TWMS Jour. Pure Appl. Math., V.3, N.1, 2012, pp.35-40

CONTINUITY OF FUZZY APPROXIMATE ADDITIVE MAPPINGS

A. ZOHRI[1], N. EGHBALI[2]

Abstract. In this paper we introduce the concept of fuzzy continuity. Then we investigate the continuity of fuzzy approximate linear maps.

Keywords: fuzzy normed space; fuzzy approximate linear map.

AMS Subject Classification: 46S40, 39B52, 39B82, 26E50, 46S50.

1. Introduction

The stability of functional equations is an interesting area of research for mathematicians, but it can be also of importance to persons who work outside of the realm of pure mathematics. For example, physicists are interested in the stability of the mathematical formulae which they use to model physical processes. More precisely, physicists and other scientists are interested in determining when a small change in an equation used to the model of a phenomenon, gives a large changes in the results predicted by the equation.

It seems that the stability problem of functional equations had been first raised by Ulam [7]: For what metric groups G is it true that an approximate additive of G is necessarily near to a strict linear map?

An answer to the above problem has been given as follows [6]. Suppose E_1 and E_2 are two real Banach spaces and $f : E_1 \to E_2$ is a mapping. If there exist $\delta \geq 0$ and $0 \leq p < 1$ such that $||f(x + y) - f(x) - f(y)|| \leq \delta(||x||^p + ||y||^p)$ for all $x, y \in E_1$, then there is a unique additive mapping $T : E_1 \to E_2$ such that $||f(x) - T(x)|| \leq 2\delta||x||^p/|2 - 2^p|$ for every $x \in E_1$.

In 1992, Gavruta [1] generalized the result of Rassias for the admissible control functions.

Moreover the approximated mappings have been studied extensively in several papers. (See for instance [2], [3]).

Fuzzy notion introduced firstly by Zadeh [8] that has been widely involved in different subjects of mathematics. Zadeh's definition of a fuzzy set characterized by a function from a nonempty set X to $[0, 1]$.

Fuzzy set theory is a powerful hand set for modelling uncertainty and vagueness in various problems arising in the field of science and engineering.

Later in 1984, Katsaras [4] defined a fuzzy norm on a linear space to construct a fuzzy vector topological structure on the space.

In this paper we introduce the new concept of continuity for functions from a normed space into a fuzzy normed space and then we study the continuity for fuzzy approximate additive mappings.

[1]Department of Mathematics, Payame Noor University-Tehran, Iran,
e-mail: alizohri@gmail.com

[2]Department of Mathematics, University of Mohaghegh Ardabili, Ardabil, Iran,
e-mail: nasrineghbali@gmail.com

Manuscript received April 2011.

2. Preliminaries

In this section, we provide a collection of definitions and related results which are essential and used in the next discussions.

Definition 2.1. [5] *Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0,1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $t, s \in \mathbb{R}$,*
(N1) $N(x,c) = 0$ for $c \leq 0$;
(N2) $x = 0$ if and only if $N(x,c) = 1$ for all $c > 0$;
(N3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x+y, s+t) \geq min\{N(x,s), N(y,t)\}$;
(N5) $N(x,.)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;
(N6) for $x \neq 0$, $N(x,.)$ is (upper semi) continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed linear space.
We give an example for a fuzzy normed space:

Example 2.1. [5] *Let $(X, ||.||)$ be a normed linear space. Then*

$$N(x,t) = \begin{cases} 0, & t \leq 0; \\ \frac{t}{||x||}, & 0 < t \leq ||x||; \\ 1, & t > ||x|| \end{cases}$$

is a fuzzy norm on X.

Definition 2.2. [5] *Let (X, N) be a fuzzy normed linear space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N-\lim_{n\to\infty} x_n = x$.*

Definition 2.3. [5] *A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.*

It is known that every convergent sequence in a fuzzy normed space is Cauchy and if each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and furthermore the fuzzy normed space is called a fuzzy Banach space.

