\mathsf{b},\mathsf{w})$ becomes a convex b-metric space.

Example 1.2. [13] *If $\mathsf{U}=\mathbb{R}$, let $\mathsf{b}:\mathsf{U}\times\mathsf{U}\to[0,+\infty)$, where $\mathsf{b}(v,o)=(v-o)^2$ be a b-metric on U (here $s=2$). Thus, $(\mathsf{U},\mathsf{b},\mathsf{w})$ forms a convex b-metric space, where $\mathsf{w}:\mathsf{U}\times\mathsf{U}\times[0,1]\to\mathsf{U}$ is defined as*

$$\mathsf{w}(v,o,\lambda)=\lambda v+(1-\lambda)o,$$

for any $v,o\in\mathsf{U}$ and $\lambda\in[0,1]$.

Theorem 1.1. [13] *Let $(\mathsf{U},\mathsf{b},\mathsf{w})$ with $s>1$ be a complete convex b -metric space and $\mathsf{F}:\mathsf{U}\to\mathsf{U}$ be a mapping. Supposing that there exists $\kappa\in[0,1)$ such that*

$$\mathsf{b}(\mathsf{F}v,\mathsf{F}o)\le\kappa\mathsf{b}(v,o). \tag{2}$$

Let $v_0\in\mathsf{U}$ be such that $\mathsf{b}(v_0,\mathsf{F}v_0)<\infty$ and the sequence $\{v_n\}$ be defined by $v_n=\mathsf{w}(v_{n-1},\mathsf{F}v_{n-1},\lambda_{n-1})$, where $0\le\lambda_{n-1}<1$ and $n\in\mathbb{N}$. Then, F has a unique fixed point provided that $\kappa<\frac{1}{s^4}$ and $0<\lambda_n<\frac{\frac{1}{s^4}-\lambda}{1-\lambda}$, for each $n\in\mathbb{N}$.

Theorem 1.2. [13] *Let $(\mathsf{U},\mathsf{b},\mathsf{w})$ with $s>1$ be a complete convex b -metric space and $\mathsf{F}:\mathsf{U}\to\mathsf{U}$ be a mapping. Supposing that there exists $\kappa\in[0,1/2)$ such that*

$$\mathsf{b}(\mathsf{F}v,\mathsf{F}o)\le\kappa[\mathsf{b}(v,\mathsf{F}v)+\mathsf{b}(\o,\mathsf{F}o)]. \tag{3}$$

Let $v_0\in\mathsf{U}$ be such that $\mathsf{b}(v_0,\mathsf{F}v_0)<\infty$ and the sequence $\{v_n\}$ be defined by $v_n=\mathsf{w}(v_{n-1},\mathsf{F}v_{n-1},\lambda_{n-1})$, where $0\le\lambda_{n-1}<1$ and $n\in\mathbb{N}$. Then, F has a unique fixed point provided that $0\le\kappa\le\frac{1}{4s^2}$ and $0<\lambda_n<\frac{1}{4s^2}$, for each $n\in\mathbb{N}$.

2. Main results

Definition 2.1. *Let* U *be a non-empty set,* $\alpha : \mathsf{U} \times \mathsf{U} \to [0, +\infty)$ *be a function and* $\mathsf{w} : \mathsf{U} \times \mathsf{U} \times [0, 1] \to \mathsf{U}$. *A mapping* $\mathsf{F} : \mathsf{U} \to \mathsf{U}$ *is called* α-w *admissible if for any* $v, o \times \mathsf{U}$,

$$\alpha(v, o) \geq 1 \Rightarrow \alpha(\mathsf{w}(v, \mathsf{F}v, \lambda_1), \mathsf{w}(o, \mathsf{F}o, \lambda_2)) \geq 1, \tag{4}$$

where$\lambda_1, \lambda_2 \in [0, 1]$.

Lemma 2.1. *Let* $\mathsf{F} : \mathsf{U} \to \mathsf{U}$ *be an* α-w-*admissible mapping,* $v_0, v_1 \in \mathsf{U}$ *such that* $\alpha(v_0, v_1) \geq 1$ *and the sequence* $\{v_n\}$ *in* U, *where*

$$v_n = \mathsf{w}(v_{n-1}, \mathsf{F}v_{n-1}, \lambda_{n-1}), \tag{5}$$

$\lambda_{n-1} \in [0, 1]$. *Then,* $\alpha(v_n, v_{n+1}) \geq 1$, *for any* $n \in \mathbb{N}$.

Proof. By the hypotheses, we have that there exist $v_0, v_1 \in \mathsf{U}$ such that $\alpha(v_0, v_1) \geq 1$. Then, since the mapping F is α-w-admissible, by (4) together with (5) we have

$$\alpha(v_0, v_1) \geq 1 \Rightarrow \alpha(\mathsf{w}(v_0, \mathsf{F}v_0, \lambda_0), \mathsf{w}(v_1, \mathsf{F}v_1, \lambda_1)) = \alpha(v_1, v_2) \geq 1,$$

where $\lambda_1, \lambda_2 \in [0, 1]$. Therefore, repeating this procedure we get that

$$\alpha(v_n, v_{n+1}) \geq 1, \text{ for any } n \in \mathbb{N}.$$

□

Theorem 2.1. *On a complete convex* b-*metric space* $(\mathsf{U}, \mathsf{b}, \mathsf{w})$ *with* $s > 1$, *let* $\mathsf{F} : \mathsf{U} \to \mathsf{U}$ *be an* α-w-*admissible mapping such that there exist* $\kappa_1, \kappa_2 \in [0, 1)$ *with the property that*

$$\alpha(v, o)\mathsf{b}(\mathsf{F}v, \mathsf{F}o) \leq \kappa_1 \frac{\mathsf{b}(v, o)\mathsf{b}(o, \mathsf{F}o)}{\mathsf{b}(v, \mathsf{F}v)} + \kappa_2 \mathsf{b}(v, o), \tag{6}$$

for all $v, o \in \mathsf{U} \setminus Fix_{\mathsf{F}}\mathsf{U}$. *Suppose that:*

(1) *there exists* $v_0 \in \mathsf{U}$ *such that* $\mathsf{b}(v_0, \mathsf{F}v_0) < \infty$ *and* $\alpha(v_0, v_1) \geq 1$, *where the sequence* $\{v_n\}$ *is defined by* $v_n = \mathsf{w}(v_{n-1}, \mathsf{F}v_{n-1}, \lambda_{n-1})$, *with* $0 \leq \lambda_{n-1} \leq 1$ *for any* $n \in \mathbb{N}$;
(2) $\kappa_1 + \kappa_2 \leq \frac{1}{4s^2}$ *and* $\lambda_n \leq \frac{1}{4s^2}$;
(3) $\alpha(v_*, v_n) \geq 1$ *for any sequence* $\{v_n\}$ *in* U *such that* $\alpha(v_n, v_{n+1}) \geq 1$ *and* $v_n \to v_*$ *as* $n \to \infty$.

