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CALDERÓN-ZYGMUND OPERATORS WITH KERNELS OF DINI'S TYPE AND THEIR MULTILINEAR COMMUTATORS ON GENERALIZED WEIGHTED MORREY SPACES

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Abstract. In this paper, we study the boundedness of the operators T and $T_{\vec{b}}$ on generalized weighted Morrey spaces $M_{p,\varphi}(w)$ with the weight function w belonging to Muckenhoupt's class $A_p(\mathbb{R}^n)$. We find the sufficient conditions on the pair (φ_1,φ_2) with $\vec{b} \in BMO^m(\mathbb{R}^n)$ and $w \in A_p(\mathbb{R}^n)$ which ensures the boundedness of the operators T and $T_{\vec{b}}$ from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $1 < p < \infty$.

Keywords: generalized weighted Morrey spaces, Calderón-Zygmund operator, A_p weights, commutator, BMO.

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1. Introduction

The theory of Calderón-Zygmund operators has played very important roles in modern harmonic analysis with lots of extensive applications in the others fields of mathematics, which has been extensively studied (see [1, 2, 3, 4, 20, 21, 29, 31, 35]). In particular, Yabuta introduced certain ω-type Calderón-Zygmund operators to facilitate his study of certain classes of pseudo-differential operators (see [34]). Let ω be a non-negative and non-decreasing function on $(0,\infty)$. We say that ω satisfies the *Dini* condition and wirte $\omega \in Dini$, if

$$\int_0^\infty \frac{\omega(t)}{t} dt < \infty. \tag{1}$$

A measurable function $K(\cdot,\cdot)$ on $\mathbb{R}^n \times \mathbb{R}^n$ is said to be a ω-type Calderón-Zygmund kernel if it satisfies

$$|K(x,y) \le C\,|x-y|^{-n} \tag{2}$$

and for all distinct $x, y \in \mathbb{R}^n$, and all z with $2|x-z| < |x-y|$, there exist positive constants C and γ such that

$$|K(x,y) - K(z,y)| + |K(y,x) - K(y,z)| \le C\omega\Big(\frac{|x-z|}{|x-y|}\Big)\,|x-y|^{-n}. \tag{3}$$

Definition 1.1. *Let T be a linear operator from $\mathcal{S}(\mathbb{R}^n)$ into its dual $\mathcal{S}'(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz class. One can say that T is a ω-type Calderón-Zygmund operator if it satisfies the following conditions:*

i) *T can be extended to be a bounded linear operator on $L_2(\mathbb{R}^n)$;*

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ii) *there is a ω-type Calderón-Zygmund kernel $K(x,y)$ such that*

$$Tf(x) = \int\limits_{\mathbb{R}^n} K(x,y)f(y)dy, \ \text{as } f \in C_c^\infty \text{ and } x \notin \operatorname{supp} f. \tag{4}$$

It is easy to see that the classical Calderón-Zygmund operator with standard kernel is a special case of ω-type operator T as $\omega(t) = t^\varepsilon$ with $0 < \varepsilon \le 1$. Given a locally integrable function b, the commutator generated by T and b is defined by

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x) = \int\limits_{\mathbb{R}^n} [b(x) - b(y)]K(x,y)f(y)dy. \tag{5}$$

Let $\vec{b} = (b_1, ..., b_m)$ and b_j, $1 \le j \le m$ be locally integrable functions when we consider multilinear commutators as defined by

$$T_{\vec{b}}f(x) = \int\limits_{\mathbb{R}^n} \prod_{j=1}^m (b_j(x) - b_j(y))K(x,y)f(y)dy. \tag{6}$$

Furthermore, if we take $b_i = b$, , $i = 1, \dots, m$, then we define the following integral equation

$$T_{\vec{b}}f(x) = \int\limits_{\mathbb{R}^n} (b(x) - b(y))^m K(x,y)f(y)dy = [b,T]^m f(x).$$

It is well known that Calderón-Zygmund operators play an important role in harmonic analysis (see [6, 7, 31]).

The classical Morrey spaces were introduced by Morrey [23] to study the local behavior of solutions to second-order elliptic partial differential equations. Moreover, various Morrey spaces are defined in the process of study. The first author, Mizuhara and Nakai [8, 24, 26] introduced generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$ (see, also [9, 10, 15, 16, 30]). Komori and Shirai [19] defined weighted Morrey spaces $L_{p,\kappa}(w)$. The first author in [11] gave a concept of the generalized weighted Morrey spaces $M_{p,\varphi}(\mathbb{R}^n, w)$ which could be viewed as extension of both $M_{p,\varphi}(\mathbb{R}^n)$ and $L_{p,\kappa}(w)$. In [11], the boundedness of the classical operators and their commutators in spaces $M_{p,\varphi}(\mathbb{R}^n, w)$ was also studied, see also [5, 13, 14, 17, 18, 27].

