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BERTRAND MATE OF BIHARMONIC CURVES IN THE SPECIAL THREE-DIMENSIONAL KENMOTSU MANIFOLD $\mathbb{K}$ WITH $\eta-$ PARALLEL RICCI TENSOR

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Abstract. In this paper, we study biharmonic curves in the special three-dimensional Kenmotsu manifold $\mathbb{K}$ with η-parallel Ricci tensor. We characterize the biharmonic curves in terms of their curvature and torsion. Moreover, we construct parametric equations of Bertrand mate of biharmonic curves in the special three-dimensional Kenmotsu manifold $\mathbb{K}$ with η-parallel Ricci tensor.

Keywords: biharmonic curve, Kenmotsu manifold, Bertrand mate.

AMS Subject Classification: 53C41, 53A10.

1. Introduction

Let (M,g) and (N,h) be manifolds and $\phi : M \longrightarrow N$ a smooth map. Denote by ∇^{ϕ} the connection of the vector bundle $\phi^{*}TN$ induced from the Levi-Civita connection ∇^{h} of (N,h). The *second fundamental form* $\nabla d\phi$ is defined by

$$(\nabla d\phi)(X,Y) = \nabla^{\phi}_{X} d\phi(Y) - d\phi(\nabla_X Y), \quad X,Y \in \Gamma(TM).$$

Here ∇ is the Levi-Civita connection of (M,g). The *tension field* $\tau(\phi)$ is a section of $\phi^{*}TN$ defined by

$$\tau(\phi) = tr\nabla d\phi. \tag{1}$$

A smooth map ϕ is said to be *harmonic* if its tension field vanishes. It is well known that ϕ is harmonic if and only if ϕ is a critical point of the *energy*:

$$E(\phi) = \frac{1}{2}\int h(d\phi, d\phi)\, dv_g$$

over every compact region of M. Now let $\phi : M \longrightarrow N$ be a harmonic map. Then the Hessian $\mathcal{H}$ of E is given by

$$\mathcal{H}_{\phi}(V,W) = \int h(\mathcal{J}_{\phi}(V), W)\, dv_g, \quad V,W \in \Gamma(\phi^{*}TN).$$

Here the *Jacobi operator* $\mathcal{J}_{\phi}$ is defined by

$$\mathcal{J}_{\phi}(V) := \overline{\Delta}_{\phi} V - \mathcal{R}_{\phi}(V), \quad V \in \Gamma(\phi^{*}TN), \tag{2}$$

$$\overline{\Delta}_{\phi} := \sum_{i=1}^{m}\left(\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i} - \nabla^{\phi}_{\nabla_{e_i}e_i}\right), \mathcal{R}_{\phi}(V) = \sum_{i=1}^{m} R^{N}(V, d\phi(e_i))\, d\phi(e_i), \tag{3}$$

where R^{N} and $\{e_i\}$ are the Riemannian curvature of N, and a local orthonormal frame field of M, respectively.

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Let $\phi : (M, g) \to (N, h)$ be a smooth map between two Lorentzian manifolds. The *bienergy* $E_2(\phi)$ of ϕ over compact domain $\Omega \subset M$ is defined by

$$E_2(\phi) = \int_\Omega h(\tau(\phi), \tau(\phi))\, dv_g.$$

A smooth map $\phi : (M, g) \to (N, h)$ is said to be *biharmonic* if it is a critical point of the $E_2(\phi)$.

The section $\tau_2(\phi)$ is called the *bitension field* of ϕ and the Euler-Lagrange equation of E_2 is

$$\tau_2(\phi) := -\mathcal{J}_\phi(\tau(\phi)) = 0. \tag{4}$$

In general, the fourth-order equation (4) is difficult to solve. Natural candidates for solutions are submanifolds of parallel mean curvature, see [10, 11]; or submanifolds with harmonic mean curvature, see [5, 6, 9]. In [1, 2], examples of biharmonic nonminimal submanifolds of spheres are given, as well as a complete classification of biharmonic curves in a sphere. Biharmonic curves on a surface are studied in [3]. We adopt a different approach here to construct biharmonic, nonharmonic maps.

