TWMS Jour. Pure Appl. Math. V.3, N.2, 2012, pp.220-230

SOLUTION OF PARAMETRIC INVERSE PROBLEM OF ATMOSPHERIC OPTICS BY MONTE CARLO METHODS*

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Abstract. In the article parametric inverse problems of atmospheric optics are considered. To solve these problems we applied algorithm "the method of dependent tests for transport theory problems" of Monte Carlo methods. The problems reduced to linear system of equations for parameters and solved by optimizations methods. The numerical solution of the optical depth of the extinction specified. The approximation error is no more than 5-10 percent, which is quite satisfactory for Monte Carlo methods.

Keywords: inverse problem, atmospheric optics, Monte Carlo methods, local estimate, transport theory problems, indicatrix, optical depth, albedo, molecular scattering coefficients.

AMS Subject Classification: 15A29, 65C05

1. Introduction

Target setting of inverse problem of atmosferic optics

Let's consider the equation in operator form:

$$\mathrm{L}f = \psi, \tag{1}$$

where $f,\, \psi \in F$. Scalar product is $\big(g,\, h\big) = \int g(x)h(x)dx$, at that integration is with respect to domain of $h \in F$, $g \in F^*$, x is the set of all variables in the problem (time, space and velocities). Along with operator L we consider adjoint operator L^*, which is defined by

$$\big(g,\, \mathrm{L}h\big) = \big(\mathrm{L}^* g,\, h\big) \tag{2}$$

for any functions g and h from the corresponding spaces F^* and F. We put into consideration inhomogeneous adjoint equation

$$\mathrm{L}^* f_p^* = p(x), \tag{3}$$

where $p(x)$ is any function for the time being, $f_p^* \in F^*$. Putting solutions f and f_p^* of theequations (1) and (3) in formula (2) instead of h and g respectively, we obtain $\big(f_p^*,\, \mathrm{L}f\big) = \big(f,\, \mathrm{L}^* f_p^*\big)$ or $\big(f_p^*,\, \psi\big) = \big(f,\, p\big)$, in other words the functional's value $I_p(f) = \big(f,\, p\big)$ can be determined in two ways; either solve equation (1) and determine the value by formula $I_p(f) = \big(f,\, p\big)$ or solve equation (3) and determine the same value by the formula $I_p(f) = I_\psi^*\big(f_p^*\big) = \big(f_p^*,\, \psi\big)$.

* Research and the writing of this paper were supported by the Grant of the Ministry Education and Science of the Republic of Kazakhstan

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Manuscript received January 2012.

Let's assume that L and L^* are uniquely defined by the set of some parameters $\{\alpha_i\}$, $i = 1, 2, \ldots, n$ and suppose that these parameters change: $\alpha_i' = \alpha_i + \delta\alpha_i$, then respectively ($\delta$ is "operator" of perturbation)

$$\mathrm{L}' = \mathrm{L} + \delta\mathrm{L}, \;\; f(x) \to f'(x), \;\; I_p(f) \to I_p' + \delta I_p.$$

We establish the link between δL and δI_p. For this purpose we consider perturbed problem

$$\mathrm{L}'f' = \big(\mathrm{L} + \delta\mathrm{L}\big)f' = \psi \tag{4}$$

and non-perturbed adjoint problem

$$\mathrm{L}^* f_p^* = p. \tag{5}$$

Multiplying scalarly (4) by f_p^* and (5) by f', deducting one from another and using the definition of adjoint operator, we obtain

$$\delta I_p = -\big(f_p^*, \, \delta\mathrm{L}f'\big), \tag{6}$$

where $\delta I_p = \big(f', \, p\big) - \big(f_p^*, \, \psi\big) = I_p(f') - I_p(f)$.

If function ψ isn't perturbed, then the equation (6) is true, but if the function ψ has a perturbation $\delta\psi$, then (6) turns into

$$\Big(f_p^*, \, \delta\mathrm{L}f' - \delta\psi\Big) = -\delta I_p. \tag{7}$$

Relations (6) or (7) are the base for problem setting and solving inverse problems in aero-optics.

Suppose, than we know solution of non-perturbed problem (1), that is we know $I_{p_k}(f)$ and the set of functionals on solution of perturbed problem is established: $I_{p_k}(f'), \; k = 1, 2, \ldots, m$, in this case the right-hand sides of the simultaneous equations are known:

$$\Big(f_{p_k}^*, \;\delta\mathrm{L}f' - \delta\psi\Big) = -\delta I_p, \;\; k = 1, 2, \ldots, m.$$

On the assumption of linear dependence of L and ψ on α_i we can state:

$$\delta\mathrm{L}f' - \delta\psi = \sum_{i=1}^{n} \delta\alpha_i \Big(A_i f' - \xi_i\Big),$$

where A_i are known operators, ξ_i are known functions.