Theorem 2.1. *Let X be a normed linear space and (Y, N) be a fuzzy Banach space. Let $\theta \geq 0$ and $q \neq 1$. Suppose that $f : X \to Y$ is a function such that*

$$\lim_{t\to\infty} N(f(x+y) - f(x) - f(y), t\theta(||x||^q + ||y||^q)) = 1$$

uniformly on $X \times X$. Then there is a unique additive mapping $T : X \to Y$ such that

$$\lim_{t\to\infty} N(T(x) - f(x), \frac{2\theta t||x||^q}{|1-2^{q-1}|}) = 1$$

uniformly on X.

Proof. [5] □

3. Continuity of fuzzy approximately additive mappings

We start our work with the definition of fuzzy continuity.

Definition 3.1. *Let X be a normed space, (Y, N) a fuzzy normed space and $T : X \to Y$ be a function. We say that $T : X \to Y$ is fuzzy continues at a point $s_0 \in X$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for each s with $||s - s_0|| < \delta$,*

$$\lim_{t\to\infty} N(T(s)-T(s_0), t\varepsilon) = 1,$$

uniformly on X.

Theorem 3.1. *Let X, Y be normed linear spaces and T be a map from X into Y. T is continuous at zero if and only if T is continuous.*

Proof. Suppose that T is continuous at zero so by definition 3.1, for all $\varepsilon > 0$ and all $s \in X$ we have $\lim_{t\to\infty} N(T(s), t\varepsilon) = 1$ for some $\delta > 0$ such that $||s|| < \delta$. Given $\epsilon > 0$ we have

$$N(T(s), t\varepsilon) \geq 1 - \epsilon$$

for all $s \in X$.

Now, for given $\varepsilon > 0$ and all $s, s_0 \in X$, we can find some $\delta > 0$ such that $||s - s_0|| < \delta$, then

$$N(T(s)-T(s_0), t\varepsilon) \geq min\{N(T(s), t\varepsilon/2), N(T(s_0), t\varepsilon/2)\} \geq 1 - \epsilon.$$

So

$$\lim_{t\to\infty} N(T(s)-T(s_0), t\varepsilon) = 1.$$

□

Theorem 3.2. *Let X, Y be normed linear spaces and T be an additive map from X into Y. T is continuous at a point if and only if T is continuous.*

Proof. Suppose that T is continuous at s_0. So by definition (3.1), for all $\varepsilon > 0$ and all $s \in X$ we have $\lim_{t\to\infty} N(T(s)-T(s_0), t\varepsilon) = 1$ for some $\delta > 0$ such that $||s - s_0|| < \delta$. We prove that T is continuous at zero.

$$\lim_{t\to\infty} N(T(s), t\varepsilon) = \lim_{t\to\infty} N(T(s)-T(s_0)+T(s_0), t\varepsilon) = \lim_{t\to\infty} N(T(s+s_0)-T(s_0), t\varepsilon) = 1$$

for all $s \in X$.

So T is continuous at zero. By Theorem (3.1) the proof is complete. □

Theorem 3.3. *Let $0 \leq q < 1$, X, Y, θ and f satisfies the conditions of Theorem (2.1). If for some $x \in X$ and all $n \in \mathbb{N}$, the mapping $g : \mathbb{R} \to Y$ defined by $g(s) = f(2^n sx)$ for every $s \in \mathbb{R}$ is fuzzy continuous. Then the mapping $s \mapsto T(sx)$ from $\mathbb{R}$ to Y is fuzzy continuous.*

