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RENORMALIZATIONS OF CIRCLE HOMEOMORPHISMS WITH A BREAK POINT*

AKHTAM DZHALILOV[1], ABDUMAJID BEGMATOV[2]

ABSTRACT. Let $f_\theta(x) = F_0(x)+\theta \pmod 1$, $x \in S^1$, $\theta \in [0,1]$ be a family of preserving orientation circle homeomorphisms with a single break point x_b, i.e. with a jump in the first derivative F_0 at the point $x = x_b$. Suppose that $F_0'(x)$ is absolutely continuous on $[x_b, x_b + 1]$ and $F_0''(x) \in L_\alpha([0,1])$ for some $\alpha > 1$. Consider f_θ with rational rotation number $\rho_\theta = \frac{p}{q}$ of rank n, i.e. $\frac{p}{q} = [k_1, k_2, ..., k_n]$. We prove that for sufficiently large n, the renormalizations of f_θ is close to certain convex linear-fractional functions in C^{1+L^1}.

Keywords: family of circle maps, break point, rotation number.

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1. Introduction

Circle homeomorphisms constitute one important class of one-dimensional dynamical systems. The investigation of their properties was initiated by Poincaré [7], who came across them in his studies of differential equations more than a century ago. Since then interest in these maps never diminished. Circle maps are also important because of their applications to natural sciences (see for instance [2]).

We identify the unit circle $\mathbb{S}^1 = \mathbb{R}^1/\mathbb{Z}^1$ with the half open interval $[0, 1)$. Consider the one-parameter families of the orientation preserving circle homeomorphisms

$$f_\theta(x) = F_0(x) + \theta \pmod 1,\ x \in \mathbb{S}^1,\ \theta \in [0; 1], \tag{1}$$

where the initial lift $F_0 : \mathbb{R}^1 \to \mathbb{R}^1$ satisfies the following conditions:
(a) F_0 is continuous and strictly increasing on $\mathbb{R}^1$;
(b) $F_0(0) = 0,\ F_0(x + 1) = F_0(x) + 1,\ x \in \mathbb{R}^1$;
(c) there is a point $x_b \in \mathbb{S}^1$ such that the one-sided derivatives $F_0'(x_b \pm 0)$ exist, are positive and $F_0'(x_b - 0) \neq F_0'(x_b + 0)$;
(d) F_0' is absolutely continuous and strictly positive on $[x_b, x_b + 1]$;
(e) $F_0'' \in \mathbb{L}^\alpha([0; 1], d\ell)$ for some $\alpha > 1$, where ℓ is Lebesque measure on the circle.

The conditions (d) and (e) are called the *Katznelson and Ornstein's smoothness conditions.* The point x_b is called *a break point* of f_θ. The ratio

$$\sigma(x_b) = \sqrt{\frac{F_0'(x_b - 0)}{F_0'(x_b + 0)}}$$

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[1] Faculty of Mathematics, Samarkand State University, Uzbekistan,
e-mail: a_dzhalilov@yahoo.com
[2] Institute of Mathematics and Information Technologies of the Academy of Sciences of Uzbekistan, Tashkent,
e-mail: begmatovms@mail.ru
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is called the *jump ratio* of f_θ at x_b or, for short, f_θ-jump ratio. Notice that the parameter $\sigma = \sigma(x_b)$ is obviously an invariant under smooth coordinate transformations and characterizes the type of the singularity.

Put $F_\theta = F_0 + \theta$, $\theta \in [0,1]$. The rotation number ρ_θ of f_θ is defined by (see [1] for details)

$$\rho_\theta = \lim_{n\to\infty} \frac{F_\theta^n(x)}{n} \ (\text{mod}1),$$

where the limit exists for all $x \in \mathbb{R}^1$ and is independent of x. Here and later, F^n denotes the $n-th$ iteration of F.

The familes like

$$A_\theta(x) = x + \frac{c}{2\pi}\sin(2\pi x) + \theta \ (\text{mod}1)$$

were studied for various constants c. For $c < 1$ the maps are diffeomorphisms and there is a result in [3], which says that the rotation number is absolutely continuous as a function of θ. When $c > 1$ the maps are non homeomorphisms and have no rotation number. In this case, both endpoints of rotation interval are rational almost everywhere w.r.t Lebesgue measure. Notice that the results are quite different if the family (1) has singularity points. Swiatek in [8] studied the family (1) with several critical points. It is proved that the set of parameter values corresponding to irrational rotation numbers has Lebesgue measure zero. In other words, the intervals on which frequency-locking occurs fill up the set of full measure. Khanin and Vul in [6] studied renormalizations and rational rotation numbers of the family (1) with single break point x_b such that $f_\theta \in C^{2+\varepsilon}(S^1\backslash\{x_b\})$. On one hand, the set of the parameter values corresponding to irrational rotation numbers has a zero measure, and the dynamics is characterized by nontrivial scaling transformations. On the other hand, similar to the case of circle diffeomorphisms (see [5]), the renormalizations group behavior of such maps is rather simple. In the renormalized coordinates, the iterations of f_θ approximated to linear-fractional transformations in the norm $\|\cdot\|_{C^2(S^1\backslash\{x_b\})}$ (see [6]).

In this paper, our purpose is to study the family (1) with a single point, but with a weaker smoothness condition for f_θ.

It is easy to see that ρ_θ is the increasing function of θ. Note that for each rational number a the set $I(a) = \{\theta : \rho_\theta = a\}$ is a nontrivial closed interval and $I(a)$ consists of only one point if a is irrational.