Recently, there has been a growing interest in the theory of biharmonic maps which can be divided in two main research directions. On the one side, constructing the examples and classification results have become important from the differential geometric aspect. The other side is the analytic aspect from the point of view of partial differential equations [2, 12, 15, 20, 21], because biharmonic maps are solutions of a fourth order strongly elliptic semilinear PDE. In differential geometry, harmonic maps, candidate minimisers of the Dirichlet energy, can be described as constraining a rubber sheet to fit on a marble manifold in a position of elastica equilibrium, i.e. without tension [7]. However, when this scheme falls through, and it can, as corroborated by the case of the two-torus and the two-sphere [8], a best map will minimise this failure, measured by the total tension, called bienergy. In the more geometrically meaningful context of immersions, the fact that the tension field is normal to the image submanifold, suggests that the most effective deformations must be sought in the normal direction.

In this paper, we study biharmonic curves in the special three-dimensional Kenmotsu manifold $\mathbb{K}$ with η-parallel Ricci tensor. We characterize the biharmonic curves in terms of their curvature and torsion. Moreover, we construct parametric equations of Bertrand mate of biharmonic curves in the special three-dimensional Kenmotsu manifold $\mathbb{K}$ with η-parallel Ricci tensor.

2. Preliminaries

Let $M^{2n+1}(\phi, \xi, \eta, g)$ be an almost contact Riemannian manifold with 1-form η, the associated vector field ξ, $(1,1)$-tensor field ϕ and the associated Riemannian metric g. It is well known that [1]

$$\phi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(\phi X) = 0, \tag{5}$$

$$\phi^2(X) = -X + \eta(X)\xi, \tag{6}$$

$$g(X, \xi) = \eta(X), \tag{7}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{8}$$

for any vector fields X, Y on M. Moreover,

$$(\nabla_X \phi) Y = -\eta(Y)\phi(X) - g(X, \phi Y)\xi, \qquad X, Y \in \chi(M), \tag{9}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{10}$$

where ∇ denotes the Riemannian connection of g, then (M,ϕ,ξ,η,g) is called an almost Kenmotsu manifold [1].

In Kenmotsu manifolds the following relations hold [1]:

$$\begin{aligned}
(\nabla_X\eta)Y &= g(\phi X,\phi Y), &(11)\\
\eta(R(X,Y)Z) &= \eta(Y)g(X,Z)-\eta(X)g(Y,Z), &(12)\\
R(X,Y)\xi &= \eta(X)Y-\eta(Y)X, &(13)\\
R(\xi,X)Y &= \eta(Y)X-g(X,Y)\xi, &(14)\\
R(\xi,X)\xi &= X-\eta(X)\xi, &(15)\\
S(\phi X,\phi Y) &= S(X,Y)+2n\eta(X)\eta(Y), &(16)\\
S(X,\xi) &= -2n\eta(X), &(17)\\
(\nabla_X R)(X,Y)\xi &= g(Z,X)Y-g(Z,Y)X-R(X,Y)Z, &(18)
\end{aligned}$$

where R is the Riemannian curvature tensor and S is the Ricci tensor. In a Riemannian manifold we also have

$$g(R(W,X)Y,Z)+g(R(W,X)Z,Y)=0 \tag{19}$$

for every vector fields X,Y,Z.

3. Special three-dimensional Kenmotsu manifold $\mathbb{K}$ with η-parallel Ricci tensor

Definition 3.1. *The Ricci tensor S of a Kenmotsu manifold is called η-parallel if it satisfies*

$$(\nabla_X S)(\phi Y,\phi Z)=0.$$

We consider the three-dimensional manifold

$$\mathbb{K}=\left\{\left(x^1,x^2,x^3\right)\in\mathbb{R}^3:\left(x^1,x^2,x^3\right)\neq(0,0,0)\right\},$$

where $\left(x^1,x^2,x^3\right)$ are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$\mathbf{e}_1=x^3\frac{\partial}{\partial x^1},\quad \mathbf{e}_2=x^3\frac{\partial}{\partial x^2},\quad \mathbf{e}_3=-x^3\frac{\partial}{\partial x^3} \tag{20}$$