Then, the mapping F *has a fixed point.*

Proof. Let v_0, v_1 be two points in U such that $\alpha(v_0, v_1) \geq 1$ and $\mathsf{b}(v_0, \mathsf{F}v_0) = K < \infty$. Thus, taking into account Lemma 2.1, letting $v = v_{n-1}$ and $o = v_n$ in (6), (where the sequence $\{v_n\}$ in U is defined by (5)) we have

$$\mathsf{b}(\mathsf{F}v_{n-1}, \mathsf{F}v_n) \leq \alpha(v_{n-1}, v_n)\mathsf{b}(\mathsf{F}v_{n-1}, \mathsf{F}v_n) \leq \kappa_1 \frac{\mathsf{b}(v_{n-1}, v_n)\mathsf{b}(v_n, \mathsf{F}v_n)}{\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1})} + \kappa_2 \mathsf{b}(v_{n-1}, v_n). \tag{7}$$

But, since the space $(\mathsf{U}, \mathsf{b}, \mathsf{w})$ is a convex b-metric space, and keeping in mind (5),

$$\begin{aligned} \mathsf{b}(v_n, v_{n+1}) &= \mathsf{b}(v_n, \mathsf{w}(v_n, \mathsf{F}v_n, \lambda_n)) \\ &\leq \lambda_n \mathsf{b}(v_n, v_n) + (1 - \lambda_n)\mathsf{b}(v_n, \mathsf{F}v_n) \\ &= (1 - \lambda_n)\mathsf{b}(v_n, \mathsf{F}v_n), \end{aligned} \tag{8}$$

for any $n \in \mathbb{N}$, where $\lambda_n \in [0,1]$. On the other hand, by (m_4).

$$\begin{aligned}\mathsf{b}(v_n, \mathsf{F}v_n) &= \mathsf{b}(\mathsf{w}(v_{n-1}, \mathsf{F}v_{n-1}, \lambda_{n-1}), \mathsf{F}v_n)\\ &\leq \lambda_{n-1}\mathsf{b}(v_{n-1}, \mathsf{F}v_n) + (1-\lambda_{n-1})\mathsf{b}(\mathsf{F}v_{n-1}, \mathsf{F}v_n)\\ &\leq s\lambda_{n-1}\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1}) + s\lambda_{n-1}\mathsf{b}(\mathsf{F}v_{n-1}, \mathsf{F}v_n) + \mathsf{b}(\mathsf{F}v_{n-1}, \mathsf{F}v_n)\\ &= s\lambda_{n-1}\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1}) + (s\lambda_{n-1}+1)\mathsf{b}(\mathsf{F}v_{n-1}, \mathsf{F}v_n)\end{aligned}$$

Thereupon, by (7) we have

$$\begin{aligned}\mathsf{b}(v_n, \mathsf{F}v_n) &\leq s\lambda_{n-1}\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1}) + (s\lambda_{n-1}+1)\left(\kappa_1 \tfrac{\mathsf{b}(v_{n-1}, v_n)\mathsf{b}(v_n, \mathsf{F}v_n)}{\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1})} + \kappa_2 \mathsf{b}(v_{n-1}, v_n)\right)\\ &\leq s\lambda_{n-1}\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1}) +\\ &\quad + (s\lambda_{n-1}+1)\left(\kappa_1 \tfrac{(1-\lambda_{n-1})\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1})\mathsf{b}(v_n, \mathsf{F}v_n)}{\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1})} + \kappa_2(1-\lambda_{n-1})\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1})\right)\\ &= s\lambda_{n-1}\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1}) + (s\lambda_{n-1}+1)\kappa_1(1-\lambda_{n-1})\mathsf{b}(v_n, \mathsf{F}v_n) +\\ &\quad + (s\lambda_{n-1}+1)\kappa_2(1-\lambda_{n-1})\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1})\\ &\leq s\lambda_{n-1}\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1}) + (s\lambda_{n-1}+1)\kappa_1\mathsf{b}(v_n, \mathsf{F}v_n) + (s\lambda_{n-1}+1)\kappa_2\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1})\end{aligned}$$

Therefore,

$$\mathsf{b}(v_n, \mathsf{F}v_n) \leq \frac{s\lambda_{n-1}(1+\kappa_2)+\kappa_2}{1-(s\lambda_{n-1}+1)\kappa_1}\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1}). \tag{9}$$

Denoting $C_n = \frac{s\lambda_{n-1}(1+\kappa_2)+\kappa_2}{1-(s\lambda_{n-1}+1)\kappa_1}$, by (2) we get $C_n < \frac{1}{s}$, when $s > 1$ and then

$$\mathsf{b}(v_n, \mathsf{F}v_n) \leq C_{n-1}\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1}) \leq ... \leq \prod_{j=0}^{n-1} C_j \mathsf{b}(v_0, \mathsf{F}v_0) = K \cdot \prod_{j=0}^{n-1} C_j < K\frac{1}{s^{n-1}}.$$

From the above inequality, on one hand we conclude that

$$\lim_{n\to\infty} \mathsf{b}(v_n, \mathsf{F}v_n) = 0 \tag{10}$$

and on the other hand, returning in (8) we have

$$\mathsf{b}(v_n, v_{n+1}) \leq (1-\lambda_n)\prod_{j=0}^{n-1} C_j \cdot K \leq \frac{1}{2s^{n+1}} \cdot K.$$

