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ON THE MEROMORPHIC EXTENSION ALONG THE COMPLEX LINES*

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Abstract. In this work studied extension properties of functions which are admitted meromorphic extension along some pencil of complex lines. Analogue of the well known Forellies' theorem is proved.

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1. Introduction

Meromorphic extension of functions primarily observed by W. Rothstein in [4] and the next analogue of well known Hartogs' lemma has been proved in the class of meromorphic functions:

Lemma 1.1. *(Rothstein). Let the functionf (z, w) be meromorphic in the domain*

$$U \times V = \{z \in \mathbf{C}^n : |z| < 1\} \times \{w \in \mathbf{C} : |w| < 1\}.$$

If for each fixed $z^0 \in U$ the function $f\left(z^0, w\right)$ can be meromorphically continued to the disc $\{|w| < R\},\ R > 1$, then the function $f(z, w)$ can be continued meromorphically to the domain $U \times \{|w| < R\}$.

For the extendability of meromorphic functions to some larger domain it is unnecessary to require meromorphic extension along all parallel sections. Similar questions were considered by M. Kazaryan [3] and recent works of A.Atamuratov [1]. So then it is known

Theorem 1.1. *Let $D \subset \mathbf{C}^n$ be a domain and the function $f(z, w)$ be meromorphic in the domain $D \times V = D \times \{w \in \mathbf{C} : |w| < r\}, r > 0$. If for each fixed z^0 from some nonpluripolar subset $E \subset D$ the function $f\left(z^0, w\right)$, of the variable w, can be continued meromorphically to the larger disc $\{w \in \mathbf{C} : |w| < R\},\ R > r$, then the function $f(z, w)$ can be continued meromorphically to the domain $D \times \left\{|w| < r^{\omega^*(z,E,D)} \cdot R^{1-\omega^*(z,E,D)}\right\}$.*

Here $\omega^*(z, E, D) -$ well known plurisubharmonic measure which is defined by the way:

$$\omega^*(z, E, D) = \overline{\lim_{\substack{\zeta \to z \\ \zeta \in D}}} \ \omega(\zeta, E, D),$$

where

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$$\omega(z,E,D)=\sup\{u\,(z)\in Psh(D):\ \ u(z)|_D\le 1,\ \ u(z)|_E\le 0\}.$$

In this paper we observe the radial analogue of the theorem 1.

In 1978 F. Forelli proved next radial version of Hartogs theorem: *if the function $f(z)$, given on unit ball $B(0,1)\subset \mathbf{C}^n$, is holomorphic in the sections of $B(0,1)$ with all complex lines passed $z=0$, and belongs to class C^∞ in the neighborhood of the origin, then it is holomorphic in* $B(0,1)$.

Proof of this theorem based on formally expansion of the given function into the power series on homogeneous polynomials and established its uniformly convergence of it. In [5] A. Sadullayev studied analyticity of formal series on homogeneous polynomials and proved next more general theorem

Theorem 1.2. *(A. Sadullayev). Let $\sum\limits_{k=0}^{\infty} Q_k(z)$ be the formal series of homogeneous polynomials Q_k and given the set $L=\{l\}$ of complex lines, passed through 0. If for each line $l\in L$ the series is convergent in the disc $l\cap B(0,1)$, then it is uniformly convergent in the domain*

$$D=\left\{z\in\mathbf{C}^n:|z|\exp V^*\left(\frac{z}{|z|},E\right)<1\right\},$$

where $E=\left(\bigcup\limits_{l\in L} l\right)\cap S(0,1)$ and V^*- extremal function of Green.

Note that extremely Green function is defined as following: in the space $\mathbf{C}^{\mathrm{n}}$ *denote the class* $L=\{u(z)\in Psh(\mathbf{C}^n):\quad u(z)\le C_u+\ln(\ 1+\|z\|\ )\ \}$, where C_u- constant depending on u. Let $K\subset\mathbf{C}^n$ be a compact and $V(z,K)=\sup\{\,u(z)\ :\ \ u(z)\in L\,,\quad u|_K\le 0\,\}$. Then regularized function $V^*(z,K)=\varlimsup\limits_{w\to z}V(w,K)$ is called extremal function of Green of the compact K.