The main idea of the renormalization group method is to study large time iterates of the original mappings in a rescaled coordinate system corresponding to some neighborhood of a given point. Let $\frac{p}{q} \in [0,1]$ be an arbitrary rational number of *rank* n, i.e $\frac{p}{q} = [k_1, k_2, ..., k_n]$, $k_n > 1$. Since the rank of $\frac{p}{q}$ equals n we put $p_n := p$ and $q_n := q$. Let us fix some $\theta \in I(\frac{p_n}{q_n})$ and denote $F = F_\theta$ and $f = f_\theta$ (we omit the parameter θ in the sequel). Let $O_f(t, q_n) = \{f^i(t), i = 0, 1, ..., q_n - 1\}$ be an arbitrary periodic orbit of f of period q_n. Denote by $[y_1, y_2]$ the closed interval formed by two consecutive points of orbit $O_f(t, q_n)$ and containing the break point x_b of f. We introduce the renormalized coordinate z on $[y_1, y_2]$ given by the formula $z = (x - y_2)/(y_1 - y_2)$. It is clear that the normalized coordinate z changes from 1 to 0, when x is moving from y_1 to y_2. Denote by d the renormalized coordinate of break point x_b, i.e. $d = (x_b - y_2)/(y_1 - y_2)$.

Now, we define the function $\bar{f}_{\rho,n}$ corresponding to F^{q_n} in this new coordinate by:

$$\bar{f}_{\frac{p_n}{q_n},n}(z) = \frac{F^{q_n}(y_2 + z(y_1 - y_2)) - y_2 - p_n}{y_1 - y_2}, \quad z \in [0,1]. \tag{2}$$

The least map is called $n-th$ *renormalization* of f on the interval $[y_1, y_2]$. Next, we define the piecewise fractional-linear function $G_{d,n}$ on $[0,1]$ by the formula:

$$G_{d,n}(z) = \begin{cases} \frac{\sigma z}{(\sigma-1)z+d(1-\sigma^2)+\sigma^2}, & \text{if } z \in [0,d], \\ \frac{\sigma^2 z+d(1-\sigma^2)}{\sigma(\sigma-1)z+d(1-\sigma^2)+\sigma}, & \text{if } z \in (d,1]. \end{cases} \tag{3}$$

The main purpose of our paper is to prove the following:

Theorem 1.1. *Let $\{f_\theta : \theta \in [0,1]\}$ be the family of circle homeomorphisms defined by (1) with the initial lift F_0 satisfying the conditions (a)-(e). Then, for any $\varepsilon > 0$ there exists $N = N(\varepsilon, F_0) > 0$ (which doesn't depend on choice of periodic orbit), such that if f belongs to this family and its rotation number $\rho = \frac{p_n}{q_n}$ is rational with rank n, $n > N$ the following estimates hold:*

$$\|\bar{f}_{\rho,n} - G_d\|_{C([0,1])} \leq \varepsilon, \quad \|\bar{f}''_{\rho,n} - G''_d\|_{L^1([0,1],\,d\ell)} \leq \varepsilon.$$

Remark 1.1. *Using the assertions of Theorem 1.1 it can easily be shown that*

$$\sup_{z\in[0,1]\setminus\{d\}} |\bar{f}'_{\rho,n} - G'_d| \leq \varepsilon.$$

2. Dynamical partitions of circle homeomorphisms with rational rotation number

Let f be an orientation preserving homeomorphism of the circle with rational rotation number $\rho = \frac{p_n}{q_n} = [k_1, k_2, ..., k_n]$. For $1 \leq m \leq n$ denote by $\frac{p_m}{q_m} = [k_1, k_2, ..., k_m]$, the convergent of $\frac{p_n}{q_n}$. Their denominators q_m satisfy $q_{m+1} = k_{m+1}q_m + q_{m-1}, 1 \leq m \leq n-1,\ q_0 = 1,\ q_1 = k_1$. Since the rotation number $\rho = \frac{p_n}{q_n}$ is rational homeomorphism f has at least one periodic orbit of period q_n (see [1]). Let $O_f(t, q_n) = \{f^i(t), i = 0, 1, ..., q_n - 1\}$ be a periodic orbit of f of period q_n. For an arbitrary point $x_0 \in O_f(t, q_n)$, denote by $\Delta_0^{(m)}(x_0)$ the closed interval with endpoints x_0 and $x_{q_m} = f^{q_m}x_0,\ \ 0 \leq m \leq n-1$. If m is odd then x_{q_m} is to the left of x_0, and to the right of x_0 if m is even. Denote by $\Delta_i^{(m)}(x_0)$ the iterates of the interval $\Delta_0^{(m)}(x_0)$ under f: $\Delta_i^{(m)}(x_0) = f^i\Delta_0^{(m)}(x_0), i \geq 1,\ \ 0 \leq m \leq n-1$. It is well known that each of the following system of intervals

$$\xi_m(x_0) = \left\{\Delta_i^{(m-1)}(x_0),\ 0 \leq i < q_m;\ \Delta_j^{(m)}(x_0),\ 0 \leq j < q_{m-1}\right\}, 1 \leq m < n,$$

$$\xi_n(x_0) = \left\{\Delta_i^{(n-1)}(x_0),\ 0 \leq i < q_n\right\}$$

cover the whole circle and that their interiors are mutually disjoint. The partition $\xi_m(x_0)$ is called the *mth dynamical partition* of the point x_0. We briefly recall the structure of the dynamical partitions. The passage from $\xi_m(x_0)$ to $\xi_{m+1}(x_0)$, $1 \leq m < n-2$ is simple: namely, all intervals of rank m are preserved and each of the intervals $\Delta_i^{(m-1)}(x_0),\ 0 \leq i < q_m$, is divided into $(k_{m+1}+1)$ intervals: $\Delta_i^{(m-1)}(x_0) = \Delta_i^{(m+1)}(x_0) \cup \bigcup_{s=0}^{k_{m+1}-1} \Delta_{i+q_{m-1}+sq_m}^{(m)}(x_0)$. Note that the endpoints of intervals $\Delta_i^{(n-1)}(x_0),\ 0 \leq i \leq q_n - 1$ are periodic points of f of period q_n. Also each interval of partition $\xi_n(x_0)$ is periodic of period q_n. The following lemma plays a key role for studying metrical properties of the homeomorphism f.