Furthermore, by Lemma 1.1 we have that $\{v_n\}$ is a Cauchy sequence on U. Thus, using the completeness of U, we get there exists $v_* \in \mathsf{U}$ such that $\lim_{n\to\infty} \mathsf{b}(v_n, v_*) = 0$. Now, supposing that $v_* \neq \mathsf{F}v_*$ and using (m_4), (6) and the assumption (3), we have

$$\begin{aligned}0 < \mathsf{b}(\mathsf{F}v_*, v_*) &\leq s[\mathsf{b}(\mathsf{F}v_*, \mathsf{F}v_n) + \mathsf{b}(\mathsf{F}v_n, v_*)]\\ &\leq s\mathsf{b}(\mathsf{F}v_*, \mathsf{F}v_n) + s^2\mathsf{b}(\mathsf{F}v_n, v_n) + s^2\mathsf{b}(v_n, v_*)\\ &\leq s\alpha(v_*, v_n)\mathsf{b}(\mathsf{F}v_*, \mathsf{F}v_n) + s^2\mathsf{b}(\mathsf{F}v_n, v_n) + s^2\mathsf{b}(v_n, v_*)\\ &\leq s[\kappa_1 \tfrac{\mathsf{b}(v_*, v_n)\mathsf{b}(v_n, \mathsf{F}v_n)}{\mathsf{b}(v_*, \mathsf{F}v_*)} + \kappa_2\mathsf{b}(v_*, v_n)] + s^2\mathsf{b}(\mathsf{F}v_n, v_n) + s^2\mathsf{b}(v_n, v_*).\end{aligned} \tag{11}$$

Letting $n \to \infty$ in the above inequality and keeping in mind (10) and (11) we get $\mathsf{b}(\mathsf{F}v_*, v_*) = 0$, which shows that v_* is a fixed point of the mapping F. □

Example 2.1. *Let* $\mathsf{U} = [0, 4]$ *and the mapping* $\mathsf{F} : \mathsf{U} \to \mathsf{U}$ *defined as*

$$\mathsf{F}v = \begin{cases} 0, & \text{for } v \in [0,1) \cup (1,2) \cup (2,4) \\ 1, & \text{for } v \in \{1,2\} \\ 2 & \text{for } v = 4 \end{cases}$$

Let $\mathsf{b} : \mathsf{U} \times \mathsf{U}[0, +\infty)$, *where* $\mathsf{b}(v, o) = (v - o)^2$ *and* $\mathsf{w} : \mathsf{U} \times \mathsf{U} \times \left\{\frac{1}{17}\right\} \to \mathsf{U}$, $\mathsf{w}(v, o) = \frac{v+16o}{17}$. *Thus, by Example 1.2, we have that the triplet* $(\mathsf{U}, \mathsf{b}, \mathsf{w})$ *forms a convex* b*-metric space.*
Let the mapping $\alpha : \mathsf{U} \times \mathsf{U} \to [0, +\infty)$, *defined as:*

$$\alpha(v, o) = \begin{cases} 2, & \text{for } (v, o) \in [0, 1] \\ 1, & \text{for } (v, o) = (2, 4) \\ 3, & \text{for } (v, o) = \left(\frac{18}{17}, \frac{36}{17}\right) \\ 0, & \text{otherwise} \end{cases} .$$

First of all, let's check that the mapping F *is* α-w *admissible.*

(1) *For* $v, o \in [0, 1]$, *we have* $\mathsf{w}(v, \mathsf{F}v, \frac{1}{17}) = \frac{v}{17} \in [0, 1]$. *So,*
$$\alpha(v, o) = 2 \Rightarrow \alpha(\mathsf{w}(v, \mathsf{F}v, \frac{1}{17}), \mathsf{w}(o, \mathsf{F}o, \frac{1}{17})) = 2;$$
(2) *For* $(v, o) = (2, 4)$, *since* $\mathsf{w}(2, \mathsf{F}2, \frac{1}{17}) = \frac{2+16}{17} = \frac{18}{17}$ *and* $\mathsf{w}(4, \mathsf{F}4, \frac{1}{17}) = \frac{4+32}{17} = \frac{36}{17}$, *we have*
$$\alpha(2, 4) = 1 \Rightarrow \alpha(\mathsf{w}(2, \mathsf{F}2, \frac{1}{17}), \mathsf{w}(4, \mathsf{F}4, \frac{1}{17})) = \alpha(\frac{18}{17}, \frac{36}{17}) = 3;$$
(3) *For* $(v, o) = (\frac{18}{17}, \frac{36}{17})$, *since* $\mathsf{w}(\frac{18}{17}, \mathsf{F}\frac{18}{17}, \frac{1}{17}) = \frac{18}{17^2} < 1$ *and* $\mathsf{w}(\frac{36}{17}, \mathsf{F}\frac{36}{17}, \frac{1}{17}) = \frac{36}{17^2} < 1$, *we have*
$$\alpha(\frac{18}{17}, \frac{36}{17}) = 3 \Rightarrow \alpha(\frac{18}{17^2}, \frac{36}{17^2}) = 2.$$

Letting $u_0 = 0$, *since* $\alpha(0, 0) = 2$ *and* $\mathsf{b}(0, \mathsf{F}(0)) = 0$, *we have* $v_1 = \frac{v_0+16\mathsf{F}v_0}{17} = 0, ..., v_n = 0$. *Consequently,* $v_n \to 0$ *as* $n \to \infty$.
Letting $v_0 = 1$, *since* $\mathsf{b}(v_0, \mathsf{F}v_0) = 0$, *we have* $v_1 = \frac{1+16}{17} = 1, ..., v_n = 1$. *Then,* $\alpha(v_0, v_1) = \alpha(1, 1) = 2$ *and* $v_n \to 1$ *as* $n \to \infty$. *Thus, the assumptions (1) and (3) hold.*
Choosing $\kappa_1 = \kappa_2 = \frac{1}{34}$, *and since* $\lambda_n = \lambda = \frac{1}{17}$, *and taking into account the definition of function* α, *we have:*