In order to determine $\delta\alpha_i$ we come to the system of equations:

$$\sum_{i=1}^{n} a_{i_k}\delta\alpha_i = -b_k, \;\; a_{i_k} = \Big(\varphi_{p_k}^*, \; A_i f' - \xi_i\Big), \;\; b_k = \delta I_{p_k}, \;\; k = 1, 2, \ldots, m.$$

Since the coefficients depend on unknown solution f' of perturbed problem, then for absolute defining of the system we could use the method of successive approximations or, if perturbations are small, then simply substitute f' for f.

It should be noticed, that the given algorithm is applicable in case of non-linear dependence of operator L and function ψ on parameters. This case require linearization [2].

2. Collision density and particle flux

$\Phi\big(\vec{r}, \,\vec{\omega}\big)$ is a particle flux (emission intensity), $f\big(\vec{r}, \,\vec{\omega}\big)$ is a collision density, $f\big(\vec{r}, \,\vec{\omega}\big) = \sigma\big(\vec{r}\big) \cdot \Phi\big(\vec{r}, \,\vec{\omega}\big)$, $\tau = \int_0^{\mathcal{L}} \sigma\big(\lambda, \,\vec{r}(l)\big)dl$, $\vec{r} = \big(x, y, z\big)$, τ is a optical layer thickness along path $\mathcal{L}$ of light beam.

We obtain an integral transport equation with respect to f. Since $f = f_0 + f_1 + \cdots + \ldots$, then $f = \sum\limits_{n=0}^{\infty} K^n \psi$ (Neumann series), where $\psi = f_0$. Consequently, we have

$$f(\vec{x}) = \int\limits_X k(\vec{x}, \vec{x'}) f(\vec{x'}) d\vec{x'} + \psi(\vec{x}) \quad \text{or} \quad f = Kf + \psi, \tag{8}$$

where $k(\vec{x'}, \vec{x}) = \frac{\sigma_s(\vec{r}) g(\mu) \exp\left(-\tau(\vec{r'}, \vec{r})\right) \sigma(\vec{r})}{\sigma(\vec{r'}) 2\pi |\vec{r} - \vec{r'}|^2} \delta\left(\vec{\omega} - \frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|}\right)$.

Here $\vec{r}$ is space point, $\vec{x} = (\vec{r}, \vec{\omega})$ is point in phase space X; $\mu = \left(\vec{\omega'}, \vec{r} - \vec{r'}\right)/|\vec{r} - \vec{r'}|$ is a cosine of scattering angle; $\vec{\omega'} \in \Omega$, $\vec{\omega'}$ is a direction of the beam, with intensify $\vec{\omega}$, $\sigma_s(\vec{r})$ is a light scattering coefficient; σ_c is an absorption factor, $\sigma(\vec{r})$ attenuation constant of the flux, with scattering indicatrix $\sigma = \sigma_s + \sigma_c$, $g(\mu, \vec{r})$ is a , $\int\limits_{-1}^{+1} g(\mu, \vec{r}) d\mu = 1$, $\tau(\vec{r'}, \vec{r})$ is an optical length of the segment $|\vec{r'} - \vec{r}|$, $\delta\left(\vec{\omega} - \frac{\vec{r} - \vec{r'}}{|\vec{r} - \vec{r'}|}\right)$ is a delta-function [1].

3. Local estimate of particle flux

It is known that local estimate of particle flux in the fixed point in phase space $\vec{x}' = \left(\vec{r}', \vec{\omega}'\right)$ have form [2]

$$\int\limits_{\Omega_i} \Phi\left(\vec{r}', \vec{\omega}'\right) d\vec{\omega}' = \int\limits_X l_i\left(\vec{x}, \vec{x}'\right) f(\vec{x}) d\vec{x} = \mathbf{E} \sum_{n=0}^{N} Q_n l_i\left(\vec{x}_n, \vec{x}'\right), \tag{9}$$

where $l_i\left(\vec{x}, \vec{x}'\right) = \frac{\exp\left(-\tau(\vec{r}, \vec{r}')\right) g(\mu')}{2\pi |\vec{r} - \vec{r}'|} \Delta_i(\vec{s}')$. Here $\vec{s}' = \frac{\vec{r}' - \vec{r}}{|\vec{r}' - \vec{r}|}$, $\mu = \left(\vec{\omega}, \vec{s}'\right)$ and $\Delta_i(\vec{s})$ is an indicator of the domain Ω_i.

4. Method of dependent tests with fixed altitude

We consider an algorithm of method of dependent tests for the case when precise physical transport process modeling is executed in "basic" system ($\lambda = \lambda_0$) [1].