Proof. Using Theorem (2.1) we deduce that, there exists a unique additive mapping T such that

$$\lim_{t\to\infty} N(T(x)-f(x), \tfrac{2\theta t||x||^q}{|1-2^{q-1}|}) = 1.$$

Given $\varepsilon > 0$, we can find some $t_0 > 0$ such that

$$N(T(x)-f(x), \frac{2\theta t||x||^q}{|1-2^{q-1}|}) \geq 1 - \varepsilon, \tag{1}$$

for all $t \geq t_0$. Since $\lim_{n\to\infty} \frac{2\theta t 2^{n(q-1)}||x||^q}{|1-2^{q-1}|} = 0$, there is some n_0 such that $\frac{2\theta t 2^{n(q-1)}||x||^q}{|1-2^{q-1}|} < t/3$ for all $n \geq n_0$. On the other hand, since N is a nondecreasing function hence for each $n \geq n_0$,

$$N(T(x) - \tfrac{f(2^n x)}{2^n}, t/3) > N(T(x) - \tfrac{f(2^n x)}{2^n}, \tfrac{2\theta t 2^{n(q-1)}||x||^q}{|1-2^{q-1}|}).$$

Note that for each $x \in X$, $t \in \mathbb{R}$ and $n \in \mathbb{N}$, by (1) we have

$$N(T(x) - \frac{f(2^n x)}{2^n}, t/3) \geq 1 - \varepsilon. \tag{2}$$

Fix $x \in X$ and $s_0 \in \mathbb{R}$. From (2) follows that

$$N(T(s_0x) - \frac{f(2^n s_0 x)}{2^n}, t/3) \geq 1 - \varepsilon.$$

By the fuzzy continuity of the mapping $t \mapsto f(2^n tx)$, there exists δ such that for each s with $0 < |s - s_0| < \delta$, we have

$$N(\frac{f(2^n sx)}{2^n} - \frac{f(2^n s_0 x)}{2^n}, t/3) \geq 1 - \varepsilon.$$

It follows that

$$N(T(sx) - T(s_0x), t) \geq$$
$$\geq min\{N(T(sx) - \frac{f(2^n sx)}{2^n}, t/3), N(\frac{f(2^n sx)}{2^n} - \frac{f(2^n s_0 x)}{2^n}, t/3), N(T(s_0x) - \frac{f(2^n s_0 x)}{2^n}, t/3)\} \geq 1 - \varepsilon,$$

for each s with $0 < |s - s_0| < \delta$. Hence, the mapping $s \mapsto T(sx)$ is fuzzy continuous.

□

Theorem 3.4. *Let $0 \leq q < 1$, X, Y and f satisfies the conditions of Theorem (2.1). If for some $x \in X$ and all $n \in \mathbb{N}$, the mapping $g : \mathbb{R} \to Y$ defined by $g(s) = f(2^n sx)$ is fuzzy continuous. Then T is real homogeneous.*

Proof. For each $q \in \mathbb{Q}$, we have $T(qx) = qT(x)$. $\mathbb{Q}$ is a dense subset of $\mathbb{R}$. Fix $r \in \mathbb{R}$ and $t > 0$. Choose a rational sequence q_n such that $q_n \to r$. Then, there exists $\delta > 0$ such that

$$N(T(rx) - rT(x), t) \geq$$
$$\geq min\{N(T(rx) - T(q_n x), t/3), N(T(q_n x) - q_n T(x), t/3), N(q_n T(x) - rT(x), t/3)\}.$$

By using the Theorem (3.3) for given $\varepsilon > 0$, we have

$$N(T(rx) - rT(x), t) \geq min\{1 - \varepsilon, 1, N(q_n T(x) - rT(x), t/3)\}.$$

By taking n tend to infinity,

$$N(T(rx) - rT(x), t) \geq 1 - \varepsilon.$$

So T is real homogeneous. □

4. A fuzzy norm for the set of fuzzy mappings

In this section we define a norm for the class of fuzzy mappings.

Definition 4.1. *Let X be a normed space, (Y, N) a fuzzy normed space and $T : X \to Y$ be a function. We define*

$$N(T, t) = \inf\{N(T(x), t) : x \in X\},$$

for all $t > 0$.