are linearly independent at each point of $\mathbb{K}$. Let g be the Riemannian metric defined by

$$\begin{aligned}
g(\mathbf{e}_1,\mathbf{e}_1) &= g(\mathbf{e}_2,\mathbf{e}_2)=g(\mathbf{e}_3,\mathbf{e}_3)=1, &(21)\\
g(\mathbf{e}_1,\mathbf{e}_2) &= g(\mathbf{e}_2,\mathbf{e}_3)=g(\mathbf{e}_1,\mathbf{e}_3)=0.
\end{aligned}$$

The characterising properties of $\chi(\mathbb{K})$ are the following commutation relations:

$$[\mathbf{e}_1,\mathbf{e}_2]=0,\quad [\mathbf{e}_1,\mathbf{e}_3]=\mathbf{e}_1,\quad [\mathbf{e}_2,\mathbf{e}_3]=\mathbf{e}_2. \tag{22}$$

Let η be the 1-form defined by

$$\eta(Z)=g(Z,\mathbf{e}_3)\text{ for any }Z\in\chi(M).$$

Let be the (1) tensor field defined by

$$\phi(\mathbf{e}_1)=-\mathbf{e}_2,\ \phi(\mathbf{e}_2)=\mathbf{e}_1,\ \phi(\mathbf{e}_3)=0.$$

Then using the linearity of ϕ and g we have

$$\eta(\mathbf{e}_3)=1, \tag{23}$$

$$\phi^2(Z)=-Z+\ \eta(Z)\mathbf{e}_3, \tag{24}$$

$$g(\phi Z,\phi W)=g(Z,W)-\eta(Z)\eta(W), \tag{25}$$

for any $Z, W \in \chi(M)$. Thus for $\mathbf{e}_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on $\mathbb{M}$, [1,16].

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g\left(\nabla_X Y, Z\right) &= Xg\left(Y,Z\right) + Yg\left(Z,X\right) - Zg\left(X,Y\right) - \\ &\quad -g\left(X,[Y,Z]\right) - g\left(Y,[X,Z]\right) + g\left(Z,[X,Y]\right), \end{aligned}$$

which is known as Koszul's formula.

Koszul's formula yields

$$\begin{aligned} \nabla_{\mathbf{e}_1}\mathbf{e}_1 &= -\mathbf{e}_3,\ \ \nabla_{\mathbf{e}_1}\mathbf{e}_2 = 0,\ \ \nabla_{\mathbf{e}_1}\mathbf{e}_3 = \mathbf{e}_1, \\ \nabla_{\mathbf{e}_2}\mathbf{e}_1 &= 0,\ \ \nabla_{\mathbf{e}_2}\mathbf{e}_2 = -\mathbf{e}_3,\ \ \nabla_{\mathbf{e}_2}\mathbf{e}_3 = \mathbf{e}_2, \\ \nabla_{\mathbf{e}_3}\mathbf{e}_1 &= 0,\ \ \nabla_{\mathbf{e}_3}\mathbf{e}_2 = 0,\ \nabla_{\mathbf{e}_3}\mathbf{e}_3 = 0. \end{aligned}$$

Moreover we put

$$R_{ijkl} = R(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_l),$$

where the indices i, j, k and l take the values $1, 2$ and 3.

$$R_{1212} = R_{1313} = R_{2323} = 1.$$

Now, we consider biharmonicity of curves in the special three-dimensional Kenmotsu manifold $\mathbb{K}$.

4. Biharmonic curves in the special three-dimensional Kenmotsu manifold $\mathbb{K}$ with η-parallel Ricci tensor

Biharmonic equation for the curve γ reduces to

$$\nabla^3_{\mathbf{T}}\mathbf{T} - R\left(\mathbf{T}, \nabla_{\mathbf{T}}\mathbf{T}\right)\mathbf{T} = 0, \tag{26}$$

that is, γ is called a biharmonic curve if it is a solution of the equation (26).