Different integral characterizations of transport process can be expressed as linear functionals of transport equation's solutions

$$I_\varphi = \left(f, \varphi\right) = \int\limits_X f(\vec{x}) \varphi(\vec{x}) d\vec{x} = \sum_{n=0}^{\infty} \left(K^n \psi, \varphi\right), \tag{10}$$

where $\vec{x}$ is a coordinate of phase space X.

It follows that for the estimation of functional I_φ by Monte Carlo methods we need in averaging the sum of values $\varphi(\vec{x})$ of different-order collision.

The method of dependent tests for transport theory problems consist in modeling of particle's trajectory in the different systems are executed with the same trajectories; arising displacements are removed by special weight coefficients.

Let length of wave, λ is a parameter of the system and since $k(\vec{x}, \vec{x}') = k(\vec{x}, \vec{x}', \lambda)$, $\varphi(\vec{x}) = \varphi(\vec{x}', \lambda) = \varphi_\lambda$. The we have $I_\varphi = \sum\limits_{n=0}^{\infty} \left(K_\lambda^n \psi, \varphi_\lambda\right)$.

Consider relation

$$\left(K_\lambda^n \psi,\, \varphi_\lambda\right) = \int \ldots \int \psi(\vec{x_0}) k(\vec{x_0},\, \vec{x_1},\, \lambda) \cdots k(\vec{x}_{n-1},\, \vec{x},\, \lambda) \varphi(\vec{x},\, \lambda) d\vec{x_0} d\vec{x_1} \cdots d\vec{x}_{n-1} d\vec{x} =$$

$$= \int \ldots \int \psi(\vec{x_0}) k(\vec{x_0},\, \vec{x_1},\, \lambda_0) \cdots k(\vec{x}_{n-1},\, \vec{x},\, \lambda_0) \times$$

$$\times \frac{k(\vec{x_0},\, \vec{x_1},\, \lambda) \cdots k(\vec{x}_{n-1},\, \vec{x},\, \lambda)}{k(\vec{x_0},\, \vec{x_1},\, \lambda_0) \cdots k(\vec{x}_{n-1},\, \vec{x},\, \lambda_0)} \cdot \varphi(\vec{x},\, \lambda) d\vec{x_0} d\vec{x_1} \cdots d\vec{x}_{n-1} d\vec{x}.$$

Hence, trajectories constructed for $\lambda = \lambda_0$ can be utilized for estimate $I_\varphi(\lambda)$, if after each passage $\vec{x} \to \vec{x}'$ auxiliary "weight" of particle is multiplied by $\dfrac{k(\vec{x},\, \vec{x}',\, \lambda)}{k(\vec{x},\, \vec{x}',\, \lambda_0)}$. Suppose that there are no points $\vec{x}$, $\vec{x}'$, such that $k(\vec{x},\, \vec{x}',\, \lambda) \neq 0$ but $k(\vec{x},\, \vec{x}',\, \lambda_0) = 0$. Practically $k(\vec{x},\, \vec{x}',\, \lambda)$ is represented as a product of conditional probability densities of elementary random values (length of free path, direction of scattering etc.). After each elementary "sample" auxiliary "weight" of particle is multiplied by the ratio of corresponding probability densities for λ and λ_0. After sampling the length of free path $l = |\vec{r}' - \vec{r}|$ "weight" should be multiplied by

$$\frac{\sigma(\vec{r}',\, \lambda) \exp\Big(- \int\limits_0^{|\vec{r}'-\vec{r}|} \sigma(\vec{r} + \vec{\omega}' l,\, \lambda) dt \Big)}{\sigma(\vec{r}',\, \lambda_0) \exp\Big(- \int\limits_0^{|\vec{r}'-\vec{r}|} \sigma(\vec{r} + \vec{\omega}' l,\, \lambda_0) dt \Big)} = \frac{\sigma(\vec{r}',\, \lambda)}{\sigma(\vec{r}',\, \lambda_0)} \cdot \exp\Big(- \big(\tau(\vec{r},\, \vec{r}',\, \lambda) - \tau(\vec{r},\, \vec{r}',\, \lambda_0) \big) \Big),$$

Thus recurring formulas for auxiliary "weights" have form

$$Q_n(\lambda) = Q_{n-1}(\lambda) \cdot \frac{\sigma(\vec{r}',\, \lambda)}{\sigma(\vec{r}',\, \lambda_0)} \cdot \exp\Big(- \big(\tau(\vec{r},\, \vec{r}',\, \lambda) - \tau(\vec{r},\, \vec{r}',\, \lambda_0) \big) \Big).$$

After sampling $\mu = (\vec{\omega},\, \vec{\omega}')$ which is cosine of scattering angle, we have

$$Q_n(\lambda) = Q_{n-1}(\lambda) \cdot \frac{g(\mu,\, \vec{r},\, \lambda)}{g(\mu,\, \vec{r},\, \lambda_0)}.$$

By means of the method of dependent tests we can estimate change of radiation field while small changes in aerosol scattering coefficient, albedo or indicatrix; for this purpose it's sufficient to change some characteristics of the model.