2. The main results

The main result of this work is

Theorem 2.1. *(A) Let $f\,(z)\in C^\infty\,(\{0\})$ and E be a subset of unit sphere $S=\{z\in\mathbf{C}^n:|z|=1\}$. If for each fixed $\xi\in E$ restrict-function $f_\xi\,(\lambda)=f\,(\lambda\xi)$ can be meromorphically continued into a disk $\{\lambda\in\mathbf{C}:|\lambda|<R\}$, then function $f\,(z)$ meromorphically extends to the domain $D=\left\{z\in\mathbf{C}^n:|z|\exp V^*\left(\frac{z}{|z|},E\right)<R\right\}$.*

From this theorem we get the next analogue of the Forellies' theorem for the class of meromorphic functions.

Corollary. *Let $f(z)\in C^\infty\,(\{0\})$. If for each fixed $z^0\neq 0$ function $f(\lambda z^0)$ of the variable λ meromorphic on $\left\{\lambda\in\mathbf{C}:\lambda z^0\in B(0,R)\right\}$, then $f(z)$ is meromorphic on $B(0,R)$.*

Indeed, according to the theorem of Forelli for holomorphic functions it follows, that there exist some ball $B(0,\rho),\ \rho>0$, where the function $f(z)$ is holomorphic.

Therefore in the accordance of theorem A the function $f(z)$ is meromorphic in the domain $D=\left\{z\in\mathbf{C}^n:|z|\exp V^*\left(\frac{z}{|z|},S\right)<R\right\}$,$S=\partial B(0,1)$, but this domain coincides with $B(0,R)$.

Before starting to prove the theorem we want to give some conceptions and preliminary results.

Radius of meromorphicity. Let $D\subset\mathbf{C}^n$ be a domain and function $f(z,w)$ is meromorphic in the domain $D\times V=D\times\{|w|<r\}\,,\ r>0\,$, and $P\subset D\times V$ is the polar set of meromorphic function $f(z,w)$. For each fixed $z^0\in D$ which is satisfy the condition $\{z^0\}\times V\not\subset P$, we denote by $R(z^0)$ radius of the greatest disk, where the function $f(z^0,w)$ of the variable w meromorphically extends. For all $z^0\in D$, where holds $\{z^0\}\times V\subset P$ we put $R(z^0)=+\infty$.

Let the analytic function $f(z)$ be determined with convergent power series in a neighborhood of origin at the complex plane, i.e.

$$f(z)=\sum_{k=1}^{\infty}c_k z^k. \tag{1}$$

Denote by R_m radius of the largest disc centered at $z=0$, to which the function $f(z)$ can be extended a meromorphic function with at most m (including multiplicity) poles. The next result for radius of $m-$ meromorphicity proved by Hadamard[2].

Theorem 2.2. *(Hadamard [2]). If the function $f(z)$ is analytic in some neighborhood of $z=0$, then for each $m>0$ it holds that*

$$R_m=\frac{l_{m-1}}{l_m},$$

where $l_0=1,\ l_m=\overline{\lim_{j\to\infty}}|A_{mj}|^{\frac{1}{j}}$ *and*

$$A_{mj}=\begin{vmatrix} c_j & c_{j+1} & \cdots & c_{j+m-1} \\ c_{j+1} & c_{j+2} & \cdots & c_{j+m} \\ \cdots & \cdots & \cdots & \cdots \\ c_{j+m-1} & c_{j+m} & \cdots & c_{j+2m-2} \end{vmatrix}.$$

(Here by convention $\frac{0}{0}=\infty$)

Using this result it is easy to get the next formula for radius of meromorphicity

$$R=\lim_{m\to\infty}R_m=\frac{1}{\lim\limits_{m\to\infty}\overline{\lim\limits_{j\to\infty}}|A_{mj}|^{\frac{1}{mj}}}. \tag{2}$$

Bernstein-Walsh inequality. If $P_m(z)$ is polynomial of the order m, then it is true that

$$\frac{1}{m}\ln|P_m(z)|\le\frac{1}{m}\ln\|P_m\|_K+V(z,K)\ , \quad z\in\mathbf{C}^n.$$

3. Proof of the theorem A.

We realize the proof in two steps.

a) Let $B(0,r)=\{z\in\mathbf{C}^n:|z|<r\},\ 0<r<1$. According to the condition of the theorem let $f(z)\in O(B(0,r))$ for some $r>0$. Denote by $R(\xi)$ radius of meromorphicity for restrict-function$f(\lambda\xi)$ on the direction $\xi\in S$. Define the function $R\left(\frac{z}{|z|}\right)$ for each $z\in\mathbf{C}^n\backslash\{0\}$ and consider its lower regularization $R_*\left(\frac{z}{|z|}\right)=\underline{\lim}_{w\to z}R\left(\frac{w}{|w|}\right),\quad z\in\mathbf{C}^n$.