(1) *For* $(v, o) \in (0, 1)$, *since* $\mathsf{F}v = 0$, *the inequality (6 is obviously satisfied.*
(2) *For* $(v, o) = (2, 4)$, *we have*
$$\mathsf{b}(2, 4) = 4,\ \mathsf{b}(\mathsf{F}2, \mathsf{F}4) = \mathsf{b}(1, 2) = 1,\ \mathsf{b}(2, \mathsf{F}2) = 1,\ \mathsf{b}(4, \mathsf{F}4) = \mathsf{b}(4, 2) = 4.$$
Then,
$$1 = \alpha(2, 4)\mathsf{b}(\mathsf{F}2, \mathsf{F}4) \leq \frac{1}{34}\frac{81}{1} + \frac{1}{34} = \kappa_1 \frac{\mathsf{b}(2, 4)\mathsf{b}(4, \mathsf{F}4)}{\mathsf{b}(2, \mathsf{F}2)} + \kappa_2 \mathsf{b}(2, 4)$$
so the inequality (6) holds.
(3) *For* $(v, o) = (\frac{18}{17}, \frac{36}{17})$, *we have* $\mathsf{b}(\mathsf{F}\frac{18}{17}, \mathsf{F}\frac{36}{17}) = 0$ *and of course, (6) holds.*

Therefore, by Theorem2.1 the mapping F *has fixed points, these are* $v = 0$ *and* $o = 1$.
We remark that, letting for example $v = 2$ *and* $o = 4$, *we have*
$$\mathsf{b}(\mathsf{F}2, \mathsf{F}4) = \mathsf{b}(1, 2) = 1 \leq 4\kappa = \kappa \mathsf{b}(2, 4)$$
gives us $\kappa \geq \frac{1}{4}$. *So Theorem (1.1) can not be applied (there is the condition* $\kappa < 1s^4 = 1/16$ *in our case.)*
Also, since from
$$\mathsf{b}(\mathsf{F}2, \mathsf{F}4) = \mathsf{b}(1, 2) = 1 \leq 5\kappa = \kappa[\mathsf{b}(2, \mathsf{F}2) + \mathsf{b}(4, \mathsf{F}4)]$$

it follows $\kappa \geq 1/5$, *neither Theorem 1.2 can not be applied (the condition* $\kappa < \frac{1}{4s^2} = \frac{1}{16}$ *is not satisfied.*

Corollary 2.1. *On a complete convex* b*-metric space* $(\mathsf{U}, \mathsf{b}, \mathsf{w})$ *with* $s > 1$, *let* $\mathsf{F} : \mathsf{U} \to \mathsf{U}$ *be a mapping such that there exist* $\kappa_1, \kappa_2 \in [0, 1)$ *such that*

$$\mathsf{b}(\mathsf{F}v, \mathsf{F}o) \leq \kappa_1 \frac{\mathsf{b}(v, o)\mathsf{b}(o, \mathsf{F}o)}{\mathsf{b}(v, \mathsf{F}v)} + \kappa_2 \mathsf{b}(v, o), \tag{12}$$

for all $v, o \in \mathsf{U} \setminus Fix_{\mathsf{F}}\mathsf{U}$. *If there exists* $v_0 \in \mathsf{U}$ *such that* $\mathsf{b}(v_0, \mathsf{F}v_0) < \infty$, *let* $\{v_n\}$ *be the sequence defined by* $v_n = \mathsf{w}(v_{n-1}, \mathsf{F}v_{n-1}, \lambda_{n-1})$, $0 \leq \lambda_{n-1} \leq 1$ *for any* $n \in \mathbb{N}$. *Then, the mapping* F *has a fixed point if* $\kappa_1 + \kappa_2 \leq \frac{1}{4s^2}$ *and* $\lambda_n \leq \frac{1}{4s^2}$.

Proof. Letting $\alpha(u, v) = 1$ in Theorem 2.1 the proof follows immediately. □

Theorem 2.2. *On a complete convex* b*-metric space* $(\mathsf{U}, \mathsf{b}, \mathsf{w})$, *let* $\mathsf{F} : \mathsf{U} \to \mathsf{U}$ *be an* α-w-*admissible mapping such that there exist* $\kappa_1, \kappa_2 \in [0, 1)$ *with the property that*

$$\alpha(v, o)\mathsf{b}(\mathsf{F}v, \mathsf{F}o) \leq \kappa_1 \frac{[\mathsf{b}(v, o) + 1]\mathsf{b}(o, \mathsf{F}o)}{\mathsf{b}(v, \mathsf{F}v) + 1} + \kappa_2 \mathsf{b}(v, o), \tag{13}$$

for all $v, o \in \mathsf{U}$. *Suppose that:*

1. *there exists* $v_0 \in \mathsf{U}$ *such that* $\mathsf{b}(v_0, \mathsf{F}v_0) < \infty$*and* $\alpha(v_0, v_1) \geq 1$, *where* $\{v_n\}$ *is the sequence defined by* $v_n = \mathsf{w}(v_{n-1}, \mathsf{F}v_{n-1}, \lambda_{n-1})$, $0 \leq \lambda_{n-1} \leq 1$ *for any* $n \in \mathbb{N}$*;*
2. $\kappa_1 + \kappa_2 \leq \frac{1}{4s^2}$ *and* $\lambda_n \leq \frac{1}{4s^2}$*;*
3. $\alpha(v_*, v_n) \geq 1$ *for any sequence* $\{v_n\}$ *in* U *such that* $\alpha(v_n, v_{n+1}) \geq 1$ *and* $v_n \to v_*$ *as* $n \to \infty$.