The main purpose of this paper is to establish a number of results concerning weighted Morrey boundedness of Calderón-Zygmund operators with kernels of mild regularity. Let T be a linear Calderón-Zygmund operator of type $\omega(t)$ with ω being nondecreasing and $\omega \in Dini$, but without assuming to be concave. We show that the ω-type Calderón-Zygmund operators T and their multlinear commutators $T_{\vec{b}}$ are bounded from one generalized weighted Morrey space $M_{p,\varphi_1}(w)$ to another $M_{p,\varphi_2}(w)$, $1 < p < \infty$. We find the sufficient conditions on the pair (φ_1, φ_2) with $\vec{b} \in BMO^m(\mathbb{R}^n)$ and $w \in A_p(\mathbb{R}^n)$ which ensures the boundedness of the operators T and $T_{\vec{b}}$ from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $1 < p < \infty$.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Generalized weighted Morrey spaces

We recall that a weight function w is in the Muckenhoupt's class $A_p(\mathbb{R}^n)$ [25], $1 < p < \infty$, if

$$[w]_{A_p} := \sup_B [w]_{A_p(B)} = \sup_B \left(\frac{1}{|B|} \int\limits_B w(x)dx \right) \left(\frac{1}{|B|} \int\limits_B w(x)^{1-p'}dx \right)^{p-1} \tag{7}$$

where the sup is taken with respect to all the balls B and $\frac{1}{p}+\frac{1}{p'}=1$. Note that, for all balls B by Hölder's inequality

$$[w]_{A_p(B)}^{1/p} = |B|^{-1}\|w\|_{L_1(B)}^{1/p}\,\|w^{-1/p}\|_{L_{p'}(B)} \geq 1. \tag{8}$$

For $p=1$, the class A_1 is defined by the condition $Mw(x) \leq Cw(x)$ with $[w]_{A_1} = \sup\limits_{x\in\mathbb{R}^n} \frac{Mw(x)}{w(x)}$, and for $p=\infty$ $A_\infty(\mathbb{R}^n) = \bigcup_{1\leq p<\infty} A_p(\mathbb{R}^n)$ and $[w]_{A_\infty} = \inf\limits_{1\leq p<\infty} [w]_{A_p}$.

We define the generalized weighed Morrey spaces as follows.

Definition 2.2. *Let $1 \leq p < \infty$, φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and w be non-negative measurable function on $\mathbb{R}^n$. We denote by $M_{p,\varphi}(w)$ the generalized weighted Morrey space, the space of all functions $f \in L_{p,w}^{\rm loc}(\mathbb{R}^n)$ with finite norm*

$$\|f\|_{M_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{L_{p,w}(B(x,r))},$$

where $L_{p,w}(B(x,r))$ denotes the weighted L_p-space of measurable functions f for which

$$\|f\|_{L_{p,w}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{L_{p,w}(\mathbb{R}^n)} = \left(\int\limits_{B(x,r)} |f(y)|^p w(y)dy\right)^{\frac{1}{p}}.$$

Furthermore, by $WM_{p,\varphi}(w)$ we denote the weak generalized weighted Morrey space of all functions $f \in WL_{p,w}^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{WL_{p,w}(B(x,r))} < \infty,$$

where $WL_{p,w}(B(x,r))$ denotes the weak $L_{p,w}$-space of measurable functions f for which

$$\|f\|_{WL_{p,w}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_{p,w}(\mathbb{R}^n)} = \sup_{t>0} t \left(\int\limits_{\{y\in B(x,r):\,|f(y)|>t\}} w(y)dy\right)^{\frac{1}{p}}.$$

Remark 2.3. *If $w \equiv 1$, then $M_{p,\varphi}(1) = M_{p,\varphi}$ is the generalized Morrey space; If $\varphi(x,r) \equiv w(B(x,r))^{\frac{\kappa-1}{p}}$, then $M_{p,\varphi}(w) = L_{p,\kappa}(w)$ is the weighted Morrey space; If $\varphi(x,r) \equiv v(B(x,r))^{\frac{\kappa}{p}} w(B(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(w) = L_{p,\kappa}(v,w)$ is the two weighted Morrey space; If $w \equiv 1$ and $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$ with $0<\lambda<n$, then $M_{p,\varphi}(w) = L_{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey space and $WM_{p,\varphi}(w) = WL_{p,\lambda}(\mathbb{R}^n)$ is the weak Morrey space; If $\varphi(x,r) \equiv w(B(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(w) = L_{p,w}(\mathbb{R}^n)$ is the weighted Lebesgue space.*

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w g(t) := \int\limits_t^\infty g(s)\, w(s)\, ds,\;\; H_w^* g(t) := \int\limits_t^\infty \left(1+\ln\frac{s}{t}\right)^m g(s)\, w(s)\, ds,\;\; 0<t<\infty,$$

where w is a weight. The following theorem was proved in [12].