We'll prove, that function $f(z)$ can be meromorphically continued into the domain

$$D'=\left\{z\in\mathbf{C}^n:|z|<R_*\left(\frac{z}{|z|}\right)\right\}, \tag{3}$$

and this domain is maximal. For this it is sufficient to show that the function $f(z)$ can be continued as a meromorphic function in some neighborhood of each $z^0\in D'$, $\quad z^0\neq 0$. Indeed, for each fixed $\varepsilon>0$, there exists neighborhood $U_\delta(z^0)$ $\left(0\notin U_\delta(z^0)\right)$ such, that for all $z\in U_\delta(z^0)$ it holds

$$R_*\left(\frac{z^0}{|z^0|}\right)-\varepsilon<R\left(\frac{z}{|z|}\right).$$

Therefore, according to definition of $R\left(\frac{z}{|z|}\right)$ for all $z \in U_\delta(z^0)$ the restrict-function $f_z(\lambda) = f(\lambda z)$ meromorphically continued into the disk

$$\left\{\lambda \in \ : |\lambda| < R_*\left(\frac{z^0}{|z^0|}\right) - \varepsilon\right\}.$$

Now we consider the function$\varphi(z,\lambda) = f(\lambda z)$. By the condition of the theorem it is meromorphic in the domain $B(0,r) \times \{\lambda \in \mathbf{C} : |\lambda| < 1\}$ and for each fixed $z \in \frac{r}{|z^0|+\delta} U_\delta\left(z^0\right)$ meromorphically continues to the disk

$$\left\{\lambda \in \ : |\lambda| < \frac{R_*\left(\frac{z^0}{|z^0|}\right) - \varepsilon}{r}\right\}.$$

According to the theorem 1. we get that function $\varphi(z,\lambda)$ meromorphically continues to the domain

$$\{\text{z} \in B(0,r)\} \times \left\{|\lambda| < 1 \cdot \left(\frac{R_*\left(\frac{z^0}{|z^0|}\right) - \varepsilon}{r}\right)^{1-\omega^* \quad z, \frac{r}{|z^0|+\delta} U_\delta\left(z^0\right), B(0,r)}\right\},$$

where ω^*- plurisubharmonic measure of the set $\frac{r}{|z^0|+\delta} U_\delta\left(z^0\right)$ relatively to domain $B(0,r)$. From this, it follows, that the function $f\left(\frac{z}{\lambda} \cdot \lambda\right) = \varphi\left(\frac{z}{\lambda}, \lambda\right)$ is meromorphic in the domain

$$\left\{(z,\lambda) \in \mathbf{C}^{n+1} : \frac{z}{\lambda} \in B(0,r), |\lambda| < \left(\frac{R_*\left(\frac{z^0}{|z^0|}\right) - \varepsilon}{r}\right)^{1-\omega^* \quad \frac{z}{|\lambda|}, \frac{r}{|z^0|+\delta} U_\delta(z^0), B(0,r)}\right\},$$

i.e. the function $f(z) = f\left(\frac{z}{\lambda}, \lambda\right)$is meromorphic in the domain $\left\{\frac{z}{\lambda} \in B(0,r)\right\}$, where $\lambda \in \mathbf{C}$ -arbitrarily number satisfying inequality

$$|\lambda| < \left(\frac{R_*\left(\frac{z^0}{|z^0|}\right) - \varepsilon}{r}\right)^{1-\omega^* \quad \frac{z}{|\lambda|}, \frac{r}{|z^0|+\delta} U_\delta(z^0), B(0,r)} . \tag{4}$$

It is easy to see, that if $z \in U_\delta(z^0)$, then

$$\omega^*\left(\frac{z}{|\lambda|}, \frac{r}{|z^0|+\delta} U_\delta(z^0), B(0,r)\right) \equiv 0$$

and inequality (4) can be written in the form $|\lambda| < \frac{R_* \quad \frac{z^0}{|z^0|} \quad -\varepsilon}{r}$. Therefore $f(z)$ is meromorphic in the ball

$$|z| < R_*\left(\frac{z^0}{|z^0|}\right) - \varepsilon.$$

It shows that, $U_\delta(z^0)$ is totally contained in the domain (3).