$I_k = I_{\varphi_k}(\lambda)$ **− derivative estimations.**

Let $I_k = I_{\varphi_k}(\lambda)$ depends on some certain parameter t. Then

$$I_k = \sum_{n=0}^{\infty} \int \ldots \int \psi(\vec{x}_0) \prod_{p=0}^{n-1} k(\vec{x}_p, \vec{x}_{p+1}, t_0, \lambda_0) \prod_{p=0}^{n-1} \frac{k(\vec{x}_p, \vec{x}_{p+1}, t, \lambda)}{k(\vec{x}_p, \vec{x}_{p+1}, t_0, \lambda_0)} \varphi_k(\vec{x}_n, t, \lambda) d\vec{x}_0 d\vec{x}_1 \ldots d\vec{x}_n.$$

Now compute $\frac{\partial I_k}{\partial t}|_{t=t_0}$. Suppose that the last series are termwise differentiable and differentiation can be done under the corresponding integral signs. Then formally we have

$$\frac{\partial}{\partial t}\left(\psi(\vec{x}_0)\prod_{p=0}^{n-1}k(\vec{x}_p,\vec{x}_{p+1},t_0,\lambda_0)\prod_{p=0}^{n-1}\frac{k(\vec{x}_p,\vec{x}_{p+1},t,\lambda)}{k(\vec{x}_p,\vec{x}_{p+1},t_0,\lambda_0)}\varphi_k(\vec{x}_n,t,\lambda)\right)_{t=t_0} =$$

$$= \psi(\vec{x}_0)\prod_{p=0}^{n-1}k(\vec{x}_p,\vec{x}_{p+1},t_0,\lambda_0)Q_n(\lambda,t_0)\varphi_k(\vec{x}_n,t_0,\lambda)\times$$

$$\times\left(\frac{\partial\ln\varphi_k(\vec{x}_n,t_0,\lambda)}{\partial t}+\frac{\partial\ln Q_n(\lambda,t)}{\partial t}\right)_{t=t_0} =$$

$$= \psi(\vec{x}_0)\prod_{p=0}^{n-1}k(\vec{x}_p,\vec{x}_{p+1},t_0,\lambda_0)Q_n(\lambda,t_0)\varphi_k(\vec{x}_n,t_0,\lambda)\Psi_n(\lambda,t_0)\Psi_n(\lambda,t) =$$

$$= \left(\frac{\partial\ln\varphi_k(\vec{x}_n,t,\lambda)}{\partial t}+\sum_{p=0}^{n-1}\frac{\partial\ln k(\vec{x}_p,\vec{x}_{p+1},t,\lambda)}{\partial t}\right)_{t=t_0}. \tag{11}$$

In order to compute derivatives of local estimates, let have the following designations: $\sigma_a(\vec{r},\lambda)$ is a section of aerosol scattering with indicatrix $g_a(\mu,\vec{r},\lambda)$, $\sigma_m(\vec{r},\lambda)$ is a section of molecular scattering with indicatrix $g_m(\mu,\vec{r},\lambda)$, $\sigma_c(\vec{r},\lambda)$ is a absorption section, $\mu(\vec{\omega},\vec{\omega}')$ is a cosine of the angle between the previous and the next direction of the particle in the collision point, $\sigma(\vec{r},\lambda)=\sigma_a(\vec{r},\lambda)+\sigma_m(\vec{r},\lambda)+\sigma_c(\vec{r},\lambda)$ is full section, $g(\mu,\vec{r},\lambda)=\frac{g_a(\mu,\vec{r},\lambda)\sigma_a(\vec{r},\lambda)+g_m(\mu,\vec{r},\lambda)\sigma_m(\vec{r},\lambda)}{|\vec{r}_n-\vec{r}_k|\sigma(\vec{r},\lambda)}$ is full indicatrix, $\tau(\vec{r}_n,\vec{r}_k,\lambda)=\int\sigma(\vec{r}_n+\vec{\omega}_k l,\lambda)dl$ is called "the optical depth", where $(\vec{\omega}_k)l = = (\frac{(\vec{r}_k-\vec{r}_n)}{|\vec{r}_k-\vec{r}_n|})l$ is called "the optical length from $\vec{r}_n$ to $\vec{r}_k$", $\vec{\omega}_k$ is unit length vector.

