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THE COMPLEMENTARY GROUP OF PROPER MOTIONS OF THE MINKOWSKI METRIC

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Abstract. It is shown that the Poincare group which is a semidirect product of the group of translations and the Lorentz group, is not a single phisicaly important group of proper motions of Minkowski metric. The complementary group of proper motions of the metric in a class of noninertial reference system has been found.

Keywords: Poincare group, Lorentz group, Minkowski metric, proper motions of metric, pseudo-orthogonal transformation.

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1. Introduction

The whole group of all proper transformations of Minkowski pseudo-Euclidean space $R^4_{1,3}$, which leaves the metric invariant, in other words the whole group of proper motions of pseudo-Euclidean space, is a semidirect product $T^4 \rtimes SO(1,3)$ of the four-dimensional group of translations T^4 and the proper pseudo-orthogonal group $SO(1,3)$. At present we identify the pseudo-orthogonal group $SO(1,3)$ with the Lorentz group of proper linear pseudo-orthogonal transformations

$$\begin{aligned} dx^0 &= dx'^0 ch\psi + dx'^i sh\psi, \\ dx^i &= dx'^0 sh\psi + dx'^i ch\psi, \end{aligned} \tag{1}$$

where $(x'^0, x'^1, x'^2, x'^3), (x^0, x^1, x^2, x^3)$ are pseudo-Euclidean coordinates of a point before and after the transformation, $i = 1, 2, 3$.

In terms of special relativity theory the Lorentz transformation provides the transition from one inertial reference system (x^0, x^1, x^2, x^3) to another one (x'^0, x'^1, x'^2, x'^3) which moves at speed v along axis x^i, and it retains the differential quadratic form $dx^{0^2} - dx^{1^2} - dx^{2^2} - dx^{3^2}$. Hence the Lorentz transformation may be rewritten in the following form:

$$dx^0 = \frac{dx'^0 + \frac{v}{c} dx'^i}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}, \quad dx^i = \frac{dx'^i + \frac{v}{c} dx'^0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}, \tag{2}$$

where $ch\psi = \dfrac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$, c is a certain constant which special relativity theory interprets as the light speed.

The semidirect product of the group of translations and the Lorentz group represents the Poincare group. Hence the Poincare group consists of a linear group of proper motions of Minkowski metric in the class of all inertial reference systems. One can extend the proper

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motion group of the Minkowski metric, if we consider not only linear transformations but also nonlinear ones preserving the Minkowski metric.

Let us define $SO(1,3)$ as a general group of all proper pseudo-orthogonal transformations in $R^4_{1,3}$, linear and nonlinear ones, preserving the differential quadratic form $dx^{0^2} - dx^{1^2} - dx^{2^2} - - dx^{3^2}$.

It would be wrong to say that the Lorentz group coincides with the group of all proper pseudo-orthogonal transformations $SO(1,3)$, or to identify the Poincare group with the $T^4 \rtimes SO(1,3)$ group. In fact, the Lorentz group is a subgroup of $SO(1,3)$. Moreover, the $SO(1,3)$ group contains another phisicaly important proper motions group of Minkowski metric in a class of noninertial frames of reference, which is surprising enough. Construction of this group provides the subject of this report.

Taking into account the wide interest in the Minkowski spaces of large dimension in, for instance, theories of strings, superstrings, supergravity etc., we consider it advisable to observe the problem for Minkowski spaces $R^{N+1}_{1,N}$ with arbitrary dimension of N.

2. Construction of a complementary group of proper pseudo-orthogonal transformations in $R^3_{1,2}$

We will start with the simple case of three-dimensional pseudo-Euclidean space, in which in addition to Lorentz group, there exists another group of proper motions of Minkowski metric.

Any proper rotation in a three-dimensional pseudo-Euclidean space $R^3_{1,2}$, i.e. any proper orthogonal or pseudo-orthogonal transformation which preserves the coordinates origin, can be decomposed into three rotations in planes $\{x^1x^2\}$, $\{x^0x^1\}$, $\{x^0x^2\}$, and one rotation in the space $R^3_{1,2} = \{x^0x^1x^2\}$ itself, which cannot be reduced to the previous ones. The first rotation modifies the space coordinates only and corresponds to the space rotations. The second and third rotations act in pseudo-Euclidean planes and correspond to the proper pseudo-orthogonal rotations or, which is the same, the Lorentz transformations of the form (1) or (2). Now we will observe in detail the extra rotation in the $R^3_{1,2}$.