Let us consider biharmonicity of curves in the special three-dimensional Kenmotsu manifold $\mathbb{K}$ with η-parallel Ricci tensor. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{27}$$

where $\kappa = |\mathcal{T}(\gamma)| = |\nabla_{\mathbf{T}}\mathbf{T}|$ is the curvature of γ and τ its torsion and

$$\begin{aligned} g\left(\mathbf{T},\mathbf{T}\right) &= 1,\ g\left(\mathbf{N},\mathbf{N}\right) = 1,\ g\left(\mathbf{B},\mathbf{B}\right) = 1, \\ g\left(\mathbf{T},\mathbf{N}\right) &= g\left(\mathbf{T},\mathbf{B}\right) = g\left(\mathbf{N},\mathbf{B}\right) = 0. \end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{T} &= T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3, \\ \mathbf{N} &= N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3, \\ \mathbf{B} &= \mathbf{T}\times\mathbf{N} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3. \end{aligned} \tag{28}$$

Theorem 4.1. *$\gamma : I \longrightarrow \mathbb{K}$ is a biharmonic curve if and only if*

$$\begin{aligned} \kappa &= \textit{constant} \neq 0, \\ \kappa^2 + \tau^2 &= 1 - B_3^2, \\ \tau' &= N_3 B_3. \end{aligned} \tag{29}$$

Proof. Using (26) and Frenet formulas (27), we have (29). □

Theorem 4.2. *Let $\gamma : I \longrightarrow \mathbb{K}$be a non-geodesic curve on the special three-dimensional Kenmotsu manifold $\mathbb{K}$ with η-parallel Ricci tensor parametrized by arc length. If κ is constant and $N_3B_3 \neq 0$, then γ is not biharmonic.*

Proof. Using Frenet formulas (27) and $\nabla_T B$, we have

$$B_3' = -\tau N_3. \tag{30}$$

Assume now that γ is biharmonic. Then, using $\tau' = N_3 B_3 \neq 0$ and from (29), we obtain

$$\tau\tau' = -B_3 B_3'$$

and

$$\tau N_3 B_3 = B_3 B_3' \ . \tag{31}$$

Substituting B_3' in equation (30), we find

$$\tau = 0. \tag{32}$$

Therefore, τ is constant and we have a contradiction. □

Theorem 4.3. *Let $\gamma : I \longrightarrow \mathbb{K}$ be a unit speed non-geodesic curve with constant curvature. Then, the parametric equations of γ are*

$$\begin{aligned} x^1(s) &= C_2 - \frac{C_1 \sin^3\varphi}{\kappa^2} e^{-\cos\varphi s}\Big(\sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}} \cos\left[\sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}} s + C\right] - \\ &\quad - \cos\varphi \sin\left[\sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}} s + C\right]\Big), \\ x^2(s) &= C_3 - \frac{C_1 \sin^3\varphi}{\kappa^2} e^{-\cos\varphi s}\Big(-\cos\varphi \cos\left[\sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}} s + C\right] + \\ &\quad + \sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}} \sin\left[\sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}} s + C\right]\Big), \\ x^3(s) &= C_1 e^{-\cos\varphi s}, \end{aligned} \tag{33}$$

where C, C_1, C_2, C_3 are constants of integration.

Proof. Since γ is biharmonic, γ is a helix. So, without loss of generality, we take the axis of γ is parallel to the vector $\mathbf{e}_3$. Then,

$$g(\mathbf{T}, \mathbf{e}_3) = T_3 = \cos\varphi, \tag{34}$$

where φ is constant angle.

The tangent vector can be written in the following form

$$\mathbf{T} = T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3. \tag{35}$$

On the other hand the tangent vector $\mathbf{T}$ is a unit vector, so the following condition is satisfied

$$T_1^2 + T_2^2 = 1 - \cos^2\varphi. \tag{36}$$

Noting that $\cos^2\varphi + \sin^2\varphi = 1$, we have

$$T_1^2 + T_2^2 = \sin^2\varphi. \tag{37}$$

The general solution of (37) can be written in the following form

$$\begin{aligned} T_1 &= \sin\varphi\sin\mu, \\ T_2 &= \sin\varphi\cos\mu, \end{aligned} \tag{38}$$

where μ is an arbitrary function of s.