Suppose that $\sigma_c(\vec{r},\lambda)$ is known. Then for estimation of the derivative of the functional $I_k = I_{\varphi_k}(\lambda)$ by $\sigma(\vec{r},\lambda)$, we need in estimations for derivatives of the aerosol and molecular scattering coefficients, that is in this case we take $t=\sigma_a(m,\lambda)$, $t_0=\sigma_a^{(0)}(m,\lambda)$, next $t=\sigma_m(m,\lambda)$, $t_0=\sigma_m^{(0)}(m,\lambda)$. After that we need to estimate intensity integral that depends on these two parameters. Final estimate of the full section $\tilde{\sigma}(\vec{r},\lambda)$ obtained from the estimates $\tilde{\sigma}(\vec{r},\lambda)=\tilde{\sigma}_a(\vec{r},\lambda)+\tilde{\sigma}_m(\vec{r},\lambda)+\sigma_c(\vec{r},\lambda)$, where $\tilde{\sigma}_a(m,\lambda)$ is an estimation of the aerosol scattering coefficient $\sigma_a(m,\lambda)$, $\tilde{\sigma}_a(m,\lambda)$ is an estimation of the molecular scattering coefficient $\sigma_m(m,\lambda)$. Now consider the process of particles' transport in the medium with indicated characteristics. Suppose that we need to estimate radiation intensity integral in the line of the given point $\vec{r}^*$. In this case

$$\varphi_k^*=\varphi_k^*(\vec{r}_n,\vec{\omega}_n^*,\lambda)=c_1\frac{\exp\big(-\tau(\vec{r}_n,\vec{r}^*,\lambda)\big)\big(g_a(\mu^*,\vec{r},\lambda)\sigma_a(\vec{r},\lambda)+g_m(\mu^*,\vec{r},\lambda)\sigma_m(\vec{r},\lambda)\big)}{\sigma(\vec{r}_n,\lambda)}$$

and

$$k(\vec{x}_p,\vec{x}_{p+1},\lambda)=c_2\tfrac{\sigma(\vec{r}_{p+1},\lambda)}{\sigma(\vec{r}_p,\lambda)}\times$$
$$\times\exp\Big(-\tau(\vec{r}_p,\vec{r}_{p+1},\lambda)\big(g_a(\mu_p,\vec{r}_p,\lambda)\sigma_a(\vec{r}_p,\lambda)+g_m(\mu_p,\vec{r}_p,\lambda)\sigma_m(\vec{r}_p,\lambda)\big)\Big),$$

where $\mu_p(\vec{\omega}_{p-1},\vec{\omega}_p)$.

Next,

$$I_k(\lambda)=\mathbf{E}\sum_{n=0}^{N}Q_n(\vec{r}_n,\lambda)\varphi_k^*(\vec{r}_n,\lambda).$$

Let the atmosphere be divided onto n_i layers and in each of them the aerosol and molecular scattering coefficients are constants, $\sigma_c(\vec{r},\lambda)$ is also fixed. Let's use the designations: $\sigma(\vec{r},\lambda) = =\sigma(m,\lambda)$, $g(\mu,\vec{r},\lambda)=g(\mu,m,\lambda)$, where $h_m<|\vec{r}|\le h_{m+1}$, $m=0,1,\dots,n_i-1$; $\psi_m^k=1$ if

$m = k$, else $\psi_m^k = 0$ if $m \neq k$; μ_n^* is a cosine of the angle between the particle's direction before the n-th collision in the point $\vec{r}_n$ and direction $\vec{\omega}^* = \frac{\vec{r}^* - \vec{r}_n}{|\vec{r}^* - \vec{r}_n|}$; $N(n)$ is a number of the domain where the collision point $\vec{r}_n$ is situated; $m(n,i)$ are the numbers of the domains, which particle intersect with between the collision points $\vec{r}_{n-1}$ and $\vec{r}_n$; $i(n)$ is a total number of such intersections; $l_{n,i}$ are the segments of such intersections; $m^*(n,i)$, $i^*(n)$ and $l^*_{n,i}$ are the same quantities from $\vec{r}_n$ to $\vec{r}_n^*$; $L_{n,m}$ is a total path length, which the particle covers in the m-th layer between $\vec{r}_{n-1}$ and $\vec{r}_n$ points; $L^{(1)}_{n,m}$ is the same quantity from $\vec{r}_n$ to $\vec{r}_n^*$.

Now let's turn to calculating the derivatives with respect to aerosol scattering coefficient. In this case $t = \sigma_a(m,\lambda)$, $t_0 = \sigma_a^{(0)}(m,\lambda)$.