Theorem 2.4. [12] *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t)\, H_w g(t) \leq C \sup_{t>0} v_1(t)\, g(t)$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0, \infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int\limits_t^{\infty} \frac{w(s)\,ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Theorem 2.5. [11] *Let v_1, v_2 and w be weights on $(0, \infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t)\, H_w^* g(t) \le C \sup_{t>0} v_1(t)\, g(t)$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0, \infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int\limits_t^{\infty} \Big(1 + \ln \frac{s}{t}\Big)^m \frac{w(s)\,ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty.$$

3. ω-type Calderón-Zygmund operators in the spaces $M_{p,\varphi}(\mathbb{R}^n, w)$

The following theorem was proved in [28].

Theorem 3.6. [28] *Let $1 \le p < \infty$, $w \in A_p(\mathbb{R}^n)$ and T be ω-type Calderón-Zygmund operator defined by (4) with ω satisfies (1). Then, the operator T is bounded on $L_{p,w}(\mathbb{R}^n)$ for $p > 1$ and bounded from $L_{1,w}(\mathbb{R}^n)$ into $WL_{1,w}(\mathbb{R}^n)$ for $p = 1$.*

The following weighted local estimates are valid (see [11]).

Theorem 3.7. *Let $1 \le p < \infty$, $w \in A_p(\mathbb{R}^n)$ and T be ω-type Calderón-Zygmund operator defined by (4) with ω satisfies (1). Then, for $p > 1$ the inequality*

$$\|Tf\|_{L_{p,w}(B)} \lesssim w(B)^{\frac{1}{p}} \int\limits_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\, w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}$$

holds for any ball $B = B(x_0, r)$ and for all $f \in L_{p,w}^{\rm loc}(\mathbb{R}^n)$.

Moreover, for $p = 1$ the inequality

$$\|Tf\|_{WL_{1,w}(B)} \lesssim w(B) \int\limits_{2r}^{\infty} \|f\|_{L_{1,w}(B(x_0,t))}\, w(B(x_0,t))^{-1} \frac{dt}{t} \tag{9}$$

holds for any ball $B = B(x_0, r)$ and for all $f \in L_{1,w}^{\rm loc}(\mathbb{R}^n)$.

Proof. Let $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r, $2B = B(x_0, 2r)$. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{\complement_{(2B)}}(y), \quad r > 0. \tag{10}$$

Then we have

$$\|Tf\|_{L_{p,w}(B)} \le \|Tf_1\|_{L_{p,w}(B)} + \|Tf_2\|_{L_{p,w}(B)}.$$

Since $f_1 \in L_p(w)$, $Tf_1 \in L_p(w)$ and from the boundedness of T in $L_p(w)$ (see Theorem 3.6) it follows that

$$\|Tf_1\|_{L_{p,w}(B)} \le \|Tf_1\|_{L_{p,w}} \le C\|f_1\|_{L_{p,w}} = C\|f\|_{L_{p,w}(2B)},$$

where constant $C > 0$ is independent of f.

It is clear that $x \in B$, $y \in {}^{\complement}(2B)$ implies $\frac{1}{2}|x_0 - y| \le |x - y| \le \frac{3}{2}|x_0 - y|$. We get

$$|Tf_2(x)| \le 2^n c_0 \int\limits_{\complement_{(2B)}} \frac{|f(y)|}{|x_0 - y|^n} dy \lesssim \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |f(y)| dy \frac{dt}{t^{n+1}}.$$

Applying Hölder's inequality, we get

$$\int\limits_{\complement_{(2B)}} \frac{|f(y)|}{|x_0-y|^n} dy \lesssim \int\limits_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\, w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}. \tag{11}$$

Moreover, for all $p \in [1,\infty)$ the inequality

$$\|Tf_2\|_{L_{p,w}(B)} \lesssim w(B)^{\frac{1}{p}} \int\limits_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\, w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t} \tag{12}$$

is valid. Thus

$$\|Tf\|_{L_{p,w}(B)} \lesssim \|f\|_{L_{p,w}(2B)} + w(B)^{\frac{1}{p}} \int\limits_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\, w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}$$

$$\lesssim w(B)^{\frac{1}{p}} \int\limits_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\, w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}.$$