So, substituting the components T_1, T_2 and T_3 in the equation (35), we have the following equation

$$\mathbf{T} = \sin\varphi\sin\mu\mathbf{e}_1 + \sin\varphi\cos\mu\mathbf{e}_2 + \cos\varphi\mathbf{e}_3. \tag{39}$$

Since $|\nabla_{\mathbf{T}}\mathbf{T}| = \kappa$, we obtain

$$\mu = \sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}}s + C, \tag{40}$$

where $C \in \mathbb{R}$.

Thus (39) and (40), imply

$$\begin{aligned} \mathbf{T} &= \sin\varphi\sin\left[\sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}}s + C\right]\mathbf{e}_1 + \\ &\quad + \sin\varphi\cos\left[\sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}}s + C\right]\mathbf{e}_2 + \cos\varphi\mathbf{e}_3. \end{aligned} \tag{41}$$

Using (20) in (41), we obtain

$$\begin{aligned} \mathbf{T} &= (x^3\sin\varphi\sin\left[\sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}}s + C\right], x^3\sin\varphi\cos\left[\sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}}s + C\right], \\ &\quad -x^3\cos\varphi). \end{aligned} \tag{42}$$

If we take integration above equation we have (33). (for details see [12]) □

We can use Mathematica, yields.

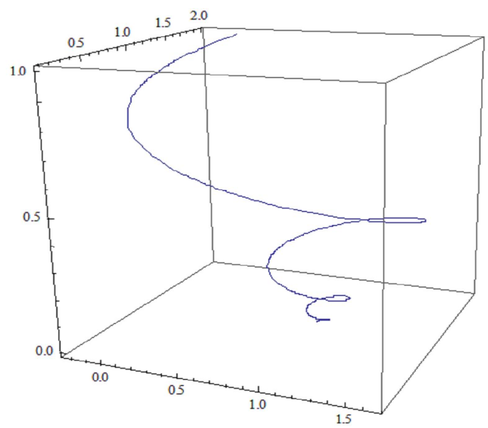

Figure 1. $\cos\varphi = \sin\varphi = \frac{\sqrt{2}}{2}$, $C = C_1 = C_2 = C_3 = \kappa = 1$.

5. Bertrand mate of biharmonic curves in the special three-dimensional Kenmotsu manifold $\mathbb{K}$ with η-parallel Ricci tensor

Definition 5.1. *A curve $\gamma : I \longrightarrow \mathbb{K}$ with $\kappa \neq 0$ is called a Bertrand curve if there exist a curve $\widetilde{\gamma} : I \longrightarrow \mathbb{K}$ such that the principal normal lines of $\gamma$and $\widetilde{\gamma}$ at $s \in I$ are equal. In this case $\tilde{\gamma}$ is called a Bertrand mate of γ.*

Theorem 5.1. *Let $\gamma : I \longrightarrow \mathbb{K}$be a Bertrand curve parametrized by arc length. A Bertrand mate of γ is as follows:*

$$\tilde{\gamma}(s) = \gamma(s) + \lambda \mathbf{N}(s), \quad \forall s \in I, \tag{43}$$

where λ is constant.

Theorem 5.2. *Let $\gamma : I \longrightarrow \mathbb{K}$be a biharmonic curve parametrized by arc length. If $\tilde{\gamma}$ is a Bertrand mate of γ, then the parametric equations of $\tilde{\gamma}$ are*