b) Now we prove the inequality

$$R_*\left(\frac{z}{|z|}\right) \geq R\,\exp\left(-V^*\left(\frac{z}{|z|}, E\right)\right).$$

Since the function $f(z)$ is holomorphic at the point $z^0 = 0$, it can be expanded into the series by homogenous polynomials, which converges at the some neighborhood $B(0,r) = \{z \in C^n : |z| < r, r < R\}$ of this point.

$$f(z) = \sum_{k=0}^{\infty} P_k(z). \tag{5}$$

Restriction of the series (5) to the complex line $l = \lambda\xi,\ \xi \in E,\ |\lambda| < R$ has the next form

$$f(\lambda\xi) = \sum_{k=0}^{\infty} P_k(\lambda\xi) = \sum_{k=0}^{\infty} \lambda^k P_k(\xi).$$

We consider the function $\varphi(\xi,\lambda) = f(\lambda\xi)$ and take $\xi = \frac{z}{|z|}$. By the condition of theorem for each fixed $\left(\frac{z}{|z|}\right) \in E \subset S$ the function $\varphi\left(\left(\frac{z}{|z|}\right), \lambda\right)$ can be meromorphically continued into the disk $|\lambda| < R$. Then from the formula for the radius of meromorphy (2) we get

$$\frac{1}{R\left(\frac{z}{|z|}\right)} = \lim_{m\to\infty}\overline{\lim_{j\to\infty}}\left|A_{mj}\left(\frac{z}{|z|}\right)\right|^{\frac{1}{mj}},$$

where

$$A_{mj}\left(\frac{z}{|z|}\right) = \begin{vmatrix} P_{j-m+1}\left(\frac{z}{|z|}\right) & P_{j-m+2}\left(\frac{z}{|z|}\right) & \dots & P_j\left(\frac{z}{|z|}\right) \\ P_{j-m+2}\left(\frac{z}{|z|}\right) & P_{j-m+3}\left(\frac{z}{|z|}\right) & \dots & P_{j+1}\left(\frac{z}{|z|}\right) \\ \dots\dots & \dots\dots & \dots\dots & \\ P_j\left(\frac{z}{|z|}\right) & P_{j+1}\left(\frac{z}{|z|}\right) & \dots & P_{j+m-1}\left(\frac{z}{|z|}\right) \end{vmatrix}$$

are holomorphic on $B(0,r)$.

As $A_{m,j}\left(\frac{z}{|z|}\right)$ are polynomials of degree mj, using Bernstein-Walsh inequality we get, that

$$\frac{1}{mj}\ln\left|A_{mj}\left(\frac{z}{|z|}\right)\right| \leq \frac{1}{mj}\ln\left\|A_{mj}\left(\frac{z}{|z|}\right)\right\|_E + V\left(\frac{z}{|z|}, E\right), \tag{6}$$

where

$$V\left(\frac{z}{|z|}, E\right) = \sup\left\{\frac{1}{mj}\ln|P_{mj}| : \|P_{mj}\|_E = 1\right\}.$$

Here $P_{mj}\left(\frac{z}{|z|}\right)$ - polynomials of degree mj.

Therefore, tending $m \to \infty,\ j \to \infty$ in inequality (6), we get

$$-\ln R\left(\frac{z}{|z|}\right) \leq -\ln R + V\left(\frac{z}{|z|}, E\right).$$

Here taking higher regularization in both part we get that

$$\left(-\ln R\left(\frac{z}{|z|}\right)\right)^* \leq \left(-\ln R + V\left(\frac{z}{|z|}, E\right)\right)^*,$$

i.e.

$$-\ln R_*\left(\frac{z}{|z|}\right) \le -\ln R + V^*\left(\frac{z}{|z|}, E\right).$$

Therefore

$$R_*\left(\frac{z}{|z|}\right) \ge R\,\exp\left(-V^*\left(\frac{z}{|z|}, E\right)\right). \tag{7}$$

Thus from the inequality (7) it follows that the function $f(z)$ meromorphically continues into the domain

$$D = \left\{z \in \mathbf{C}^n : |z| \exp\left(V^*\left(\frac{z}{|z|}, E\right)\right) < R\right\}.$$

Note, that the domain D, described with this inequality is a domain of holomorphy. It means, that D is the maximal domain to where each function satisfying in conditions of the theorem admits meromorphic extension. Theorem A is proved.

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