Let $p = 1$. From the weak $(1,1)$ boundedness of T it follows that:

$$\begin{aligned} \|Tf_1\|_{WL_{1,w}(B)} &\le \|Tf_1\|_{WL_1(w)} \lesssim \|f_1\|_{L_{1,w}} = \|f\|_{L_{1,w}(2B)} \\ &\lesssim w(B) \int\limits_{2r}^{\infty} \|f\|_{L_{1,w}(B(x_0,t))}\, w(B(x_0,t))^{-1} \frac{dt}{t}. \end{aligned} \tag{13}$$

Then by (12) and (13) we get the inequality (9). □

Theorem 3.8. *Let $1 \le p < \infty$, $w \in A_p(\mathbb{R}^n)$, T be ω-type Calderón-Zygmund operator defined by (4) with ω satisfies (1), and (φ_1,φ_2) satisfy the condition*

$$\int\limits_{r}^{\infty} \frac{\operatorname*{ess\,inf}\limits_{t<s<\infty} \varphi_1(x,s) w(B(x,s))^{1/p}}{w(B(x,t))^{1/p}} \le C\varphi_2(x,r), \tag{14}$$

where C does not depend on x and r. Then the operator T is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $p > 1$ and from $M_{1,\varphi_1}(w)$ to $WM_{1,\varphi_2}(w)$ for $p = 1$.

Proof. For $p > 1$ from Theorem 2.4 and Theorem 3.7 we get

$$\|Tf\|_{M_{p,\varphi_2}(w)} \lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int\limits_{r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\, w(B(x,t))^{-\frac{1}{p}} \frac{dt}{t}$$

$$\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} w(B)^{-\frac{1}{p}} \|f\|_{L_{p,w}(B)} = \|f\|_{M_{p,\varphi_1}(w)}$$

and for $p = 1$

$$\|Tf\|_{WM_{1,\varphi_2}(w)} \lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int\limits_{r}^{\infty} \|f\|_{L_{1,w}(B(x_0,t))}\, w(B(x_0,t))^{-1} \frac{dt}{t}$$

$$\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} w(B)^{-1} \|f\|_{L_{1,w}(B)} = \|f\|_{M_{1,\varphi_1}(w)}.$$

□

Remark 3.9. *Let $0 \le \kappa < 1$. Assume that ψ is a positive increasing function defined in $(0,\infty)$ and satisfies the following $\mathcal{D}_\kappa$ condition :*

$$\frac{\psi(t_2)}{t_2^\kappa} \le C\frac{\psi(t_1)}{t_1^\kappa}, \text{ for any } 0 < t_1 < t_2 < \infty,$$

where $C > 0$ is a constant independent of t_1 and t_2. If $\varphi_1(x,r) = \varphi_2(x,r) = \psi(w(x,r))$ and ψ satisfy the $\mathcal{D}_\kappa$ condition, Theorems 3.7 and 3.8 were proved in [32]*. Also, in the case $\omega(t) = t^\varepsilon$ with $0 < \varepsilon \le 1$, Theorems 3.7 and 3.8 were proved in* [11].

4. Commutators of ω-type Calderón-Zygmund operators in the spaces $M_{p,\varphi}(\mathbb{R}^n, w)$

We recall the definition of the space of $BMO(\mathbb{R}^n)$.

Definition 4.10. *Suppose that $b \in L_1^{\text{loc}}(\mathbb{R}^n)$, and let*

$$\|b\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |b(y) - b_{B(x,r)}|dy < \infty,$$

where

$$b_{B(x,r)} = \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} b(y)dy.$$

Define

$$BMO(\mathbb{R}^n) = \{b \in L_1^{\text{loc}}(\mathbb{R}^n) \; : \; \|b\|_* < \infty\}.$$

Modulo constants, the space $BMO(\mathbb{R}^n)$ is a Banach space with respect to the norm $\|\cdot\|_*$.

The following lemma is proved in [11].

Lemma 4.11. [11]

(1) *Let $w \in A_\infty$ and $b \in BMO(\mathbb{R}^n)$. Let also $1 \le p < \infty$, $x \in \mathbb{R}^n$, $k > 0$ and $r_1, r_2 > 0$. Then,*

$$\Big(\frac{1}{w(B(x,r_1))} \int\limits_{B(x,r_1)} |b(y) - b_{B(x,r_2),w}|^{kp} w(y)dy\Big)^{\frac{1}{p}} \le C\Big(1 + \Big|\ln\frac{r_1}{r_2}\Big|\Big)^k \|b\|_*^k,$$

where $C > 0$ is independent of f, w, x, r_1 and r_2.