The desired transformation should leave invariant the differential quadratic form

$$dx^{0^2} - dx^{1^2} - dx^{2^2}, \tag{3}$$

or the form equivalent to it, in the polar coordinate system

$$dx^{0^2} - dr^2 - r^2 d\varphi^2, \tag{4}$$

where $x^1 = r\cos\varphi$, $x^2 = r\sin\varphi$.

Let (x^0, r, φ) be coordinates of a point s in the three-dimensional pseudo-Euclidean subspace $\{x^0x^1x^2\}$. We require that the transformations from the desired group $\mathbf{G}$ leave unchanged the radial coordinate r of the point s. A subset in $\{x^0x^1x^2\}$ which consists of all point of $\mathbf{G}s$ type, forms an orbit of element s with respect to the group $\mathbf{G}$ and is denoted $G_r(s)$. If s_1 and s_2 are two points from $\{x^0x^1x^2\}$ which have the same parameter r, then $G_r(s_1) = G_r(s_2)$, i.e. orbits of two different points are only defined by the radial coordinate r of these points. Hence we will use symbol G_r to denote an orbit. If $r_1 \neq r_2$, then $G_{r_1} \cap G_{r_2} \subset \oslash$, which means that the orbits of two different points either coincide, or do not intersect. A Pseudo-Euclidean subspace $\{x^0x^1x^2\}$ is a union of pairwise not intersecting orbits $G_r : \{x^0x^1x^2\} = \cup_{0<r<\infty} G_r$.

Let f_r be the homomorphism of a group $\mathbf{G}$ into group $\mathbf{G}_r$ effective in orbit $G_r : f_r(\mathbf{G}) = \mathbf{G}_r$. According to the definition of effectiveness of a group $\mathbf{G}$ in an orbit G_r, we have $\mathbf{G}s = \mathbf{G}_r s$, when $s \in G_r$.

Now we directly construct the group $\mathbf{G}_r$. Let the differentials of coordinates x^0, φ of a point $s(x^0, r, \varphi)$ be subject to linear transformation $a_\omega : (dx^0, d\varphi) \to (dx'^0, d\varphi')$ of the form:

$$dx^0 = \frac{dx'^0 + \dfrac{r^2\omega}{c} d\varphi'}{\sqrt{1 - (\dfrac{r\omega}{c})^2}}, \tag{5}$$

$$d\varphi = \frac{d\varphi' + \dfrac{\omega}{c} dx'^0}{\sqrt{1 - \left(\dfrac{r\omega}{c}\right)^2}},$$

where ω is the angular velocity of a circle of radius r in a plane $\{x^1 x^2\}$ relative to the origin of coordinates, $|\omega| < \dfrac{c}{r}$.

It is easy to see that the transformations (5) leaves invariant the differential quadratic form (4) and is the element of the group $\mathbf{G}_r$. Obviously, the element a_0 corresponds to unit I of the group $\mathbf{G}_r$. The element $a_{-\omega}$ is identified as an element inverse to $a_\omega \in \mathbf{G}_r$, i.e. $(a_\omega)^{-1} = a_{-\omega}$. If $a_{\omega_1}, a_{\omega_2} \in \mathbf{G}_r$, then their group product may be determined as

$$a_{\omega_1} \cdot a_{\omega_2} = a_{\frac{\omega_1 + \omega_2}{1 + (\frac{r}{c})^2 \omega_1 \omega_2}}.$$

If $|\omega_1|, |\omega_2| < \dfrac{c}{r}$, it immediately follows from the group multiplication that

$$\left|\frac{\omega_1 + \omega_2}{1 + \left(\frac{r}{c}\right)^2 \omega_1 \omega_2}\right| < \frac{c}{r}.$$

The group $\mathbf{G}_r$ can formally be supplemented by adding two elements $a_{-\frac{c}{r}}$ and $a_{\frac{c}{r}}$.

The replenished group will be stationary relative to the associated elements $a_{-\frac{c}{r}}$ and $a_{\frac{c}{r}}$. Indeed, for any a_ω, $|\omega| \leq \dfrac{c}{r}$ we have $a_\omega \cdot a_{\pm\frac{c}{r}} = a_{\pm\frac{c}{r}}$.

In terms of special relativity theory it means that the linear velocity of a circular motion cannot exceed the light speed c and that the light speed is identical in all steadily rotating noninertial reference systems.