Lemma 2.1. *Let f be a circle homeomorphism with lift F and rational rotation number $\rho_f = \frac{p_n}{q_n}$ of rank n. Let the finite derivatives $F'(x_b \pm 0) > 0$ exist and let $F \in C^1([x_b, x_b+1])$ and $var_{[x_b,x_b+1]} \log F' = \bar{v} < \infty$. We write*

$$v = \bar{v} + |\log F'(x_b - 0) - \log F'(x_b + 0)| = \bar{v} + 2\log\sigma(x_b).$$

In this case, the inequality

$$e^{-v} \leq \prod_{s=0}^{q_k-1} F'(x_s) \leq e^{v} \tag{4}$$

holds for any $1 \leq k \leq n$ *and* $x_0 \in S^1$ *such that* $x_i \neq x_b, i = 0, 1, 2, ..., n.$

The last inequality is called the *Denjoy inequality.* The proof of Lemma 2.1 is just like that of the similar assertion for diffeomorphism (see for instance [5]). Using Lemma 2.1 it can easily be shown that the lengths of the intervals of the dynamical partition ξ_n are exponentially small.

Corollary 2.1. *Suppose that* $\Delta^{(k)} \subset \Delta^{(l)} \in \xi_l(x_0)$, $\Delta^{(k)} \in \xi_k(x_0)$, $1 \leq l < k \leq n$. *Then for some constant* $M_0 > 0$

$$l(\Delta^{(k)}) \leq M_0 \lambda^{k-l} l(\Delta^{(l)}), \tag{5}$$

where $\lambda = (1 + e^{-2v})^{-1/2} < 1.$

3. Proof of Theorem 1.1

Consider the dynamical partition generated by periodic orbit $O_f(t, q_n)$. By assumption $[y_1, y_2]$ is the closed interval formed by two consecutive points of $O_f(t, q_n)$ and containing the break point x_b. We put $x_0 = y_1$. Consider the partition $\xi_n(x_0)$. It is clear that $\Delta_0^{(n-1)}(x_0) = [y_1, y_2]$ and $f^{q_n}\Delta_0^{(n-1)}(x_0) = \Delta_0^{(n-1)}(x_0)$. It follows from Corollary 2.1, that the intervals of the dynamical partition $\xi_n(x_0)$ have exponentially small length, i.e. $\ell(\Delta_0^{(n-1)}(x_0)) \leq const\lambda^n,\ \lambda \in (0, 1)$. Note that the function $\bar{f}_{\rho,n}(z)$ can be represented as the superposition of two functions, $\bar{f}_1$ and $\bar{f}_2$, which correspond to the mappings $f : \Delta_0^{(n-1)}(x_0) \to \Delta_1^{(n-1)}(x_0)$, $f^{q_n-1} : \Delta_1^{(n-1)}(x_0) \to \Delta_{q_n}^{(n-1)}(x_0) = \Delta_0^{(n-1)}(x_0)$, respectively. We introduce relative coordinates $z_i,\ i = 0, 1, ..., q_n - 1$, in the intervals $\Delta_i^{(n-1)}(x_0)$

$$z_i = (f^i(x) - f^i(y_2))/(f^i(y_1) - f^i(y_2)),\ \ x \in \Delta_0^{(n-1)}(x_0).$$

Then the functions $\bar{f}_1$ and $\bar{f}_2$ can be written as

$$\bar{f}_1(z_0) = \frac{f(y_2 + (y_1 - y_2)z_0) - f(y_2)}{f(y_1) - f(y_2)}, \tag{6}$$

$$\bar{f}_2(z_1) = \frac{f^{q_n-1}(f(y_2)) + (f(y_1) - f(y_2))z_1) - y_2}{y_1 - y_2}. \tag{7}$$

It is clear that $\bar{f}_{\rho,n}(z) = \bar{f}_2(\bar{f}_1(z))$. Define the following functions:

$$g(z_1) = \frac{\sigma z_1}{1 + z_1(\sigma - 1)}, \quad R_d(z_0) = \begin{cases} \frac{z_0}{\sigma^2(1-d)+d}, & \text{if } z_0 \in [0, d], \\ \frac{\sigma^2 z_0 + d(1-\sigma^2)}{\sigma^2(1-d)+d}, & \text{if } z_0 \in (d, 1]. \end{cases} \tag{8}$$

We put $M_n = \exp\{\sum_{i=1}^{q_n-2} \int_{\Delta_i^{(n-1)}} \frac{f''(y)}{2f'(y)} dy\}$. From now on we shall denote by K constants that depend only on the original family f_θ. Next, we formulate two necessary lemmas.

Lemma 3.1. *For any* $\varepsilon > 0$, *the following relation holds for sufficiently large* n

$$z_{q_n-1}(z_1) = \frac{z_1 M_n \exp \tau_n(z_1)}{1 + z_1(M_n \exp \tau_n(z_1) - 1)}, \tag{9}$$

where the function $\tau_n(z_1)$ *and its derivatives satisfies the following inequalities:*

$$\max_{0 \leq z_1 \leq 1} |\tau_n(z_1)| \leq \varepsilon, \quad \max_{0 \leq z_1 \leq 1} |(z_1 - z_1^2)\tau_n'(z_1)| \leq \varepsilon, \tag{10}$$

$$\|\tau_n'(z_1)\|_{L^1([0,1],\,d\ell)} \le \varepsilon, \quad \|(z_1 - z_1^2)\tau_n''(z_1)\|_{L^1([0,1],\,d\ell)} \le \varepsilon. \tag{11}$$

Lemma 3.2. *The following estimates hold for sufficiently large n*

$$\|\bar{f}_1 - R_d\|_{C([0,1])} \le K\lambda^{\frac{n}{\beta}}, \quad \|\bar{f}''_1 - R_d''\|_{L^1([0,1],\ d\ell)} \le K\lambda^{\frac{n}{\beta}}, \tag{12}$$

where λ is the same as in Corollary 2.1 and $\beta = \frac{\alpha}{\alpha-1}$.