Calculating the derivatives with respect to molecular scattering coefficient is the same procedure. For that, due to the symmetry, it's sufficient to substitute index a for m. Denote $\bar{\sigma}_a(m,\lambda) = \sigma_a(m,\lambda) - \sigma_a^{(0)}(m,\lambda)$. We obtain

$$\frac{\partial I_k}{\partial t}\Big|_{t=t_0} = \frac{\partial I_k}{\partial \sigma_a(m,\lambda)}\Big|_{\sigma_a(m,\lambda)=\sigma_a^{(0)}(m,\lambda)} = \frac{\partial I_k}{\partial \bar{\sigma}_a(m,\lambda)}\Big|_{\bar{\sigma}_a(m,\lambda)=0}.$$

Then

$$\varphi_k^*(\vec{r}_n, \vec{\omega}_n^*, \sigma_a, \lambda) =$$
$$= c_1 \frac{\sigma_a(N(n),\lambda) g_m(\mu_n^*, N(n),\lambda) + \big(\sigma_a^{(0)}(N(n),\lambda) + \psi^m_{N(n)} \bar{\sigma}_a^{(0)}(N(n),\lambda)\big) g_a(\mu_n^*, N(n),\lambda)}{\sigma_m^{(0)}(N(n),\lambda) + \sigma_a^{(0)}(N(n),\lambda) + \psi^m_{N(n)} \bar{\sigma}_a^{(0)}(m,\lambda)} \times$$
$$\times \exp\Bigg(- \sum_{i=1}^{i^*(n)} l^*_{n,i} \Big(\sigma_m^{(0)}(m^*(n,i),\lambda) + \sigma_a^{(0)}(m^*(n,i),\lambda) + \psi^m_{m^*(n,i)} \sigma_a(m,\lambda)\Big)\Bigg).$$

So,

$$\frac{\partial \ln \varphi_k^*}{\partial \bar{\sigma}_a(m,\lambda)} =$$
$$= -L^{(1)}_{n,m} + \frac{\psi^m_{N(n)} g_a\big(\mu_n^*, N(n),\lambda\big)}{\sigma_m^{(0)}(N(n),\lambda) g_m\big(\mu_n^*, N(n),\lambda\big) + \sigma_a^{(0)}(N(n),\lambda) g_a\big(\mu_n^*, N(n),\lambda\big)} -$$
$$- \frac{\psi^m_{N(n)}}{\sigma_m^{(0)}(N(n),\lambda) + \sigma_a^{(0)}(N(n),\lambda)}, \tag{12}$$

since $\sum_{i=1}^{i^*(n)} l^*_{n,i} \psi^m_{m^*(n,i)} = L^{(1)}_{n,m}$.

Calculate

$$\frac{\partial \ln Q_n(\lambda)}{\partial \bar{\sigma}_a} = \sum_{p=1}^{n-1} \frac{\partial \ln k(\vec{x}_p, \vec{x}_{p+1}, \sigma_a, \lambda)}{\partial \bar{\sigma}_a(m,\lambda)} =$$
$$= \frac{\partial \ln \big(\sigma_m^{(0)}(N(n),\lambda) + \sigma_a^{(0)}(N(n),\lambda) + \psi^m_{N(n)} \bar{\sigma}_a(N(n),\lambda)\big)}{\partial \bar{\sigma}_a(m,\lambda)} -$$
$$- \sum_{p=1}^{n} \frac{\partial}{\partial \bar{\sigma}_a(m,\lambda)} \Big(\sum_{i=1}^{i(p)} l_{p,i} \big(\bar{\sigma}_m(m(p,i),\lambda) + \psi^m_{m(p,i)} \bar{\sigma}_a(m,\lambda)\big)\Big) +$$
$$+ \sum_{p=1}^{n} \frac{\partial \ln \Big(\sigma_m^{(0)}(N(p),\lambda) g_m(\mu_p, N(p),\lambda) + \big(\sigma_a^{(0)}(N(p),\lambda) + \psi^m_{N(p)} \bar{\sigma}_a(m,\lambda)\big) g_a(\mu_p, N(p),\lambda)\Big)}{\partial \bar{\sigma}_a(m,\lambda)}. \tag{13}$$

At last,

$$\frac{\partial \ln Q_n(\lambda)}{\partial \bar{\sigma}_a(m,\lambda)}\Big|_{\bar{\sigma}_a(m,\lambda)=0} = \frac{\psi^m_{N(n)}}{\sigma_m^{(0)}(N(n),\lambda)+\sigma_a^{(0)}(N(n),\lambda)} - \sum_{p=1}^{n} L_{p,m} +$$

$$+\sum_{p=1}^{n-1} \frac{\psi^m_{N(p)} g_a(\mu_p,N(p),\lambda)}{\sigma_m^{(0)}(N(p),\lambda)g_m(\mu_p,N(p),\lambda)+\sigma_a^{(0)}(N(p),\lambda)g_a(\mu_p,N(p),\lambda)}. \quad (14)$$