$$\begin{aligned} \tilde{x}^1(s) &= C_2 - \frac{C_1 \sin^3 \varphi}{\kappa^2} e^{-\cos\varphi s}(\cos[s+C] - \cos\varphi \sin[s+C]) + \\ &+ \frac{\lambda}{\kappa}\left(\sin\varphi\cos[s+C] + \cos\varphi\sin\varphi\sin[s+C]\right)(\bar{C}_1 e^{\sin\varphi s} + \bar{C}_2 e^{-\sin\varphi s}), \\ \tilde{x}^2(s) &= C_3 - \frac{C_1 \sin^3 \varphi}{\kappa^2} e^{-\cos\varphi s}(-\cos\varphi\cos[s+C] + \sin[s+C]) + \\ &+ \frac{\lambda}{\kappa}\left(-\sin\varphi\sin[s+C] + \cos\varphi\sin\varphi\cos[s+C]\right)(\bar{C}_1 e^{\sin\varphi s} + \bar{C}_2 e^{-\sin\varphi s}), \\ \tilde{x}^3(s) &= C_1 e^{-\cos\varphi s} + \frac{\lambda}{\kappa}(\bar{C}_1 e^{\sin\varphi s} + \bar{C}_2 e^{-\sin\varphi s}), \end{aligned} \tag{44}$$

where C, $\overline{C}_1$, $\overline{C}_2$, C_1, C_2, C_3 are constants of integration and $= \sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}}$.

Proof. Using first equation of (28), we have

$$\nabla_{\mathbf{T}}\mathbf{T} = \left(T_1' + T_1 T_3\right)\mathbf{e}_1 + \left(T_2' + T_2 T_3\right)\mathbf{e}_2 + \left(T_3' - T_2^2 - T_1^2\right)\mathbf{e}_3. \tag{45}$$

From (20) and (41), we get

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \sin\varphi\left(\cos[s+C] + \cos\varphi\sin[s+C]\right)\mathbf{e}_1 + \\ &+ \sin\varphi\left(-\sin[s+C] + \cos\varphi\cos[s+C]\right)\mathbf{e}_2 - \sin^2\varphi\mathbf{e}_3, \end{aligned} \tag{46}$$

where $= \sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}}$.

By the use of Frenet formulas (27), we get

$$\begin{aligned} \mathbf{N} &= \frac{1}{\kappa}\nabla_{\mathbf{T}}\mathbf{T} = \\ &= \frac{1}{\kappa}[(\sin\varphi\cos[s+C] + \cos\varphi\sin\varphi\sin[s+C])\mathbf{e}_1 + \\ &+ (-\sin\varphi\sin[s+C] + \cos\varphi\sin\varphi\cos[s+C])\mathbf{e}_2 - \sin^2\varphi\mathbf{e}_3]. \end{aligned} \tag{47}$$

Substituting (20) in (47), we have

$$\begin{aligned} \mathbf{N} &= \frac{1}{\kappa}((\sin\varphi\cos[s+C] + \cos\varphi\sin\varphi\sin[s+C])(\bar{C}_1 e^{\sin\varphi s} + \bar{C}_2 e^{-\sin\varphi s}), \\ &(-\sin\varphi\sin[s+C] + \cos\varphi\sin\varphi\cos[s+C])(\bar{C}_1 e^{\sin\varphi s} + \bar{C}_2 e^{-\sin\varphi s}), \\ &(\bar{C}_1 e^{\sin\varphi s} + \bar{C}_2 e^{-\sin\varphi s})). \end{aligned} \tag{48}$$

Next, we substitute (33) and (48) into (43), we get (44). The proof is completed. □

Similarly, we can use Mathematica in above theorem, yields.

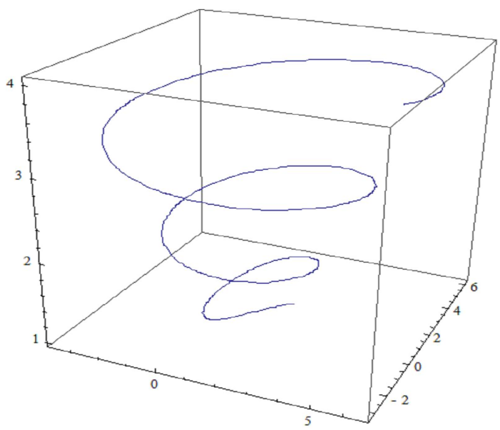

Figure 2. $\cos\varphi = \sin\varphi = \frac{\sqrt{2}}{2}$, $\bar{C}_1 = \bar{C}_2 = C = C_1 = C_2 = C_3 = \lambda = \kappa = 1$.