(2) *Let $w \in A_p$ and $b \in BMO(\mathbb{R}^n)$. Let also $1 < p < \infty$, $x \in \mathbb{R}^n$, $k > 0$ and $r_1, r_2 > 0$. Then,*

$$\Big(\frac{1}{w^{1-p'}(B(x,r_1))} \int\limits_{B(x,r_1)} |b(y) - b_{B(x,r_2),w}|^{kp'} w(y)^{1-p'} dy\Big)^{\frac{1}{p'}} \le C\Big(1 + \Big|\ln\frac{r_1}{r_2}\Big|\Big)^k \|b\|_*^k,$$

where $C > 0$ is independent of b, w, x, r_1 and r_2.

Since linear commutator has a greater degree of singularity than the corresponding ω-type Calderón-Zygmund operator, we need a slightly stronger version of condition

$$\int\limits_0^1 \frac{\omega(t)}{t}\Big(1 + \log\frac{1}{t}\Big)^m dt < \infty. \tag{15}$$

The following weighted endpoint estimate for commutator $T_{\vec{b}}$ of the ω-type Calderón-Zygmund operator was established in [33] under a stronger version of condition (15) assumed on ω, if $\vec{b} \in BMO^m(\mathbb{R}^n)$ (for the unweighted case, see [22]).

The following theorem was proved in [33].

Theorem 4.12. [33] *Let T be linear ω-CZO and $\vec{b} \in BMO^m(\mathbb{R}^n)$. If ω satisfies condition* (15) *and $w \in A_p(\mathbb{R}^n)$, $1 < p < \infty$, then there exists a constant $C > 0$ such that*

$$\|T_{\vec{b}} f\|_{L_{p,w}} \leq C\, \|\vec{b}\|_* \, \|f\|_{L_{p,w}},$$

where $\|\vec{b}\|_* = \prod_{j=1}^m \|b_j\|_*$.

The following weighted local estimates are valid (see [11]).

Theorem 4.13. *Let T be linear ω-CZO and $\vec{b} \in BMO^m(\mathbb{R}^n)$. Let also ω satisfies condition* (15) *and $w \in A_p(\mathbb{R}^n)$, $1 < p < \infty$. Then*

$$\|T_{\vec{b}} f\|_{L_{p,w}(B)} \leq C\, \|\vec{b}\|_* \, w(B)^{\frac{1}{p}} \int\limits_{2r}^{\infty} \ln^m \Big(e + \frac{t}{r}\Big) \|f\|_{L_{p,w}(B(x_0,t))}\, w(B(x_0,t))^{-1/p} \frac{dt}{t}$$

holds for any ball $B = B(x_0, r)$ and for all $f \in L^{\rm loc}_{p,w}(\mathbb{R}^n)$, where C does not depend on f, $x_0 \in \mathbb{R}^n$ and $r > 0$.

Proof. Let $p \in (1, \infty)$. For arbitrary $x_0 \in \mathbb{R}^n$ and $r > 0$, set $B = B(x_0, r)$. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement_{(2B)}}$. For all $f \in L^{\rm loc}_p(\mathbb{R}^n, w)$ we define

$$T_{\vec{b}} f(x) := T_{\vec{b},0} f_1(x) + \int\limits_{\mathbb{R}^n} \prod_{j=1}^m (b_j(x) - b_j(y)) K(x,y) f_2(y) dy, \tag{16}$$

here $T_{\vec{b},0}$ denotes the commutator as a bounded linear operator on $L_{p,w}(\mathbb{R}^n)$ with $1 \leq p < \infty$ and $w \in A_p(\mathbb{R}^n)$ (see [33]). It is easy to check that the definition of $T_{\vec{b}} f(x)$ does not depend on the choice of the ball B. First we show that $T_{\vec{b},0} f(x)$ is well-defined *a.e.* x and independent of the choice of B containing x. As $T_{\vec{b},0}$ is bounded on $L_{p,w}(\mathbb{R}^n)$ provided by Theorem 4.12 and $f_1 \in L_{p,w}(\mathbb{R}^n)$, $T_{\vec{b},0} f_1$ is well-defined.

Next, we show that the second-term of the right-hand side defining $T_{\vec{b}} f(x)$ converges absolutely for any $f \in L_{p,w}(\mathbb{R}^n)$ and almost every $x \in \mathbb{R}^n$.