Then, the mapping F *has a fixed point. Moreover, if* $\alpha(o_*, v_*) \geq 1$ *for every* $o_*, v_* \in Fix_{\mathsf{F}}(\mathsf{U})$, *then the fixed point of* F *is unique.*

Proof. Let $v_0, v_1 \in \mathsf{U}$ satisfying the conditions in (1). As in the previous proof, we construct the sequence $\{v_n\}$ in U as

$$v_n = \mathsf{w}(v_{n-1}, \mathsf{F}v_{n-1}, \lambda_{n-1}),$$

where $\lambda_{n-1} \in [0, 1]$,for any$n \in \mathbb{N}$. Thus, since $\mathsf{b}(v_n, v_{n+1}) \leq (1 - \lambda_n)\mathsf{b}(v_n, \mathsf{F}v_n)$, for any $n \in \mathbb{N}$, we have

$$\begin{aligned}
\mathsf{b}(v_n, \mathsf{F}v_n) &= \mathsf{b}(\mathsf{w}(v_{n-1}, \mathsf{F}v_{n-1}, \lambda_{n-1}), \mathsf{F}v_n) \\
&\leq \lambda_{n-1}\mathsf{b}(v_{n-1}, \mathsf{F}v_n) + (1 - \lambda_{n-1})\mathsf{b}(\mathsf{F}v_{n-1}, \mathsf{F}v_n) \\
&\leq s\lambda_{n-1}\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1}) + s\lambda_{n-1}\mathsf{b}(\mathsf{F}v_{n-1}, \mathsf{F}v_n) + \mathsf{b}(\mathsf{F}v_{n-1}, \mathsf{F}v_n) \\
&= s\lambda_{n-1}\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1}) + (s\lambda_{n-1} + 1)\mathsf{b}(\mathsf{F}v_{n-1}, \mathsf{F}v_n) \\
&\leq s\lambda_{n-1}\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1}) + (s\lambda_{n-1} + 1)\left(\kappa_1 \tfrac{[(1-\lambda_{n-1})\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1})+1]\mathsf{b}(v_n, \mathsf{F}v_n)}{\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1})+1} + \right. \\
&\quad \left. + \kappa_2(1 - \lambda_{n-1})\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1})\right) \\
&= s\lambda_{n-1}\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1}) + (s\lambda_{n-1} + 1)\kappa_1(1 - \lambda_{n-1})\mathsf{b}(v_n, \mathsf{F}v_n) + \\
&\quad + (s\lambda_{n-1} + 1)\kappa_2(1 - \lambda_{n-1})\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1}) \\
&\leq s\lambda_{n-1}\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1}) + (s\lambda_{n-1} + 1)\kappa_1\mathsf{b}(v_n, \mathsf{F}v_n) + \\
&\quad + (s\lambda_{n-1} + 1)\kappa_2\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1}).
\end{aligned}$$

Therefore,

$$\mathsf{b}(v_n, \mathsf{F}v_n) \leq \frac{s\lambda_{n-1}(1+\kappa_2)+\kappa_2}{1-(s\lambda_{n-1}+1)\kappa_1}\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1}).$$

Consequently, by a verbatim repetition of the lines of the previous proof we obtain that $\lim_{n\to\infty}(\mathsf{b}(v_n, \mathsf{F}v_n) = 0$ and also, the sequence $\{v_n\}$ is Cauchy on a complete convex b-metric space, so, there exists $v_* \in \mathsf{U}$ such that $v_n \to v_*$ as $n \to \infty$.

We claim that $v_* \in Fix_{\mathsf{F}}(\mathsf{U})$. Supposing on the contrary,

$$\begin{aligned} 0 < \mathsf{b}(\mathsf{F}v_*, v_*) &\leq s[\mathsf{b}(\mathsf{F}v_*, \mathsf{F}v_n) + \mathsf{b}(\mathsf{F}v_n, v_*)] \\ &\leq s\mathsf{b}(\mathsf{F}v_*, \mathsf{F}v_n) + s^2\mathsf{b}(\mathsf{F}v_n, v_n) + s^2\mathsf{b}(v_n, v_*) \\ &\leq s\alpha(v_*, v_n)\mathsf{b}(\mathsf{F}v_*, \mathsf{F}v_n) + s^2\mathsf{b}(\mathsf{F}v_n, v_n) + s^2\mathsf{b}(v_n, v_*) \\ &\leq s[\kappa_1\tfrac{[\mathsf{b}(v_*, v_n)+1]\mathsf{b}(v_n, \mathsf{F}v_n)}{\mathsf{b}(v_*, \mathsf{F}v_*)+1} + \kappa_2\mathsf{b}(v_*, v_n)] + s^2\mathsf{b}(\mathsf{F}v_n, v_n) + s^2\mathsf{b}(v_n, v_*). \end{aligned}$$

Since the right part o this inequality tends to $\mathsf{b}(\mathsf{F}v_*, v_*)$, as $n \to \infty$, we get $\mathsf{b}(\mathsf{F}v_*, v_*) = 0$. To prove the uniqueness of the fixed point, we assume by contradiction, that there exist $o_*, v_* \in Fix_{\mathsf{F}}(\mathsf{U})$, with $o_* \neq v_*$. Using the supplementary condition, $\alpha(o_*, v_*) \geq 1$ for any $o_*, v_* \in Fix_{\mathsf{F}}(\mathsf{U})$, by (6) we have

$$\begin{aligned} 0 < \mathsf{b}(o_*, v_*) &\leq \alpha(o_*, v_*)\mathsf{b}(\mathsf{F}o_*, \mathsf{F}v_*) \leq \kappa_1\tfrac{(\mathsf{b}(o_*, v_*)+1)\mathsf{b}(v_*, \mathsf{F}v_*)}{\mathsf{b}(o_*, \mathsf{F}o_*)+1} + \kappa_2\mathsf{b}(o_*, v_*) \\ &= \kappa_2\mathsf{b}(o_*, v_*) < \mathsf{b}(o_*, v_*), \end{aligned}$$

which is a contradiction. Therefore, $o_* = v_*$. □

Example 2.2. *Let* $\mathsf{U} = [0, 8]$, *the* b-*metric* $\mathsf{b} : \mathsf{U}\times\mathsf{U} \to [0, +\infty)$, $\mathsf{b}(v-o) = (v-o)^2$, *the function* $\mathsf{w} : \mathsf{U}\times\mathsf{U}\times\left\{\frac{1}{17}\right\}$ *and a mapping* $\mathsf{F} : \mathsf{U} \to \mathsf{U}$, *where*

$$\mathsf{F}v = \begin{cases} 2, \text{ if } v \in [0,5) \\ \frac{v^2+1}{13}, \text{ if } v \in [5,6) \\ \frac{4v}{7}, \text{ if } v \in [6,8] \end{cases}$$