For an easy flow of our presentation, we shall prove these two Lemmas at the end of this section. So we continue our proof of Theorem 1.1. It is not hard to show that $\bar{f}_2(z_1) = z_{q_n-1}(z_1)$. Using the last relation and Lemma 3.1, we obtain

$$\|\bar{f}_2(z_1) - \frac{M_n z_1}{1 + z_1(M_n - 1)}\|_{C^1([0,1])} \le \varepsilon, \tag{13}$$

$$\|\bar{f}''_2(z_1) - \frac{2M_n(1 - M_n)}{(1 + z_1(M_n - 1))^3}\|_{L^1([0,1],\ d\ell)} \le \varepsilon. \tag{14}$$

It is clear that

$$\ln M_n = \sum_{i=1}^{q_n-2} \int\limits_{\Delta_i^{(n-1)}(x_0)} \frac{f''(y)}{2f'(y)}dy = \ln\sigma - \int\limits_{\Delta_0^{(n-1)}(x_0)} \frac{f''(y)}{2f'(y)}dy - \int\limits_{\Delta_{q_n-1}^{(n-1)}(x_0)} \frac{f''(y)}{2f'(y)}dy. \tag{15}$$

Thus, we have

$$\left| \int\limits_{\Delta_k^{(n-1)}(x_0)} \frac{f''(y)}{2f'(y)}dy \right| \le K\|f''\|_\alpha \lambda^{n/\beta}, \ \text{for}, \ \ k = 0, q_n - 1.$$

Together with relations (13)-(15) this implies that

$$\|\bar{f}_2 - g\|_{C^1([0,1])} \le \varepsilon, \quad \|\bar{f}''_2 - g''\|_{L^1([0,1],\ d\ell)} \le \varepsilon.$$

So, the relation $\overline{f}_{\rho,n}(z) = \bar{f}_2(\bar{f}_1(z))$ and Lemma 3.2 imply the proof of Theorem 1.1.

Proof. Lemma 3.1. Denote $a_i = f^i(y_1), \ \ b_i = f^i(y_2), \ \ x_i = f^i(x), \ \ i = 1, 2, ..., q_n - 1$. Then we get

$$z_{i+1} = (x_{i+1} - b_{i+1})/(a_{i+1} - b_{i+1}). \tag{16}$$

It is easy to check that

$$x_{i+1} = f(x_i) = f(a_i) + f'(a_i)(x_i - a_i) + \int\limits_{a_i}^{x_i} f''(y)(x_i - y)dy,$$

$$b_{i+1} = f(b_i) = f(a_i) + f'(a_i)(b_i - a_i) + \int\limits_{a_i}^{b_i} f''(y)(b_i - y)dy,$$

by definition $a_{i+1} = f(a_i)$. Substituting this into (16) we get

$$z_{i+1} = z_i(1 + A_i(z_i - 1)), \ \ i = 1, 2, ..., q_n - 1, \tag{17}$$

where

$$A_i = -\frac{\frac{1}{f'(a_i)(x_i - a_i)} \int\limits_{a_i}^{x_i} f''(y)(y - a_i)dy + \frac{1}{f'(a_i)(b_i - x_i)} \int\limits_{x_i}^{b_i} f''(y)(b_i - y)dy}{1 + \frac{1}{f'(a_i)(b_i - a_i)} \int\limits_{a_i}^{b_i} f''(y)(b_i - y)dy}. \tag{18}$$

We denote

$$\tau_n(z_1)=\sum_{i=1}^{q_n-2}\psi_i,$$

where

$$\chi_i=\int_{a_i}^{b_i}\frac{f''(y)}{2f'(y)}dy,\quad \psi_i=-\chi_i-\ln\left(\frac{1+A_iz_i}{1+A_i(z_i-1)}\right),\ i=1,2,...,q_n-1.$$

Using (17) we obtain

$$\frac{1-z_{i+1}}{z_{i+1}}=\frac{1-z_i}{z_i}\frac{1+A_iz_i}{1+A_i(z_i-1)}=\frac{1-z_i}{z_i}\exp\{-\chi_i\}\exp\{-\psi_i\}. \tag{19}$$

Taking iteration of (19) we get

$$\frac{1-z_{q_n-1}}{z_{q_n-1}}=\frac{1-z_1}{z_1}\exp\{-\sum_{i=1}^{q_n-2}\chi_i\}\exp\{-\sum_{i=1}^{q_n-2}\psi_i\}=\frac{1-z_1}{z_1}\frac{1}{M_n\exp\tau_n(z_1)}. \tag{20}$$

Solving equation (20) with respect to z_{q_n-1} we obtain the relation (9).

Let us estimate $\tau_n(z_1)$. First we estimate A_i. Denote by V_i the second term of the denominator of (18). Since $f''(x)\in L_\alpha([0,1],d\ell)$ applying the Holder inequality we obtain

$$|V_i|\leq\frac{1}{f'(a_i)(b_i-a_i)}\int_{a_i}^{b_i}|f''(y)|(y-a_i)dy\leq\frac{\|f''\|_\alpha(b_i-a_i)^{1+\frac{1}{\beta}}}{f'(a_i)(b_i-a_i)(1+\beta)}\leq K(b_i-a_i)^{\frac{1}{\beta}}. \tag{21}$$