Hence

$$\|T_{\vec{b}} f\|_{L_{p,w}(B)} \leq \|T_{\vec{b}} f_1\|_{L_{p,w}(B)} + \|T_{\vec{b}} f_2\|_{L_{p,w}(B)}.$$

From the boundedness of $T_{\vec{b}}$ in $L_{p,w}(\mathbb{R}^n)$ (see Theorem 4.12) it follows that:

$$\|T_{\vec{b}} f_1\|_{L_{p,w}(B)} \leq \|T_{\vec{b}} f_1\|_{L_{p,w}} \lesssim \|\vec{b}\|_* \, \|f_1\|_{L_{p,w}} = \|\vec{b}\|_* \, \|f\|_{L_{p,w}(2B)}.$$

For the term $\|T_{\vec{b}} f_2\|_{L_{p,w}(B)}$, without loss of generality, we can assume $m = 2$. Thus, the operator $T_{\vec{b}} f_2$ can be divided into four parts

$$\begin{aligned}
T_{\vec{b}} f_2(x) &= \big(b_1(x) - (b_1)_{B,w}\big)\big(b_2(x) - (b_2)_{B,w}\big) \int\limits_{\mathbb{R}^n} K(x,y) f_2(y) dy \\
&+ \int\limits_{\mathbb{R}^n} K(x,y) \big(b_1(y) - (b_1)_{B,w}\big)\big(b_2(y) - (b_2)_{B,w}\big) f_2(y) dy \\
&- \big(b_1(x) - (b_1)_{B,w}\big) \int\limits_{\mathbb{R}^n} K(x,y) \big(b_2(y) - (b_2)_{B,w}\big) f_2(y) dy \\
&- \big(b_2(x) - (b_2)_{B,w}\big) \int\limits_{\mathbb{R}^n} K(x,y) \big(b_1(y) - (b_1)_{B,w}\big) f_2(y) dy \\
&= I_1(x) + I_2(x) + I_3(x) + I_4(x).
\end{aligned}$$

For $x \in B$ we have

$$
\begin{aligned}
|T_{\vec{b}} f_2(x)| &\le |I_1(x) + |I_2(x)| + |I_3(x)| + |I_4(x)| \\
&\lesssim \left|b_1(x) - \left(b_1\right)_{B,w}\right| \left|b_2(x) - \left(b_2\right)_{B,w}\right| \int\limits_{\complement_{(2B)}} \frac{|f(y)|}{|x_0 - y|^n} dy \\
&+ \int\limits_{\complement_{(2B)}} \left|b_1(y) - \left(b_1\right)_{B,w}\right| \left|b_2(y) - \left(b_2\right)_{B,w}\right| \frac{|f(y)|}{|x_0 - y|^n} dy \\
&+ \left|b_1(x) - \left(b_1\right)_{B,w}\right| \int\limits_{\complement_{(2B)}} \left|b_2(y) - \left(b_2\right)_{B,w}\right| \frac{|f(y)|}{|x_0 - y|^n} dy \\
&+ \left|b_2(x) - \left(b_2\right)_{B,w}\right| \int\limits_{\complement_{(2B)}} \left|b_1(y) - \left(b_1\right)_{B,w}\right| \frac{|f(y)|}{|x_0 - y|^n} dy.
\end{aligned}
$$

Then

$$
\begin{aligned}
\|T_{\vec{b}} f_2\|_{L_{p,w}(B)} &\lesssim \Big(\int\limits_B \Big(\int\limits_{\complement_{(2B)}} \frac{\prod\limits_{j=1}^{2} \left|b_i(y) - \left(b_i\right)_{B,w}\right|}{|x_0 - y|^n} |f(y)| dy \Big)^p w(x) dx \Big)^{\frac{1}{p}} \\
&+ \left(\int\limits_B \left|b_1(x) - \left(b_1\right)_{B,w}\right| \left(\int\limits_{\complement_{(2B)}} \frac{\left|b_2(y) - \left(b_2\right)_{B,w}\right|}{|x_0 - y|^n} |f(y)| dy \right)^p w(x) dx \right)^{\frac{1}{p}} \\
&+ \left(\int\limits_B \left|b_2(x) - \left(b_2\right)_{B,w}\right| \left(\int\limits_{\complement_{(2B)}} \frac{\left|b_1(y) - \left(b_1\right)_{B,w}\right|}{|x_0 - y|^n} |f(y)| dy \right)^p w(x) dx \right)^{\frac{1}{p}} \\
&+ \Big(\int\limits_B \Big(\int\limits_{\complement_{(2B)}} \frac{\prod\limits_{j=1}^{2} \left|b_i(x) - \left(b_i\right)_{B,w}\right|}{|x_0 - y|^n} |f(y)| dy \Big)^p w(x) dx \Big)^{\frac{1}{p}} \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

Let us estimate I_1.

$$
\begin{aligned}
I_1 &= w(B)^{\frac{1}{p}} \int\limits_{\complement_{(2B)}} \frac{\prod\limits_{j=1}^{2} \left|b_i(y) - \left(b_i\right)_{B,w}\right|}{|x_0 - y|^n} |f(y)| dy \\
&\approx w(B)^{\frac{1}{p}} \int\limits_{2r}^{\infty} \int\limits_{2r \le |x_0 - y| \le t} \prod_{j=1}^{2} \left|b_i(y) - \left(b_i\right)_{B,w}\right| |f(y)| dy \frac{dt}{t^{n+1}} \\
&\lesssim w(B)^{\frac{1}{p}} \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} \prod_{j=1}^{2} \left|b_i(y) - \left(b_i\right)_{B,w}\right| |f(y)| dy \frac{dt}{t^{n+1}}.
\end{aligned}
$$