Theorem 4.1. *Let X be a normed space, (Y, N) a fuzzy normed space. Then $N(T, t)$ is defined in Definition (4.1) is a norm.*

Proof. We check the items in Definition (2.1). It is easy to see that (N1), (N2), (N3), (N5) and (N6) are true. We consider (N4):

$$N(T + S, t + s) = \inf\{N((T + S)(x), t + s) : x \in X\} \geq \inf\{min\{N(T(x), t), N(S(x), s)\} : x \in X\} = \inf\{\inf\{N(T(x), t), N(S(x), s)\} : x \in X\} = \inf\{\inf\{N(T(x), t) : x \in X\}, \inf\{N(S(x), s) : x \in X\}\} = min\{\inf\{N(T(x), t) : x \in X\}, \inf\{N(S(x), s) : x \in X\}\}$$

(N5)$N(T,.)$ is a non-decreasing function on $\mathbb{R}$ and $\lim\limits_{t\to\infty} N(T,t) = 1$;

(N6) for $x \neq 0$, $N(T,t)$ is (upper semi) continuous on $\mathbb{R}$.

□

Let X be a normed space, (Y,N) a fuzzy normed space. By associating to each T the number $N(T,t)$ makes the class of functions from X into Y a normed space.

Theorem 4.2. *Let X be a normed space, (Y,N) a fuzzy normed space. If Y is a Banach space, so the class of functions from X into Y is complete, too.*

Proof. Assume that Y is complete and that $\{T_n\}$ is a cauchy sequence. For given $\varepsilon > 0$,

$$N(T_{n+p} - T_n, t) = \inf\{N(T_{n+p}(x) - T_n(x), t) : x \in X\} \geq 1 - \varepsilon.$$

So $N(T_{n+p}(x) - T_n(x), t) \geq 1 - \varepsilon$ for every $x \in X$. Therefore $\{T_n(x)\}$ is Cauchy in Y. Now, there exists $T : X \to Y$ such that $\lim\limits_{t\to\infty} N(T_n(x) - T(x), t) = 1$. On the other hand we have $N(T_n - T) = \inf\{N(T_n(x) - T(x), t) : x \in X\}$. By taking t tend to infinity we have $\lim\limits_{t\to\infty} N(T_n - T, t) = 1$.

□

References

[1] Gavruta, P., (1994), A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl., 184, pp.431-436.

[2] Hyers, D.H., Isac, G., Rassias, Th.M., (1998), Stability of Functional Equations in Several Variables, Birkhäuser, Basel.

[3] Hyers, D.H., Rassias, Th.M., (1992), Approximate homomorphisms, Aequationes Math., 44(2-3), pp.125-153.

[4] Katsaras, A.K., (1984), Fuzzy topological vector spaces II, Fuzzy Sets and Systems, 12, pp.143-154.

[5] Mirmostafaee, A.K., Moslehian, M.S., (2007), Fuzzy almost quadratic functions, arXiv:0710.0106v1 [math.FA], 30 Sep.

[6] Rassias, Th.M., (1978), On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc., 72, pp.297–300.

[7] Ulam, S.M., (1960), Problems in Modern Mathematics, Chap. VI, Science Editions John Wiley and Sons, New York.

[8] Zadeh, L.A., (1965), Fuzzy sets, Inform. and Control, 8, pp.338-353.

Ali Zohri was born in 1961 in Ardabil in North Western Iran. He graduated from Tabriz University with a M.Sc. degree in Mathematics in 1989 and got his Ph.D. in Functional Analysis from Guilan University in 2009. He has been scientific member of Payame Noor University since 1991.

Nasrin Eghbali was born in 1978 in Ardebil, Iran. She received her M.Sc. and Ph.D. degrees from Guilan University in 2004 and 2008 respectively. Her main researches are Functional Analysis, Approximate mappings, Stability.