Analogously, it can be shown that the absolute values of both terms in (18) are not greater than $K(b_i-a_i)^{\frac{1}{\beta}}$. Let us recall that $[a_i,b_i]\in\xi_n(x_0)$ and $\ell([a_i,b_i])\leq K\lambda^n,\ \ i=0,1,...,q_n-2$. This, together with the expression for A_i imply that $|A_i|\leq Const\lambda^{\frac{n}{\beta}}$. Next, we rewrite $\tau_n(z_1)$ in the form

$$\tau_n(z_1)=-\sum_{i=1}^{q_n-2}\chi_i-\sum_{i=1}^{q_n-2}\ln\left(\frac{1+A_iz_i}{1+A_i(z_i-1)}\right)=-\ln M_n-\sum_{i=1}^{q_n-2}A_i-\sum_{i=1}^{q_n-2}0(A_i^2). \tag{22}$$

We estimate the last sum in (22). Note that each term of (18) containing an integral is not greater than $\int_{a_i}^{b_i}|f''(y)|dy$. Using the estimate for A_i, it can easily be shown that

$$\sum_{i=1}^{q_n-2}O(A_i^2)\leq K\lambda^{\frac{n}{\beta}}. \tag{23}$$

We rewrite the second to the last sum in (22) in the following form

$$\begin{aligned}\sum_{i=1}^{q_n-2}A_i=&-\sum_{i=1}^{q_n-2}\int_{a_i}^{b_i}\frac{f''(y)}{2f'(y)}dy-\sum_{i=1}^{q_n-2}\left[\frac{1}{f'(a_i)(x_i-a_i)}\int_{a_i}^{x_i}f''(y)(y-a_i)dy-\frac{1}{2}\int_{a_i}^{x_i}\frac{f''(y)}{2f'(y)}dy\right]-\\&-\sum_{i=1}^{q_n-2}\left[\frac{1}{f'(a_i)(b_i-x_i)}\int_{x_i}^{b_i}f''(y)(b_i-y)dy-\frac{1}{2}\int_{x_i}^{b_i}\frac{f''(y)}{2f'(y)}dy\right]+\\&+\sum_{i=1}^{q_n-2}\frac{V_i}{1+V_i}\left[\frac{1}{f'(a_i)(x_i-a_i)}\int_{a_i}^{x_i}f''(y)(y-a_i)dy+\frac{1}{f'(a_i)(b_i-x_i)}\int_{x_i}^{b_i}f''(y)(b_i-y)dy\right].\end{aligned} \tag{24}$$

The first after the sign of equality sum equals to $(-\ln M_n)$. Since $|V_i| \leq K\lambda^{\frac{n}{\beta}}$, the last sum is not greater than $K\lambda^{\frac{n}{\beta}}$. Together with relations (22)-(24), the last inequality implies that

$$\tau_n(z_1) = -\sum_{i=1}^{q_n-2}\left[\frac{1}{f'(a_i)(x_i-a_i)}\int_{a_i}^{x_i} f''(y)(y-a_i)dy - \frac{1}{2}\int_{a_i}^{x_i}\frac{f''(y)}{2f'(y)}dy\right] - \tag{25}$$

$$-\sum_{i=1}^{q_n-2}\left[\frac{1}{f'(a_i)(b_i-x_i)}\int_{x_i}^{b_i} f''(y)(b_i-y)dy - \frac{1}{2}\int_{x_i}^{b_i}\frac{f''(y)}{2f'(y)}dy\right] + O(\lambda^{\frac{n}{\beta}}).$$

Denote by S_n and $\bar{S}_n$ the last two sums in (25) respectively. Then, we show that for any $\varepsilon > 0$, the following estimates hold for sufficiently large n :

$$|S_n|,\ |\bar{S}_n| \leq K\varepsilon. \tag{26}$$

We prove only the estimate for S_n, the one for $\bar{S_n}$ is quite similar. Rewrite the sum S_n as

$$S_n = \sum_{i=1}^{q_n-2}\int_{a_i}^{x_i}\frac{f''(y)}{f'(a_i)}\left(\frac{y-a_i}{x_i-a_i}-\frac{1}{2}\right)dy+ \tag{27}$$

$$+\sum_{i=1}^{q_n-2}\int_{a_i}^{x_i}\left(\frac{f''(y)}{2f'(a_i)f'(y)}\int_{a_i}^{y} f''(t)dt\right)dy \equiv S_n^{(1)} + S_n^{(2)}.$$

Using the condition $f''(x) \in L_\alpha(S^1, d\ell)$, $\alpha > 1$, and the Hölder inequality, it can easily be shown that

$$|S_n^{(2)}| \leq K\sum_{i=1}^{q_n-2}\left(\int_{a_i}^{x_i}|f''(y)|dy\right)^2 \leq K\lambda^{\frac{n}{\beta}}. \tag{28}$$

Let us estimate the sum $S_n^{(1)}$. Fix an arbitrary $\varepsilon > 0$. Since $f''(x) \in L_\alpha(S^1, d\ell)$, it can be written in the form

$$f''(x) = h_\varepsilon(x) + r_\varepsilon(x),\ x \in S^1, \tag{29}$$

where $h_\varepsilon(x)$ is a continuous function on S^1 and $\|r_\varepsilon\|_{L^1} < \varepsilon$. Substituting (29) in expression for $S_n^{(1)}$, we obtain

$$|S_n^{(1)}| \leq \left|\sum_{i=1}^{q_n-2}\frac{1}{f'(a_i)}\int_{a_i}^{x_i} h_\varepsilon(y)(\frac{y-a_i}{x_i-a_i}-\frac{1}{2})dy\right| + \tag{30}$$

$$+\left|\sum_{i=1}^{q_n-2}\frac{1}{f'(a_i)}\int_{a_i}^{x_i} r_\varepsilon(y)(\frac{y-a_i}{x_i-a_i}-\frac{1}{2})dy\right| \equiv P_n + Q_n.$$