If we introduce the variable $y = r\varphi$, the transformations (5) may be rewritten in the following form:

$$dx^0 = \frac{dx'^0 + \frac{v}{c} dy'}{\sqrt{1 - (\frac{v}{c})^2}}, \; dy = \frac{dy' + \frac{v}{c} dx'^0}{\sqrt{1 - (\frac{v}{c})^2}}, \tag{6}$$

where $v = r\omega$.

If we compare the Lorentz transformations (2) with the transformations (6), it is not hard to observe their formal structural similarity, written in different reference systems. However, there is much difference between them. The Lorentz transformations describes the transition between inertial reference systems which move uniformly and straight relative to each other, yet the transformations (6) describes the transition between noninertial systems rotating uniformly in circles with different angular velocities. This difference becomes clear when we go over from the polar coordinate system to the pseudo-Euclidean one, a considerably nonlinear character of the transformations emerges. Indeed, if we go over to the pseudo-Euclidean coordinate system x^0, x^1, x^2, then the transformations (6) assumes the form

$$dx^0 = \frac{1}{\sqrt{1 - (\frac{v}{c})^2}} dx'^0 - \frac{\omega}{c} \frac{x'^2}{\sqrt{1 - (\frac{v}{c})^2}} dx'^1 + \frac{\omega}{c} \frac{x'^1}{\sqrt{1 - (\frac{v}{c})^2}} dx'^2,$$

$$dx^1 = -\frac{v}{c}\frac{\sin\varphi}{\sqrt{1-(\frac{v}{c})^2}}dx'^0 + \frac{\omega}{v}Adx'^1 + \frac{\omega}{v}Bdx'^2, \tag{7}$$

$$dx^2 = \frac{v}{c}\frac{\cos\varphi}{\sqrt{1-(\frac{v}{c})^2}}dx'^0 + \frac{\omega}{v}Cdx'^1 + \frac{\omega}{v}Ddx'^2,$$

where $\varphi = \dfrac{\arccos\frac{x'^1}{\sqrt{x'^{1^2}+x'^{2^2}}} + \frac{\omega}{c}x'^0}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$, $A = x'^1\cos\varphi + \frac{x'^2\sin\varphi}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$, $B = x'^2\cos\varphi - \frac{x'^1\sin\varphi}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$, $C = x'^1\sin\varphi - \frac{x'^2\cos\varphi}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$, $D = x'^2\sin\varphi + \frac{x'^1\cos\varphi}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$.

The whole group of all proper pseudo-orthogonal transformations $SO(1,2)$ in a space $R^3_{1,2}$ is thus generated by elements from the Lorentz group (2) and by transformations of form (7) from the group $\mathbf{G}_r$ in the pseudo-Euclidean coordinate system.

3. The complementary group of proper pseudo-orthogonal transformations in $R^{2n+1}_{1,2n}$

In the previous paragraph we have reviewed a simple case of odd-dimensional Minkowski space $R^{2n+1}_{1,2n}$ with $n = 1$, in which the construction of additional motion group was favoured by a lucky choice of coordinate system. The analogous construction may also be done in the general case.

Any element A of the maximum subgroup $SO(2n)$, constituent of $SO(1,2n)$ group, according to the known result in linear algebra [1] is represented as a block-diagonal matrix

$$A = \begin{pmatrix} \square & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & \square & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \square \end{pmatrix}, \tag{8}$$

where the kthdiagonal block has the form $\begin{pmatrix} \cos\varphi_k & \sin\varphi_k \\ -\sin\varphi_k & \cos\varphi_k \end{pmatrix}$.

On the basis of (8) we will introduce a biharmonic coordinate system [2] $r, \varphi_1, \dots, \varphi_n, \theta_1, \dots, \theta_{n-1}$, which divides all Cartesian coordinates $x^1, \dots, x^{2n}$ from Euclidean subspace R^{2n} into pairs $\left(z^k, y^k\right)$, where $z^k = x^{2k-1}$, $y^k = x^{2k}$, $k = 1, \dots, n$, the x^0 coordinate remains unchanged.

Suppose that

$$\begin{aligned}
z^1 &= r\cos\varphi_1\sin\theta_1\dots\sin\theta_{n-2}\sin\theta_{n-1}, \\
y^1 &= r\sin\varphi_1\sin\theta_1\dots\sin\theta_{n-2}\sin\theta_{n-1}, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
z^k &= r\cos\varphi_k\sin\theta_1\dots\sin\theta_{n-k}\cos\theta_{n-k+1}, \\
y^k &= r\sin\varphi_k\sin\theta_1\dots\sin\theta_{n-k}\cos\theta_{n-k+1}, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
z^n &= r\cos\varphi_n\cos\theta_1, \\
y^n &= r\sin\varphi_n\cos\theta_1,
\end{aligned} \tag{9}$$

where $k = 2, \dots, n$.