Let also, $\alpha : \mathsf{U}\times\mathsf{U} \to [0, +\infty)$,

$$\alpha(v, o) = \begin{cases} 2, \text{ if } v, o \in [0,5) \\ 1, \text{ if } (v, o) \in \{(2,7), (2,5)\} \\ 0, \text{ otherwise} \end{cases}$$

We can easily check the α-w-*admissibility of the mapping* F. *Indeed, for* $v, o \in [0, 5)$ *we have*

$$\mathsf{w}(v, \mathsf{F}v, \frac{1}{17}) = \frac{v+32}{17} < 1,$$

so

$$\alpha(v, o) = 2 \geq 1 \Rightarrow \alpha(\mathsf{w}(v, \mathsf{F}v, \frac{1}{17}), \mathsf{w}(o, \mathsf{F}o, \frac{1}{17}) = 2 \geq 1.$$

For $(v, u) = (2, 7)$, $\mathsf{w}(2, \mathsf{F}2, \frac{1}{17}) = \frac{2+32}{17} = 2$ *and* $\mathsf{w}(7, \mathsf{F}7, \frac{1}{17}) = \frac{7+32}{17} = \frac{71}{17}$. *Thus,*

$$\alpha(2, 7) = 1 \Rightarrow \alpha(\mathsf{w}(2, \mathsf{F}2, \frac{1}{17}), \mathsf{w}(4, \mathsf{F}4, \frac{1}{17})) = \alpha(2, \frac{71}{17}) = 2 \geq 1.$$

For $(v, u) = (2, 5)$, $\mathsf{w}(2, \mathsf{F}2, \frac{1}{17}) = 2$ *and* $\mathsf{w}(5, \mathsf{F}5, \frac{1}{17}) = \frac{5+32}{17} = \frac{37}{17}$. *Thus,*

$$\alpha(2, 5) = 1 \Rightarrow \alpha(\mathsf{w}(2, \mathsf{F}2, \frac{1}{17}), \mathsf{w}(5, \mathsf{F}5, \frac{1}{17})) = \alpha(2, \frac{37}{17}) = 2 \geq 1.$$

Next, choosing $v_0 = 2$, *we have* $\alpha(2,2) = \alpha(2, \mathsf{F}2) = 2$, $\mathsf{b}(2, \mathsf{F}2) = 0$ *and the sequence*

$$\begin{aligned} v_1 &= \tfrac{v_0+16\mathsf{F}v_0}{17} = 2; \\ v_2 &= \tfrac{v_1+16\mathsf{F}v_1}{17} = 2; \\ &\dots \\ v_{n-1} &= \tfrac{v_n+16\mathsf{F}v_n}{17} = 2. \end{aligned}$$

Moreover, $v_n \to 2$ *as* $n \to \infty$ *and* $\alpha(2, v_n) = 2 \geq 1$. *As a last step, we have to check (13). Taking into account the definitions of* F *and* α *we will discuss just the following two cases.*

(1) *For* $(v, o) = [0, 5) \cup \{(2, 5)\}$, *we have* $\mathsf{b}(\mathsf{F}v, \mathsf{F}o) = \mathsf{b}(2, 2) = 0$ *and then (ref1T2) holds;*
(2) *For* $(v, o) = (2, 7)$, *we have*

$$\mathsf{b}(2,7) = 25,\ \mathsf{b}(\mathsf{F}2, \mathsf{F}7) = \mathsf{b}(2,4) = 4,\ \mathsf{b}(2, \mathsf{F}2) = \mathsf{b}(2,2) = 0,\ \mathsf{b}(7, \mathsf{F}7) = \mathsf{b}(7,4) = 9.$$

Then,

$$\alpha(2,7)\mathsf{b}(\mathsf{F}2, \mathsf{F}7) = 4 \leq \frac{259}{34} = \kappa_1 + 25\kappa_2 = \kappa_1 \frac{(\mathsf{b}(2,7)+1)\mathsf{b}(7, \mathsf{F}7)}{\mathsf{b}(2, \mathsf{F}2)+1} + \kappa_2 \mathsf{b}(2,7)$$

(we choose $\kappa_1 = \kappa_2 = \frac{1}{34}$.*) Consequently, all the assumption of Theorem 2.2 are satisfied and* $v = 2$ *is the unique fixed point of* F.
We can also mention that for example, when $v = 2$ *and* $o = 7$ *the Theorem 1.1 respectively 1.2 can not be applied.*

Corollary 2.2. *On a complete convex* b*-metric space* $(\mathsf{U}, \mathsf{b}, \mathsf{w})$ *with* $s > 1$, *let* $\mathsf{F} : \mathsf{U} \to \mathsf{U}$ *be a mapping such that there exist* $\kappa_1, \kappa_2 \in [0, 1)$ *such that*

$$\mathsf{b}(\mathsf{F}v, \mathsf{F}o) \leq \kappa_1 \frac{[\mathsf{b}(v, o)+1]\mathsf{b}(o, \mathsf{F}o)}{\mathsf{b}(v, \mathsf{F}v)+1} + \kappa_2 \mathsf{b}(v, o), \tag{14}$$

for all $v, o \in \mathsf{U}$. *If there exists* $v_0 \in \mathsf{U}$ *such that* $\mathsf{b}(v_0, \mathsf{F}v_0) < \infty$, *let* $\{v_n\}$ *be the sequence defined by* $v_n = \mathsf{w}(v_{n-1}, \mathsf{F}v_{n-1}, \lambda_{n-1})$, $0 \leq \lambda_{n-1} \leq 1$ *for any* $n \in \mathbb{N}$. *Then, the mapping* F *has a unique fixed point if* $\kappa_1 + \kappa_2 \leq \frac{1}{4s^2}$ *and* $\lambda_n \leq \frac{1}{4s^2}$.