First, we estimate the sum P_n. Denote by t_i the middle of the interval $[a_i, x_i]$ i.e. $t_i = \frac{x_i+a_i}{2}$. We rewrite the sum P_n in the following form

$$P_n = \left|\sum_{i=1}^{q_n-2}\frac{1}{f'(a_i)}\int_{a_i}^{t_i} h_\varepsilon(y)(\frac{y-a_i}{x_i-a_i}-\frac{1}{2})dy + \sum_{i=1}^{q_n-2}\frac{1}{f'(a_i)}\int_{t_i}^{x_i} h_\varepsilon(y)(\frac{y-a_i}{x_i-a_i}-\frac{1}{2})dy\right|.$$

Applying the Mean Value Theorem we obtain

$$P_n = \left|\sum_{i=1}^{q_n-2}\frac{h_\varepsilon(\xi_1^i)}{f'(a_i)}\int_{a_i}^{t_i}(\frac{y-a_i}{x_i-a_i}-\frac{1}{2})dy + \sum_{i=1}^{q_n-2}\frac{h_\varepsilon(\xi_2^i)}{f'(a_i)}\int_{t_i}^{x_i}(\frac{y-a_i}{x_i-a_i}-\frac{1}{2})dy\right| = \tag{31}$$

$$= \sum_{i=1}^{q_n-2} \frac{x_i - a_i}{16f'(a_i)} |h_\varepsilon(\xi_2^i) - h_\varepsilon(\xi_1^i)| \leq \sum_{i=1}^{q_n-2} \frac{x_i - a_i}{16f'(a_i)} \omega(\lambda^n, h_\varepsilon, [a_i, b_i]) \leq K \max_{1\leq i\leq q_n-2} \omega(\lambda^n, h_\varepsilon, [a_i, b_i]),$$

where $\omega(\lambda^n, h_\varepsilon, [a_i, b_i]) = \sup |h_\varepsilon(\xi_2^i) - h_\varepsilon(\xi_1^i)|$ is the "modulus of continuity" of h_ε. Since $\lambda \in (0, 1)$, we have $\omega(\lambda^n, h_\varepsilon) \to 0$, as $n \to\propto$. Next, we estimate the sum Q_n. It is easy to see that

$$Q_n \leq K \sum_{i=1}^{q_n-2} \int_{a_i}^{x_i} |r_\varepsilon(y)| dy \leq K \int_{S^1} |r_\varepsilon(y)| dy \leq K\varepsilon.$$

Hence, the relations in (26) are proved. Then, summing (22)-(26), we obtain the first relation in (10).

Let us prove the second relation in (10). Note that there exists a constant $C_2 > 0$ such that the following inequalities hold for all i, $i = 1, 2, ..., q_n - 2$,

$$\frac{1}{C_2} \leq \frac{z_1(1 - z_1)}{z_i(1 - z_i)} \leq C_2, \quad \frac{1}{C_2} \leq \frac{dz_i}{dz_1} \leq C_2. \tag{32}$$

Notice that the function $\frac{d\psi_i}{dz_i}$ is defined almost everywhere. Using (18), we calculate the derivative of ψ_i by z_i :

$$\frac{d\psi_i}{dz_i} = \frac{A_i^2 - A_i'}{(1 + A_i z_i)(1 + A_i(z_i - 1))}, \tag{33}$$

where

$$A_i' = \frac{dA_i}{dz_i} = \frac{dA_i}{dx_i}\frac{dx_i}{dz_i} = (b_i - a_i)\frac{dA_i}{dx_i},$$

$$\frac{dA_i}{dx_i} = \frac{\frac{1}{f'(a_i)(x_i-a_i)^2} \int_{a_i}^{x_i} f''(y)(y - a_i)dy - \frac{1}{f'(a_i)(b_i-x_i)^2} \int_{x_i}^{b_i} f''(y)(b_i - y)dy}{1 + \frac{1}{f'(a_i)(b_i-a_i)} \int_{a_i}^{b_i} f''(y)(b_i - y)dy}. \tag{34}$$

Using (23), (28), (32)-(34) we obtain

$$|(z_1 - z_1^2)\tau_n'(z_1)| = \left|(z_1 - z_1^2) \sum_{i=1}^{q_n-2} \frac{d\psi_i}{dz_i}\frac{dz_i}{dz_1}\right| \leq \tag{35}$$

$$\leq K \left|\sum_{i=1}^{q_n-2} (z_i - z_i^2)(b_i - a_i) \left[\int_{a_i}^{x_i} f''(y)\frac{y - a_i}{(x_i - a_i)^2}dy - \int_{x_i}^{b_i} f''(y)\frac{b_i - y}{(b_i - x_i)^2}dy\right]\right| + O(\lambda^{\frac{n}{\beta}}).$$

Denote by E_n the last sum in (35). Using relation (29) we rewrite E_n in the following form

$$E_n = \left|\sum_{i=1}^{q_n-2} (z_i - z_i^2)(b_i - a_i) \left[\int_{a_i}^{x_i} h_\varepsilon(y)\frac{y - a_i}{(x_i - a_i)^2}dy - \int_{x_i}^{b_i} h_\varepsilon(y)\frac{b_i - y}{(b_i - x_i)^2}dy\right]\right| +$$

$$+ \left|\sum_{i=1}^{q_n-2} \left[z_i \int_{a_i}^{x_i} r_\varepsilon(y)\frac{y - a_i}{x_i - a_i}dy - (1 - z_i)\int_{x_i}^{b_i} r_\varepsilon(y)\frac{b_i - y}{b_i - x_i}dy\right]\right| \equiv E_n^{(1)} + E_n^{(2)}. \tag{36}$$

First, we estimate the sum $E_n^{(1)}$. Applying the Mean Value Theorem again we get

$$E_n^{(1)} \leq K \sum_{i=1}^{q_n-2} (b_i - a_i)|h_\varepsilon(\xi_1^i) - h_\varepsilon(\xi_2^i)| \leq \tag{37}$$

$$\leq K\sum_{i=1}^{q_n-2}(b_i-a_i)\omega(\lambda^n,h_\varepsilon)\leq K\max_{1\leq i\leq q_n-2}\omega(\lambda^n,h_\varepsilon,[a_i,b_i]).$$