The differential quadratic form

$$dx^{0^2} - dx^{1^2} - \ldots - dx^{2n^2} = dx^{0^2} - dz^{1^2} - dy^{1^2} - \ldots - dz^{n^2} - dy^{n^2},$$

which remains invariant by the Lorentz group of transformations, taking the following form in biharmonic coordinate system

$$dx^{0^2} - dr^2 - r_1^2 d\varphi_1^2 - \ldots - r_n^2 d\varphi_n^2 -$$

$$-r^2 \left(d\theta_1^2 + \sin^2\theta_1 \left(d\theta_2^2 + \sin^2\theta_2 \left(\ldots + \sin^2\theta_{n-2} d\theta_{n-1}^2\right)\right)\ldots\right), \tag{10}$$

where $r_1 = r\sin\theta_1 \ldots \sin\theta_{n-1}, \ldots,\ r_k = r\sin\theta_1 \ldots \sin\theta_{n-k}\cos\theta_{n-k+1},\ k = 2, \ldots, n$, i.e. the following condition is fulfilled

$$r_1^2 + \ldots + r_n^2 = r^2. \tag{11}$$

Let $r_1, \ldots, r_n$ be fixed, then the differential quadratic form (10) remains invariant relative to n one-parameter subgroups of proper pseudo-orthogonal transformations, where the kth subgroup consists of the following transformations

$$dx^0 = \frac{dx'^0 + \frac{r_k^2\omega}{c} d\varphi'_k}{\sqrt{1 - \left(\frac{r_k\omega}{c}\right)^2}},\ \ d\varphi_k = \frac{d\varphi'_k + \frac{\omega}{c} dx'^0}{\sqrt{1 - \left(\frac{r_k\omega}{c}\right)^2}}. \tag{12}$$

Obviously, the required invariability for parameters $r_1, \ldots, r_n$ by pseudo-orthogonal transformations (12) is equivalent to the requirement of constancy for $\theta_1, \ldots, \theta_{n-1}$, which provides the invariance for the differential form

$$r^2 \left(d\theta_1^2 + \sin^2\theta_1 \left(\ldots + \sin^2\theta_{n-2} d\theta_{n-1}^2\right)\ldots\right).$$

The invariance of the quadratic form $dx^{0^2} - r_1^2 d\varphi_1^2 - \ldots - r_n^2 d\varphi_n^2$ is due to the very type of transformations (12). Hence the group of all proper pseudo-orthogonal transformations G_r, which leaves the radial parameter r unchanged, is generated by all possible pseudo-orthogonal transformations of form (12) from n subgroups $G_{r_1}, \ldots, G_{r_n}$ which correspond to fixed arbitrary sets $r_1, \ldots, r_n$, satisfying the condition (11).

The expansional group of proper pseudo-orthogonal transformations in $R_{1,2n}^{2n+1}$ is thus generated by the Lorentz group and group G_r.

4. The complementary group of the proper pseudo-orthogonal transformations in $R_{1,2n+1}^{2(n+1)}$

In the case of even-dimensional Minkowski space, any element A of the maximum orthogonal subgroup $SO(2n+1)$ from group $SO(1, 2n+1)$ may be represented in a block-diagonal form

$$A = \begin{pmatrix} \square & 0 & 0 & 0 \\ 0 & \ldots & 0 & 0 \\ 0 & 0 & \square & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{13}$$

where the kth diagonal block has the same form as in the relation (8).

Dividing the Cartesian coordinates $x^1, \ldots, x^{2n}$, as in the previous case, into pairs (z^k, y^k), where $x^{2k-1} = z^k$, $x^{2k} = y^k$, $k = 1, \ldots, n$, and assuming $x^{2n+1} = z^{n+1}$, we will introduce the biharmonic coordinate system $r, \varphi_1, \dots, \varphi_n, \theta_1, \dots, \theta_n$ so that

$$\begin{aligned} z^1 &= r\cos\varphi_1 \sin\theta_1 \dots \sin\theta_n, \\ y^1 &= r\sin\varphi_1 \sin\theta_1 \dots \sin\theta_n, \\ &\dots\dots\dots\dots\dots\dots \\ z^k &= r\cos\varphi_k \sin\theta_1 \dots \sin\theta_{n-k+1}\cos\theta_{n-k+2}, \\ y^k &= r\sin\varphi_k \sin\theta_1 \dots \sin\theta_{n-k+1}\cos\theta_{n-k+2}, \\ &\dots\dots\dots\dots\dots\dots \\ z^n &= r\cos\varphi_n \sin\theta_1 \cos\theta_2, \\ y^n &= r\sin\varphi_n \sin\theta_1 \cos\theta_2, \\ z^{n+1} &= r\cos\theta_1. \end{aligned} \tag{14}$$