Combining (12), (13) (14), we obtain the expression for derivative estimate

$$\frac{\partial I_k}{\partial \bar{\sigma}_a(m,\lambda)}\Big|_{\bar{\sigma}_a(m,\lambda)=0} = \mathbf{E}\sum_{n=1}^{N} Q_n(\vec{r}_n,\sigma_a^{(0)},\lambda)\varphi^*(\vec{r}_n,\vec{\omega}_n^*,\sigma_a^{(0)},\lambda)\Psi_n(\lambda), \quad (15)$$

where

$$\Psi_n(\lambda) = -L^{(1)}_{n,m} + \frac{\psi^m_{N(n)} g_a(\mu_n^*,N(n),\lambda)}{\sigma_m^{(0)}(N(n),\lambda)g_m(\mu_n^*,N(n),\lambda)+\sigma_a^{(0)}(N(n),\lambda)g_a(\mu_n^*,N(n),\lambda)} -$$

$$-\sum_{p=1}^{n} L_{p,m} + \sum_{p=1}^{n-1} \frac{\psi^m_{N(p)} g_a(\mu_n,N(p),\lambda)}{\sigma_m^{(0)}(N(p),\lambda)g_m(\mu_p,N(p),\lambda)+\sigma_a^{(0)}(N(p),\lambda)g_a(\mu_p,N(p),\lambda)}.$$

For convergence of Neumann's series it's sufficient that $\|K\| = q < 1$. After termwise formal differentiation for arbitrary σ_a value from $\big(\sigma_a^{(0)}-\varepsilon,\, \sigma_a^{(0)}+\varepsilon\big)$ we have

$$\frac{\partial I_k(\sigma_a)}{\partial \sigma_a} = \mathbf{E}\sum_{n=1}^{N} Q_n(\vec{r}_n,\sigma_a,\lambda)\varphi^*(\vec{r}_n,\vec{\omega}_n^*,\sigma_a,\lambda)\Psi_n(\lambda).$$

On the assumption of bounded medium (therefore, $|\vec{r}_{n-1}-\vec{r}_n| < c_1 < \infty$) and $\sigma(\vec{r},\lambda) < c_2 < \infty$, it's easy to have

$$Q_n(\vec{r}_n,\sigma_a,\lambda)\varphi^*(\vec{x}_n,\sigma_a,\lambda)\Psi_n(\lambda) \le c q_\varepsilon^n \cdot n Q_n(\vec{r}_n,\sigma_a^{(0)},\lambda)\varphi^*(\vec{r}_n,\sigma_a^{(0)},\lambda,$$

where $q_\varepsilon \to 1$ while $\varepsilon \to 0$.

If $\|K\| < 1$ and ε is such that $\|K\| \cdot q_\varepsilon < 1$, then the derivative is majorizable by function that is not depended on any parameter, whose expected value is limited. Therefore derivative integral with respect to probability measure converges while $|\sigma_a - \sigma_a^{(0)}| < \varepsilon$. So, differentiability of (11) and estimate (15) were proved. We consider

$$I_p(f) = \int_D \int_{\Delta\Omega} f \cdot \xi \cdot \delta(\vec{r}-\vec{r}_0)\cdot d\vec{r}\cdot d\vec{\Omega}$$

as the functionals of the problem, ξ is a known function.

Suppose we look for full section $\sigma(\vec{r},\lambda)$ $\big(\sigma(\vec{r},\lambda) = \sigma_a(\vec{r},\lambda)+\sigma_m(\vec{r},\lambda)+\sigma_c(\vec{r},\lambda)$ is full section$\big)$.

While we look for constant coefficients in different layers, that is $\vec{\sigma}_a \equiv \vec{\sigma} = (\sigma_1,\ldots,\sigma_n)$, ψ is not perturbed (ψ is external source). $\delta \mathrm{L} f = \delta\big(f - Kf\big)$ is represented as a Taylor's series with respect to $\vec{\sigma}$ near $\vec{\sigma}_0$

$$\delta\big(f-Kf\big) \approx \sum_{i=1}^{n} \frac{\partial}{\partial \sigma_i}\big(f-K'f\big)\Big|_{\vec{\sigma}=\vec{\sigma}_0}\Delta\sigma_i. \quad (16)$$