Let us estimate $E_n^{(2)}$. It is easy to see that

$$E_n^{(2)}\leq\frac{1}{2}\sum_{i=1}^{q_n-2}\left[\int_{a_i}^{x_i}|r_\varepsilon(y)|dy+\int_{x_i}^{b_i}|r_\varepsilon(y)|dy\right]\leq\frac{1}{2}\int_{S^1}|r_\varepsilon(y)|dy<\frac{\varepsilon}{2}.$$

This, together with (35)-(37) imply the second relation in (10). Now, we prove the first relation in (11). Using the same arguments as in (35), we can show that

$$\int_0^1|\tau_n'(z_1)|dz_1\leq \tag{38}$$

$$\leq K\int_0^1\left|\sum_{i=1}^{q_n-2}(b_i-a_i)\left[\int_{a_i}^{x_i}f''(y)\frac{y-a_i}{(x_i-a_i)^2}dy-\int_{x_i}^{b_i}f''(y)\frac{b_i-y}{(b_i-x_i)^2}dy\right]\right|dz_1+O(\lambda^{\frac{n}{\beta}}).$$

Using relations (32), it is easy to see that

$$\int_0^1|\tau_n'(z_1)|dz_1\leq \tag{39}$$

$$\leq K\sum_{i=1}^{q_n-2}\int_{a_i}^{b_i}\left|\int_{a_i}^{x_i}f''(y)\frac{y-a_i}{(x_i-a_i)^2}dy-\int_{x_i}^{b_i}f''(y)\frac{b_i-y}{(b_i-x_i)^2}dy\right|dx_i+O(\lambda^{\frac{n}{\beta}}).$$

We denote by I_n the last sum in (39) and estimate it. Using the representation (29), we get

$$I_n\leq\sum_{i=1}^{q_n-2}\int_{a_i}^{b_i}\left|\int_{a_i}^{x_i}h_\varepsilon(y)\frac{y-a_i}{(x_i-a_i)^2}dy-\int_{x_i}^{b_i}h_\varepsilon(y)\frac{b_i-y}{(b_i-x_i)^2}dy\right|dx_i+ \tag{40}$$

$$+\sum_{i=1}^{q_n-2}\int_{a_i}^{b_i}\left|\int_{a_i}^{x_i}r_\varepsilon(y)\frac{y-a_i}{(x_i-a_i)^2}dy\right|dx_i+\sum_{i=1}^{q_n-2}\int_{a_i}^{b_i}\left|\int_{x_i}^{b_i}r_\varepsilon(y)\frac{b_i-y}{(b_i-x_i)^2}dy\right|dx_i.$$

It can easily be shown that the first sum in (40) is not greater than

$$\max_{1\leq i\leq q_n-2}\omega(\lambda^n,h_\varepsilon,[a_i,b_i]). \tag{41}$$

Denote by $I_n^{(1)}$ the second to the last sum in (40). Applying the Hölder inequality we obtain

$$I_n^{(1)}=\sum_{i=1}^{q_n-2}\int_{a_i}^{b_i}\left|\frac{1}{(x_i-a_i)^2}\int_{a_i}^{x_i}r_\varepsilon(y)(y-a_i)dy\right|dx_i\leq$$

$$\leq K\sum_{i=1}^{q_n-2}\int_{a_i}^{b_i}(x_i-a_i)^{\frac{1}{\beta}-1}\left(\int_{a_i}^{x_i}|r_\varepsilon(y)|^\alpha\,dy\right)^{\frac{1}{\alpha}}dx_i\leq$$

$$\leq K\sum_{i=1}^{q_n-2}\left(\int_{a_i}^{b_i}|r_\varepsilon(y)|^\alpha\,dy\right)^{\frac{1}{\alpha}}(b_i-a_i)^{\frac{1}{\beta}}\leq K\left[\sum_{i=1}^{q_n-2}\int_{a_i}^{b_i}|r_\varepsilon(y)|^\alpha\,dy\right]^{\frac{1}{\alpha}}\leq K\varepsilon^{\frac{1}{\alpha}}.$$

Analogously, it can be shown that the last sum in (40), is also not greater than $K\varepsilon^{\frac{1}{\alpha}}$. Together with (38)-(41), this implies the first relation in (11).

Let us prove the second inequality in (11). It is not too hard to show that there exists a constant $C_3 > 0$ such that for all $i, 1 \le i \le q_n - 2$

$$\frac{1}{C_3} \le \int_0^1 |\frac{d^2 z_i}{dz_1^2}| dz_1 \le C_3. \tag{42}$$

Note that the function $\frac{d^2\psi_i}{dz_i^2}$ is defined almost everywhere. By differentiating (33) we get

$$\frac{d^2\psi_i}{dz_i^2} = \frac{2A_iA_i' - A_i''}{(1+A_iz_i)(1+A_i(z_i-1))} - \frac{2(A_i'z_i + a_i)}{1+A_iz_i} \cdot \frac{d\psi_i}{dz_i} - (\frac{d\psi_i}{dz_i})^2, \tag{43}$$

where

$$A_i'' = \frac{d^2A_i}{dz_i^2} = (b_i - a_i)^2 \frac{d^2A_i}{dx_i^2}. \tag{44}$$

Finally, differentiating (34) gives

$$\frac{d^2A_i}{dx_i^2} = \frac{\frac{2}{f'(a_i)(x_i-a_i)^2}\int\limits_{a_i}^{x_i}(f''(x_i)-f''(y))(y-a_i)dy + \frac{2}{f'(a_i)(b_i-x_i)^2}\int\limits_{x_i}^{b_i}(f''(x_i)-f''(y))(b_i-y)dy}{1+\frac{1}{f'(a_i)(b_i-a_i)}\int\limits_{a_i}^{b_i} f''(y)(y-a_i)dy}. \tag{45}$$