The differential quadratic form

$$dx^{0^2} - dx^{1^2} - \dots - dx^{2n+1^2} = dx^{0^2} - dz^{1^2} - dy^{1^2} - \dots - dz^{n^2} - dy^{n^2} - dz^{n+1^2}$$

is invariant relative to the Lorentz group of transformations and in the biharmonic coordinate system (14) it assumes the form

$$dx^{0^2} - dr^2 - r_1^2 d\varphi_1^2 - \dots - r_n d\varphi_n^2 - r^2\left(d\theta_1^2 + \sin^2\theta_1\left(d\theta_2^2 + \sin^2\theta_2\left(\dots + \sin^2\theta_{n-1} d\theta_n^2\right)\right)\dots\right),$$

where $r_1 = r\sin\theta_1 \dots \sin\theta_n, \dots, r_k = r\sin\theta_1 \sin\theta_{n-k+1}\cos\theta_{n-k+2}$, $k = 2, \dots, n$, i.e. the following condition is fulfilled

$$r_1^2 + \dots + r_n^2 = r^2 \sin^2\theta_1. \tag{15}$$

The form remains invariant relative to any pseudo-orthogonal transformations of the form (12) and is identical with the form (10) with the only difference that parameters $r_1, \dots, r_n$ satisfy the condition (15) and not (12) as in the case of the odd-dimensional spaces.

These results be summed up as the following statement:

Theorem 4.1. *The subgroup of proper pseudo-orthogonal transformations from $SO(1, N)$ of Minkowski space $R_{1,N}^{N+1}$ is generated by various transformations from the Lorentz group of the form*

$$dx^0 = \frac{dx'^0 + \frac{v}{c}dx'^i}{\sqrt{1 - (\frac{v}{c})^2}}, \ dx^i = \frac{dx'^i + \frac{v}{c}dx'^0}{\sqrt{1 - (\frac{v}{c})^2}}, \ i = 1, \dots, N,$$

relative to the pseudo-Euclidean coordinate system $x^0, \dots, x^N$, and by various transformations relative to the biharmonic coordinate system $x^0, r, \varphi_1, \dots, \varphi_{[\frac{N}{2}]}, \theta_1, \dots, \theta_{[\frac{N}{2}]}$ from group G_r of the form

$$dx^0 = \frac{dx'^0 + \frac{r_k^2\omega}{c}d\varphi'_k}{\sqrt{1 - (\frac{r_k\omega}{c})^2}}, \ d\varphi_k = \frac{d\varphi'_k + \frac{\omega}{c}dx'^0}{\sqrt{1 - (\frac{r_k\omega}{c})^2}}, \ k = 1, \dots, [\frac{N}{2}],$$

with fixed $r_1, \dots, r_{[\frac{N}{2}]}, r$, satisfying the following condition $r_1^2 + \dots + r_{[\frac{N}{2}]}^2 = r^2$ if N is even and $r_1^2 + \dots + r_{[\frac{N}{2}]}^2 = r^2 \sin^2\theta_1$ if N is odd.

5. The Maxwell equations invariance relative to the motion group G_r

We will show that the Maxwell equations

$$\frac{\partial F_{ij}}{\partial x^k} + \frac{\partial F_{ki}}{\partial x^j} + \frac{\partial F_{jk}}{\partial x^i} = 0, \tag{16}$$

$$\frac{\partial F^{ij}}{\partial x^j} = J^i, \quad i, j = 0, 1, ..., 3 \tag{17}$$

appear invariant relative to the complementary transformation group G_r in the four-dimentional Minkowski space $R^4_{1,3}$, where

$$\begin{aligned} F^{ij} &= F_{ij} \text{ for } i, j = 1, 2, 3 \\ F^{oj} &= -F_{oj} \text{ for } j = 1, 2, 3. \end{aligned} \tag{18}$$

That is, we want to show that an arbitrary transformation of coordinates x^0, x^1, x^2, x^3 from the group G_r of kind (7) in new coordinates x'^0, x'^1, x'^2, x'^3, leaves invariant the Maxwell equations (16), (17) relative to transformed tensors $F_{i'j'} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} F_{ij}$, $F^{i'j'} = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^{j'}}{\partial x^j} F^{ij}$, and that the condition (18) is fulfilled.