Applying Hölder's inequality and by Lemma 4.11, we get

$$I_1 \lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \prod_{j=1}^{2} \Bigg(\int_{B(x_0,t)} \big|b_i(y) - \big(b_i\big)_{B,w}\big|^{2p'} w(y)^{1-2p'} dy \Bigg)^{\frac{1}{2p'}} \|f\|_{L_{p,w}(B(x_0,t))} \frac{dt}{t^{n+1}}$$

$$\lesssim \prod_{j=1}^{2} \|b_j\|_* \, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \Big(1 + \ln \frac{t}{r}\Big)^2 \|w^{-1/p}\|_{L_{p'}(B(x_0,t))} \|f\|_{L_{p,w}(B(x_0,t))} \frac{dt}{t^{n+1}}$$

$$\lesssim \|\vec{b}\|_* \, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \ln^2 \Big(e + \frac{t}{r}\Big) \|f\|_{L_{p,w}(B(x_0,t))} \, w(B(x_0,t))^{-1/p} \, \frac{dt}{t}.$$

Let us estimate I_2.

$$I_2 = \Bigg(\int_B \big|b_1(x) - \big(b_1\big)_{B,w}\big|^p w(x) dx\Bigg)^{\frac{1}{p}} \int_{\complement_{(2B)}} \frac{\big|b_2(y) - \big(b_2\big)_{B,w}\big|}{|x_0 - y|^n} |f(y)| dy$$

$$\lesssim \|b_1\|_* \, w(B)^{\frac{1}{p}} \int_{\complement_{(2B)}} \big|b_2(y) - \big(b_2\big)_{B,w}\big| \, |f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}} dy$$

$$\approx \|b_1\|_* \, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \int_{2r \le |x_0-y| \le t} \big|b_2(y) - \big(b_2\big)_{B,w}\big| \, |f(y)| dy \frac{dt}{t^{n+1}}$$

$$\lesssim \|b_1\|_* \, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \int_{B(x_0,t)} \big|b_2(y) - \big(b_2\big)_{B,w}\big| \, |f(y)| dy \frac{dt}{t^{n+1}}.$$

Applying Hölder's inequality and by Lemma 4.11, we get

$$I_2 \lesssim \|b_1\|_* \, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \Big(\int_{B(x_0,t)} \big|b_2(y) - \big(b_2\big)_{B,w}\big|^{p'} w(y)^{1-p'} dy\Big)^{\frac{1}{p'}} \|f\|_{L_{p,w}(B(x_0,t))} \frac{dt}{t^{n+1}}$$

$$\lesssim \prod_{j=1}^{2} \|b_j\|_* \, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \Big(1 + \ln \frac{t}{r}\Big) \|w^{-1/p}\|_{L_{p'}(B(x_0,t))} \|f\|_{L_{p,w}(B(x_0,t))} \frac{dt}{t^{n+1}}$$

$$\lesssim \|\vec{b}\|_* \, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \ln^2 \Big(e + \frac{t}{r}\Big) \|f\|_{L_{p,w}(B(x_0,t))} \, w(B(x_0,t))^{-1/p} \, \frac{dt}{t}.$$

In the same way, we shall get the result of I_3

$$I_3 \lesssim \|\vec{b}\|_* \, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \ln^2 \Big(e + \frac{t}{r}\Big) \|f\|_{L_{p,w}(B(x_0,t))} \, w(B(x_0,t))^{-1/p} \, \frac{dt}{t}.$$

In order to estimate I_4 note that

$$I_4 = \left(\int_B \prod_{j=1}^{2} \left| b_i(x) - \left(b_i \right)_{B,w} \right|^p w(x) dx \right)^{\frac{1}{p}} \int_{\complement_{(2B)}} \frac{|f(y)|}{|x_0 - y|^n} dy$$
$$\leq \prod_{j=1}^{2} \left(\int_B \left| b_i(x) - \left(b_i \right)_{B,w} \right|^{2p} w(x) dx \right)^{\frac{1}{2p}} \int_{\complement_{(2B)}} \frac{|f(y)|}{|x_0 - y|^n} dy.$$

By Lemma 4.11, we get

$$I_4 \lesssim \|\vec{b}\|_* \, w(B)^{\frac{1}{p}} \int_{\complement_{(2B)}} \frac{|f(y)|}{|x_0 - y|^n} dy.$$