$$\mathsf{b}(\mathsf{F}v, \mathsf{F}o) \leq \kappa_1 \frac{[\mathsf{b}(v, o)+1]\mathsf{b}(o, \mathsf{F}o)}{\mathsf{b}(v, \mathsf{F}v)+1} + \kappa_2 \mathsf{b}(v, o), \tag{15}$$

for all $v, o \in \mathsf{U}$, *then the mapping* F *has a unique fixed point.*

Proof. Let $\alpha(v, o) = 1$ in Theorem 2.2. □

Theorem 2.3. *On a complete convex* b*-metric space* $(\mathsf{U}, \mathsf{b}, \mathsf{w})$, *let* $\mathsf{F} : \mathsf{U} \to \mathsf{U}$ *be an* α-w-*admissible mapping such that there exists* $\kappa \in [0, 1)$ *with the property that*

$$\alpha(v, o)\mathsf{b}(\mathsf{F}v, \mathsf{F}o) \leq \kappa \frac{\mathsf{b}(v, \mathsf{F}o)\mathsf{b}(v, \mathsf{F}v) + \mathsf{b}(o, \mathsf{F}v)\mathsf{b}(o, \mathsf{F}o)}{s \cdot \max\{\mathsf{b}(v, \mathsf{F}v), \mathsf{b}(o, \mathsf{F}o)\}}, \tag{16}$$

for all $v, o \in \mathsf{U} \setminus Fix_{\mathsf{F}}(\mathsf{U})$. *Suppose that:*

(1) *there exists* $v_0 \in \mathsf{U}$ *such that* $\mathsf{b}(v_0, \mathsf{F}v_0) < \infty$ *and* $\alpha(v_0, v_1) \geq 1$, *where* $\{v_n\}$ *is the sequence defined by* $v_n = \mathsf{w}(v_{n-1}, \mathsf{F}v_{n-1}, \lambda_{n-1})$ *for any* $n \in \mathbb{N}$;
(2) $\kappa \leq \frac{1}{4s^2}$ *and* $\lambda_n \leq \frac{1}{4s^2}$;
(3) $\alpha(v_*, v_n) \geq 1$ *for any sequence* $\{v_n\}$ *in* U *such that* $\alpha(v_n, v_{n+1}) \geq 1$ *and* $v_n \to v_*$ *as* $n \to \infty$.

Then, the mapping F *has a fixed point.*

Proof. As in the previous consideration, starting with two given points $v_0, v_1 \in \mathsf{U}$ such that $\mathsf{b}(v\_0, \mathsf{F}v_0) < \infty$ and, also $\alpha(v_0, v_1) \geq 1$, we consider the sequence $\{v_n\}$ in U, where $v_n = \mathsf{w}(v_{n-1}, \mathsf{F}v_{n-1}, \lambda_{n-1})$, for $\lambda_{n-1} \in [0,1]$, $n \in \mathbb{N}$. Since by Lemma 2.1 we know that $\alpha(v_n, v_{n+1}) \geq 1$ for any $n \in \mathbb{N}$, taking $v = v_{n-1}$ and $o = v_n$ in (16) we get

$$\begin{aligned}
\mathsf{b}(\mathsf{F}v_{n-1}, \mathsf{F}v_n) &\leq \alpha(v_{n-1}, v_n)\mathsf{b}(\mathsf{F}v_{n-1}, \mathsf{F}v_n) \\
&\leq \kappa \tfrac{\mathsf{b}(v_{n-1},\mathsf{F}v_n)\mathsf{b}(v_{n-1},\mathsf{F}v_{n-1})+\mathsf{b}(v_n,\mathsf{F}v_{n-1})\mathsf{b}(v_n,\mathsf{F}v_n)}{s\max\{\mathsf{b}(v_{n-1},\mathsf{F}v_{n-1}),\mathsf{b}(v_n,\mathsf{F}v_n)\}} \leq \kappa \tfrac{\mathsf{b}(v_{n-1},\mathsf{F}v_n)+\mathsf{b}(v_n,\mathsf{F}v_{n-1})}{s} \\
&\leq \kappa \tfrac{s\mathsf{b}(v_{n-1},v_n)+s\mathsf{b}(v_n,\mathsf{F}v_n)+\mathsf{b}(\mathsf{w}(v_{n-1},\mathsf{F}v_{n-1},\lambda_{n-1}),\mathsf{F}v_{n-1})}{s} \\
&\leq \kappa \tfrac{s\mathsf{b}(v_{n-1},v_n)+s\mathsf{b}(v_n,\mathsf{F}v_n)+\lambda_{n-1}\mathsf{b}(\mathsf{w}(v_{n-1},\mathsf{F}v_{n-1})}{s} \\
&\leq \kappa[(1-\lambda_{n-1})\mathsf{b}(v_{n-1},\mathsf{F}v_{n-1}) + \mathsf{b}(v_n, \mathsf{F}v_n) + \lambda_{n-1}\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1})] \\
&\leq \kappa[\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1}) + \mathsf{b}(v_n, \mathsf{F}v_n)].
\end{aligned} \tag{17}$$

On the other hand,

$$\begin{aligned}
\mathsf{b}(v_n, \mathsf{F}v_n) &= \mathsf{b}(\mathsf{w}(v_{n-1}, \mathsf{F}v_{n-1}, \lambda_{n-1}), \mathsf{F}v_n) \\
&\leq \lambda_{n-1}\mathsf{b}(v_{n-1}, \mathsf{F}v_n) + (1-\lambda_{n-1})\mathsf{b}(\mathsf{F}v_{n-1}, \mathsf{F}v_n) \\
&\leq s\lambda_{n-1}\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1}) + s\lambda_{n-1}\mathsf{b}(\mathsf{F}v_{n-1}, \mathsf{F}v_n) + \mathsf{b}(\mathsf{F}v_{n-1}, \mathsf{F}v_n) \\
&= s\lambda_{n-1}\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1}) + (s\lambda_{n-1}+1)\mathsf{b}(\mathsf{F}v_{n-1}, \mathsf{F}v_n) \\
&\leq s\lambda_{n-1}\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1}) + (s\lambda_{n-1}+1)\kappa\,[\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1}) + \mathsf{b}(v_n, \mathsf{F}v_n)]
\end{aligned}$$

and then

$$\mathsf{b}(v_n, \mathsf{F}v_n) \leq \frac{s\lambda_{n-1}(1+\kappa)+\kappa}{1-(s\lambda_{n-1}+1)\kappa}\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1}).$$