6. Summary

Biharmonic curves are utilized in many physical situations, particularly in fluid dynamics and elasticity problems. Most important applications of the theory of functions of a complex variable were obtained in the plane theory of elasticity and in the approximate theory of plates subject to normal loading.

Therefore, we study biharmonic curves in the special three-dimensional Kenmotsu manifold $\mathbb{K}$ with η-parallel Ricci tensor. We characterize the biharmonic curves in terms of their curvature and torsion. Moreover, we construct parametric equations of Bertrand mate of biharmonic curves in the special three-dimensional Kenmotsu manifold $\mathbb{K}$ with η-parallel Ricci tensor.

References

[1] Blair, D.E., (1976), Contact Manifolds in Riemannian Geometry, Lecture Notes in Mathematics, Springer-Verlag 509, Berlin-New York.

[2] Caddeo, R., Montaldo, S., (2001), Biharmonic submanifolds of $\mathbb{S}^3$, Internat. J. Math., 12(8), pp.867–876.

[3] Caddeo, R., Montaldo, S., Oniciuc, C., Biharmonic submanifolds of $\mathbb{S}^n$, Israel J. Math., to appear.

[4] Caddeo, R., Montaldo, S., Piu, P., Biharmonic curves on a surface, Rend. Mat., to appear.

[5] Chen, B.Y., (1991), Some open problems and conjectures on submanifolds of finite type, Soochow J. Math., 17, pp.169–188.

[6] Dimitric, I., (1992), Submanifolds of $\mathbb{E}^m$ with harmonic mean curvature vector, Bull. Inst. Math. Acad. Sinica, 20, pp.53–65.

[7] Eells, J., Lemaire, L., (1978), A report on harmonic maps, Bull., London Math. Soc., 10, pp.1–68.

[8] Eells, J., Sampson, J.H., (1964), Harmonic mappings of Riemannian manifolds, Amer. J. Math., 86, pp.109–160.

[9] Hasanis, T., Vlachos, T., (1995), Hypersurfaces in $\mathbb{E}^4$ with harmonic mean curvature vector field, Math. Nachr., 172, pp.145–169.

[10] Jiang, G.Y., (1986), 2-harmonic isometric immersions between Riemannian manifolds, Chinese Ann. Math. Ser., A., 7(2), pp.130–144.

[11] Jiang, G.Y., (1986), 2-harmonic maps and their first and second variational formulas, Chinese Ann. Math. Ser. A., 7(4), pp.389–402.

[12] Körpınar, T., Turhan, E., Biharmonic curves in the special three-dimensional Kentmotsu manifold K with η-parallel Ricci tensor, Revista Notas de Matemática (in press).

[13] Loubeau, E., Oniciuc, C., (2007), On the biharmonic and harmonic indices of the Hopf map, Transactions of the American Mathematical Society, 359(11), pp.5239–5256.

[14] O'Neill, B., (1983), Semi-Riemannian Geometry, Academic Press, New York.

[15] Strzelecki, P., (2003), On biharmonic maps and their generalizations, Calc. Var., 18, pp.401-432.
[16] Shukla, S.S., Shukla, M.K., (2009), On $\phi-$Ricci Symmetric Kenmotsu manifolds, Novi Sad J. Math., 39(2), pp.89-95.
[17] Turhan, E., Körpınar, T., (2009), Characterize on the Heisenberg Group with left invariant Lorentzian metric, Demonstratio Mathematica, 42(2), pp.423-428.
[18] Turhan, E., Körpınar, T., (2010), On Characterization of Timelike Horizontal Biharmonic Curves in the Lorentzian Heisenberg Group Heis^3, Zeitschrift für Naturforschung A- A. Journal of Physical Sciences, 65a, pp.641-648.
[19] Wang, C., (2004), Biharmonic maps from $\mathbb{R}^4$ into a Riemannian manifold, Math. Z., 247, pp.65-87.
[20] Wang, C., (2004), Remarks on biharmonic maps into spheres, Calc. Var., 21, pp.221-242.

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