In fact, having in mind that the Minkowski metric remains invariant during the pseudo-orthogonal transformations of kind (7), the correlation (18) remain unchanged for the transformed tensors $F^{i'j'}$ and $F_{i'j'}$.

Equation (16) remains invariant after any continuously differentiable nondegenerate transformation of coordinates, which is following from the structure of this equation, which represents the Bianchi identity. Further, counting that $\frac{\partial}{\partial x^j}$ is transformed as a vector during any pseudo-orthogonal coordinate transformations, of kind (7) in particular, we are having $\frac{\partial}{\partial x^j} = \frac{\partial x^{m'}}{\partial x^j} \frac{\partial}{\partial x^{m'}}$. Then the left side of the Eq (17) can be represented in the following way:

$$\frac{\partial F^{ij}}{\partial x^j} = \frac{\partial x^{m'}}{\partial x^j} \frac{\partial}{\partial x^{m'}} \frac{\partial x^i}{\partial x^{k'}} \frac{\partial x^j}{\partial x^{l'}} F^{k'l'} = \frac{\partial x^i}{\partial x^{k'}} \frac{\partial F^{k'l'}}{\partial x^{l'}}$$

and the equation (17) assumes the following form in the new coordinates:

$$\frac{\partial F^{k'l'}}{\partial x^{l'}} = J^{k'}, \text{ where } J^{k'} = \frac{\partial x^{k'}}{\partial x^i} J^i.$$

Thus, we consider the Eq (17) invariance to be proven during the pseudo-orthogonal thanformations of kind (7).

Using the previos constraction we come to a general conclusion that the Maxwell equations are invariant relative to all proper motions of Minkowski metric. Strictly speaking, this conclusion immediately follows from the proof of Maxwell equations invariance relative to the group G_r, generalized to the whole group of metric proper motions.

Various combinations of Lorentz transformations, and transformations from the group G_r lead to a rather wide set of possible motions of non-inertial reference systems, relative to which Maxwell equations retain their form. Such motions are represented by the uniform cycloidal motion, or a motion in a helical line along a coordinate axis, or better a uniform spiral motion on a torus. Thus, every proper transformation from the group $SO(1,3)$ has a corresponding continuous motion of a coordinate system, relative to which Maxwell equations retain their form. For example, relativistic motion in the spiral line at constant speed V along X^3 axis and at constant speed v in plane $\{x^1, x^2\}$ in a circle of radius r, is corresponding to the pseudo-orthogonal coordinate transformation from the proper motions group of Minkowski metric

$$\begin{pmatrix} dx^0 \\ dr \\ d\varphi \\ dx^3 \end{pmatrix} = \begin{pmatrix} \xi\eta & 0 & \frac{vr}{c}\xi\eta & \frac{V}{c}\xi \\ 0 & 1 & 0 & 0 \\ \frac{v}{cr}\eta & 0 & \eta & 0 \\ \frac{V}{c}\xi\eta & 0 & \frac{Vvr}{c^2}\xi\eta & \xi \end{pmatrix} \begin{pmatrix} dx'^0 \\ dr' \\ d\varphi' \\ dx'^3 \end{pmatrix},$$

where $\xi = \frac{1}{\sqrt{1-\left(\frac{V}{c}\right)^2}}$, $\eta = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$.

This explains why the Maxwell equations are invariant relative to this thanformation.

6. Discussion and conclusions

The existence of the extra group of proper motions of the Minkowski metric in the class of noninertial reference frames gives us a new fundamental group of Minkowski space symmetry. It occures a question if this fact can lead us, in the case of four-dimensional Minkowski space, to a revision of the fundamentals of special relativity theory. Should we require invariance for all physical theories in respect to the new symmetry group? The fact that the light speed and the Maxwell electrodynamic equations, as it was shown above, result invariant relative to various uniformly rotating non-inertial reference systems makes us feel that such requirement may be justified. On the other hand, the fact that we can distinguish, among reference systems uniformly rotating around a common center, a reference system in a state of complete rest relative to this center, contradicts the very spirit of special relativity theory.

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