Letting $\mathcal{C}_n = \frac{s\lambda_{n-1}(1+\kappa)+\kappa}{1-(s\lambda_{n-1}+1)\kappa}$, for any $n \in \mathbb{N}$, under the assumption (2), we can observe that $\mathcal{C}_n < \frac{1}{s}$. Therefore, $\lim_{n\to\infty} \mathsf{b}(v_n, \mathsf{F}v_n) = 0$ and moreover, since

$$\mathsf{b}(v_n, v_{n-1}) \leq (1-\lambda_{n-1})\mathsf{b}(v_{n-1}, \mathsf{F}v_{n-1}) \leq \beta_{n-1}\prod_{i=0}^{n-1}\mathcal{C}_i \cdot \mathsf{b}(v_0, \mathsf{F}v_0),$$

by Lemma 1.1 it follows that $\{v_n\}$ is a Cauchy sequence on a complete convex$\mathsf{b}$-metric space, so that it is convergent (here $\beta_n = 1-\lambda_n$). Let $v_* \in \mathsf{U}$ be the limit of the sequence $\{v_n\}$. We claim that this point is in fact a fixed point of F. Indeed, if it is not, then keeping in mind the assumption (3),

$$\begin{aligned}
0 < \mathsf{b}(\mathsf{F}v_*, v_*) &\leq s\mathsf{b}(\mathsf{F}v_*, \mathsf{F}v_n) + s^2\mathsf{b}(\mathsf{F}v_n, v_n) + s^2\mathsf{b}(v_n, v_*) \\
&\leq s\alpha(v_*, v_n))\mathsf{b}(\mathsf{F}v_*, \mathsf{F}v_n) + s^2\mathsf{b}(\mathsf{F}v_n, v_n) + s^2\mathsf{b}(v_n, v_*) \\
&\leq s\kappa\tfrac{\mathsf{b}(v_*,\mathsf{F}v_n)\mathsf{b}(v_*,\mathsf{F}v_*)+\mathsf{b}(v_n,\mathsf{F}v_*)\mathsf{b}(v_n,\mathsf{F}v_n)}{s\cdot\max\{\mathsf{b}(v_n,\mathsf{F}v_n)\mathsf{b}(v_*,\mathsf{F}v_*)\}} + s^2\mathsf{b}(\mathsf{F}v_n, v_n) + s^2\mathsf{b}(v_n, v_*) \\
&\leq s\kappa\tfrac{\mathsf{b}(v_*,\mathsf{F}v_n)+\mathsf{b}(v_n,\mathsf{F}v_*)}{s} + s^2\mathsf{b}(\mathsf{F}v_n, v_n) + s^2\mathsf{b}(v_n, v_*) \\
&\leq s\kappa[\mathsf{b}(v_*, v_n) + \mathsf{b}(v_n, \mathsf{F}v_n) + \mathsf{b}(v_n, v_*) + \mathsf{b}(v_*, \mathsf{F}v_*)] + \\
&\qquad + s^2\mathsf{b}(\mathsf{F}v_n, v_n) + s^2\mathsf{b}(v_n, v_*).
\end{aligned}$$

Letting $n \to \infty$ in the above inequality, we get

$$0 < \mathsf{b}(\mathsf{F}v_*, v_*) \leq s\kappa \mathsf{b}(\mathsf{F}v_*, v_*) < \frac{1}{4s}\mathsf{b}(\mathsf{F}v_*, v_*),$$

which is a contradiction. Thereupon, $v_* = \mathsf{F}v_*$, that is $v_* \in Fix_{\mathsf{F}}(\mathsf{U})$.
The uniqueness of the fixed point it follows as in the previous proof. □

Corollary 2.3. *On a complete convex* b*-metric space* $(\mathsf{U}, \mathsf{b}, \mathsf{w})$ *with* $s > 1$*, let* $\mathsf{F} : \mathsf{U} \to \mathsf{U}$ *be a mapping such that there exists* $\kappa \in [0, 1)$ *such that*

$$\mathsf{b}(\mathsf{F}v, \mathsf{F}o) \leq \kappa \frac{\mathsf{b}(v, \mathsf{F}o)\mathsf{b}(v, \mathsf{F}v) + \mathsf{b}(o, \mathsf{F}v)\mathsf{b}(o, \mathsf{F}o)}{s \cdot \max\{\mathsf{b}(v, \mathsf{F}v), \mathsf{b}(o, \mathsf{F}o)\}}, \tag{18}$$

for all $v, o \in \mathsf{U} \setminus Fix_{\mathsf{F}}\mathsf{U}$*. If there exists* $v_0 \in \mathsf{U}$ *such that* $\mathsf{b}(v_0, \mathsf{F}v_0) < \infty$*, let* $\{v_n\}$ *be the sequence defined by* $v_n = \mathsf{w}(v_{n-1}, \mathsf{F}v_{n-1}, \lambda_{n-1})$*,* $0 \leq \lambda_{n-1} \leq 1$ *for any* $n \in \mathbb{N}$*. Then, the mapping* F *has a fixed point provided that* $\kappa \leq \frac{1}{4s^2}$ *and* $\lambda_n \leq \frac{1}{4s^2}$*.*

Proof. Let $\alpha(v, o) = 1$ in Theorem 2.3. □

3. Conclusion

In this paper, we discuss the existence and uniqueness of a fixed point of certain operators that providing inequalities with rational expressions in the setting of b-convex metric spaces. Although the notion of convexity has been considered in the metric structure, it is rarely used in the b-metric structure. Another interesting contribution of the paper is the usage of admissible mappings. This consideration is a candidate to initiate the new trends in the metric fixed point theory.

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