Applying Hölder's inequality, we get

$$\begin{aligned} \int_{\complement_{(2B)}} \frac{|f(y)|}{|x_0 - y|^n} dy &\lesssim \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \, \|w^{-1/p}\|_{L_{p'}(B(x_0,t))} \, \frac{dt}{t^{n+1}} \\ &\leq [w]_{A_p}^{1/p} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \, w(B(x_0,t))^{-1/p} \, \frac{dt}{t}. \end{aligned} \tag{17}$$

Thus, by (17)

$$I_4 \lesssim \|\vec{b}\|_* \, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \, w(B(x_0,t))^{-1/p} \, \frac{dt}{t}.$$

Summing up I_1 and I_4, for all $p \in [1, \infty)$ we get

$$\|T_{\vec{b}} f_2\|_{L_{p,w}(B)} \lesssim \|\vec{b}\|_* \, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \ln^2 \left(e + \frac{t}{r} \right) \|f\|_{L_{p,w}(B(x_0,t))} \, w(B(x_0,t))^{-1/p} \, \frac{dt}{t}. \tag{18}$$

On the other hand,

$$\begin{aligned} \|f\|_{L_{p,w}(2B)} &\lesssim |B| \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \frac{dt}{t^{n+1}} \\ &\leq w(B)^{\frac{1}{p}} \|w^{-1/p}\|_{L_{p'}(B)} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \frac{dt}{t^{n+1}} \\ &\leq w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \, \|w^{-1/p}\|_{L_{p'}(B(x_0,t))} \, \frac{dt}{t^{n+1}} \\ &\leq [w]_{A_p}^{1/p} \, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \, w(B(x_0,t))^{-1/p} \, \frac{dt}{t}. \end{aligned} \tag{19}$$

Finally,

$$\|T_{\vec{b}}f\|_{L_{p,w}(B)} \lesssim \|\vec{b}\|_* \, \|f\|_{L_{p,w}(2B)} + \|\vec{b}\|_* \, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \ln^m \Big(e+\frac{t}{r}\Big) \|f\|_{L_{p,w}(B(x_0,t))} \, w(B(x_0,t))^{-1/p} \, \frac{dt}{t},$$

and the statement of Theorem 4.13 follows by (19). □

Theorem 4.14. *Let T be linear ω-CZO and $\vec{b} \in BMO^m(\mathbb{R}^n)$. Let also ω satisfies condition (15), $w \in A_p(\mathbb{R}^n)$, $1<p<\infty$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^{\infty} \ln^m \Big(e+\frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) w(B(x,s))^{1/p}}{w(B(x,t))^{1/p}} \le C\varphi_2(x,r), \tag{20}$$

where C does not depend on x and r. Then the operator $T_{\vec{b}}$ is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$. Moreover,

$$\|T_{\vec{b}}f\|_{M_{q,\varphi_2}(w)} \lesssim \|\vec{b}\|_* \, \|f\|_{M_{p,\varphi_1}(w)}.$$

Proof. Using the Theorem 2.5 and the Theorem 4.13 we have

$$\|T_{\vec{b}}f\|_{M_{p,\varphi_2}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|T_{\vec{b}}f\|_{L_{p,w}B(x,r)}$$

$$\lesssim \|\vec{b}\|_* \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} \ln^m \Big(e+\frac{t}{r}\Big) \|f\|_{L_{p,w}(B(x,t))} \, w(B(x,t))^{-1/p} \, \frac{dt}{t}$$

$$\lesssim \|\vec{b}\|_* \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{L_{p,w}(B(x,r))} = \|\vec{b}\|_* \, \|f\|_{M_{p,\varphi_1}(w)}.$$

□

Remark 4.15. *Note that, if $\varphi_1(x,r) = \varphi_2(x,r) = \psi(w(x,r))$ and ψ satisfy the $\mathcal{D}_\kappa$ condition, Theorems 4.13 and 4.14 were proved in* [32]. *Also, in the case $m = 1$ and $\omega(t) = t^\varepsilon$ with $0 < \varepsilon \le 1$, Theorems 4.13 and 4.14 were proved in* [11].

5. Conclusion

In this paper, we obtain that the ω-type Calderón-Zygmund operators T and their multlinear commutators $T_{\vec{b}}$ are bounded from one generalized weighted Morrey space $M_{p,\varphi_1}(w)$ to another $M_{p,\varphi_2}(w)$, $1<p<\infty$. We find the sufficient conditions on the pair (φ_1, φ_2) with $\vec{b} \in BMO^m(\mathbb{R}^n)$ and $w \in A_p(\mathbb{R}^n)$ which ensures the boundedness of the operators T and $T_{\vec{b}}$ from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $1<p<\infty$.

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