Using the relations (10), (11), (43) and (45) it can easily be shown that

$$\int_0^1 |(z_1 - z_1^2)\tau_n''(z_1)|dz_1 = \int_0^1 |(z_1 - z_1^2)\sum_{i=1}^{q_n-2}\left[\frac{d^2\psi_i}{dz_i^2}(\frac{dz_i}{dz_1})^2 + \frac{d\psi_i}{dz_i}\frac{d^2z_i}{dz_1^2}\right]|dz_1 \le$$

$$\le K\int_0^1 \left|(z_1 - z_1^2)\sum_{i=1}^{q_n-2}(b_i-a_i)^2\frac{d^2A_i}{dx_i^2}\right| + K\varepsilon \le$$

$$\le K\int_0^1 \left|(z_1 - z_1^2)\sum_{i=1}^{q_n-2}\left(\frac{b_i-a_i}{x_i-a_i}\right)^2\int_{a_i}^{x_i}[f''(x_i)-f''(y)]\frac{y-a_i}{x_i-a_i}dy\right|dz_1 +$$

$$+K\int_0^1 \left|(z_1 - z_1^2)\sum_{i=1}^{q_n-2}\left(\frac{b_i-a_i}{x_i-a_i}\right)^2\int_{x_i}^{b_i}[f''(x_i)-f''(y)]\frac{b_i-y}{b_i-x_i}dy\right|dz_1 + K\varepsilon.$$

The proof of the second relation in (11) proceeds now exactly as in the previous case. This concludes the proof of Lemma 3.1. □

Proof. Lemma 3.2. It is easy to check, that

$$f(x) - f(y_2) = f'(x_b + 0)(x - y_2) + \int_x^{y_2} f''(y)(y - y_2)dy,\ x_b < x < y_2,$$

$$f(x) - f(x_b) = f'(x_b - 0)(x - x_b) + \int_x^{x_b} f''(y)(y - x_b)dy,\ y_1 < x < x_b,$$

$$f(y_1) - f(x_b) = f'(x_b - 0)(y_1 - x_b) + \int_{y_1}^{x_b} f''(y)(y - x_b)dy,$$

$$f(x_b) - f(y_2) = f'(x_b + 0)(x_b - y_2) + \int_{x_b}^{y_2} f''(y)(y - y_2)dy.$$

This together with (6) imply that

$$\bar{f}_1(z_0) = \begin{cases} \frac{z_0 + H_1(x)}{\sigma^2(1-d)+d+H_3+H_4}, & z_0 \in [0, d], \\ \frac{\sigma^2 z_0 + d(1-\sigma^2) + H_2(x) + H_4}{\sigma^2(1-d)+d+H_3+H_4}, & z_0 \in (d, 1], \end{cases} \tag{46}$$

where

$$H_1(x) = \frac{1}{f'(x_b + 0)(y_1 - y_2)} \int_{x}^{y_2} f''(y)(y - y_2)dy, \;\; x \in [x_b, y_2],$$

$$H_2(x) = \frac{1}{f'(x_b + 0)(y_1 - y_2)} \int_{x}^{x_b} f''(y)(y - x_b)dy, \;\; x \in [y_1, x_b],$$

$$H_3 = \frac{1}{f'(x_b + 0)(y_1 - y_2)} \int_{y_1}^{x_b} f''(y)(y - x_b)dy, \;\; H_4 = \frac{1}{f'(x_b + 0)(y_1 - y_2)} \int_{x_b}^{y_2} f''(y)(y - y_2)dy.$$

Because $\ell(\Delta_0^{(n-1)}) \leq \lambda^n$, using the condition (d) and Hölder inequality we find that the relation

$$|H_1(x)| \leq \frac{1}{f'(x_b + 0)(y_1 - y_2)} \int_{x}^{y_2} |f''(y)(y - y_2)|dy \leq K\lambda^{\frac{n}{\beta}} \tag{47}$$

holds for all $x \in [y_1, x_b]$. Analogously, it can be shown that the following inequalities

$$|H_2(x)|, \; |H_3|, \; |H_4| \leq K\lambda^{\frac{n}{\beta}} \tag{48}$$

for all $x \in (x_b, y_2]$. Summing (46)-(48), we get the first relation in (12). We have

$$\bar{f''}_1(z_0) = \frac{f''(y_2 + z_0(y_1 - y_2))(y_1 - y_2)^2}{f(y_1) - f(y_2)} = \frac{1}{f'(x_b + 0)} \frac{f''(y_2 + z_0(y_1 - y_2))(y_1 - y_2)}{\sigma^2(1 - d) + d + H_3 + H_4}$$

for almost all z_0. Since the inequalities

$$\int_0^d |F_1''(z_0)|dz_0, \;\; \int_d^1 |F_1''(z_0)|dz_0 \leq K\lambda^{\frac{n}{\beta}}$$

hold, also in the case (47) this proves the second relation in (12). Lemma 3.2 is completely proved. □

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Akhtam Dzhalilov was born in 1956. He graduated from Lomonosov Moscow State University in 1979. He got Ph.D. degree in Physics and Mathematics in Lomonosov Moscow State University in 1987 and Doctor of Sciences degree in 2000 in Institute of Mathematics of the Academy of Sciences of Uzbekistan (Tashkent). His research interests include dynamical systems, statistical mechanics and probability theory. Presently he is the Head and Professor of the Department of Probability Theory and Teaching Methods of Mathematics (Mathematics and Mechanics Faculty of Samarkand State University, Samarkand, Uzbekistan).

Abdumajid Begmatov was born in 1982 in Djizzak (Uzbekistan). He is a Ph.D. student of Institute of Mathematics and Information Technologies of Academy of Sciences of Uzbekistan.