Put (16) into (6),

$$\big(f^*_{p_k},\, \delta \mathrm{L} f\big) = \left(f^*_{p_k},\, \sum_{i=1}^{n} \frac{\partial}{\partial \sigma_i}\big(f-K'f\big)\Big|_{\vec{\sigma}=\vec{\sigma}_0}\Delta\sigma_i\right) =$$

$$= \sum_{i=1}^{n}\left(\left(f^*_{p_k},\, \frac{\partial}{\partial \sigma_i}\big(f-K'f\big)\right)\Big|_{\vec{\sigma}=\vec{\sigma}_0}\Delta\sigma_i\right), \quad k=1,\ldots,n.$$

Or

$$\sum_{i=1}^{n} \frac{\partial I_k}{\partial \sigma_i} \Delta\sigma_i = -\delta I_{p_k} \quad \Longrightarrow \quad A\Delta\vec{\sigma} = \vec{b}. \tag{17}$$

Obtained system (17) is inconsistent ($m > n$), therefore we need in most convenient values for $\Delta\sigma_i$. We do it by minimizing the following expression $\|A\Delta\vec{\sigma} - \vec{b}\|^2 = \delta^2$ and obtain: $A^*A\Delta\vec{\sigma} = A^*\vec{b}$. Demand normal distribution of errors for I_p measurements and a little difference between them. Otherwise it's possible to use statistical weights

$$A^*WA\Delta\vec{\sigma} = A^*W\vec{b}. \tag{18}$$

This system was solved by method of minimal errors. If the system (18) would be ill-conditioned we can apply regularization methods [12]. According to this method we'll minimize the following expression

$$\left|\left|A^*WA\Delta\vec{\sigma} - A^*W\vec{b}\right|\right| + \alpha Q\big[\Delta\vec{\sigma}\big] = \delta,$$

where $Q\big[\Delta\vec{\sigma}\big] = \big(\Delta\vec{\sigma},\, Q\Delta\vec{\sigma}\big)$, Q is approximate value in matrix form for

$$\int_0^H \left| \sum_{i=1}^{m} q_i(x) \frac{d^i\sigma}{dx^i} \right| dx$$

Here H is a height of atmosphere, $q_i(x) > 0$. The last means that there are additional constraints imposed on solution class. Usually $i = 1, \quad q_i(x) \equiv 1$, that is, limited derivative for $\Delta\vec{\sigma}$ is demanded. Regularization coefficient α can be found approximately by different means. For example, in [3] the following value is presented:

$$\alpha = \frac{n}{\big(\Delta\vec{\sigma},\, Q\Delta\vec{\sigma}\big)}, \tag{19}$$

where n is a number of dimensions. But as α depends on unknown solution, we suggest put the right hand sides of equation (19) instead of $\Delta\vec{\sigma}$. Then

$$\alpha = \frac{n}{\big(A^*W\vec{b},\, QA^*W\vec{b}\big)}. \tag{20}$$

It's relevant to notice, that if dependence of L (L is operator) from σ_i is not linear (that is $I_k = I_k(\sigma_1, \ldots, \sigma_n)$ is nonlinear functional), then the problem can be solved by method of successive approximations, using the formula of small perturbations; coefficients a_{i_k} are the partial derivatives

$$a_{i_k} = \frac{\partial I_k}{\partial \sigma_i}, \;\; k = 1, 2, \ldots, n_0, \;\; i = 1, 2, \ldots, n.$$

In this case the numeric calculations of derivatives are implemented by Monte Carlo methods. [1].

Let's consider another approach to numeric solving of inverse problem. Target setting doesn't change. Let we have a set of measured functionals $\tilde{I}_k(\sigma_1, \ldots, \sigma_n) = \big(f,\, \varphi_k\big), \;\; k = 1, 2, \ldots, n_\theta$, where f is a solution of transport equation $\mathrm{L}f = \psi$. It's required $\big(\sigma_1, \ldots, \sigma_n\big)$.

If $\big(\sigma_1^{(0)}, \ldots, \sigma_n^{(0)}\big)$ are some prognostic values of these parameters, then applying perturbation theory we arrive at system of equations

$$\sum_{i=1}^{n} a_{i_k} \delta\sigma_i = \tilde{I}_k - I_k^{(0)}$$

on conditions that L depends on σ_i linearly; here $I_k^{(0)} = I_k\big(\sigma_1^{(0)}, \dots, \sigma_n^{(0)}\big)$. If the indicated dependence is not linear then problem can be solved by method of successive approximations, using the formula of small perturbations; coefficients a_{i_k} are the partial derivatives $a_{i_k} = \frac{\partial I_k}{\partial \sigma_i}$, $k = 1, 2, \dots, n_\theta$, $i = 1, 2, \dots, n$. Calculation of derivatives by Monte Carlo methods was described above.

The weights are inversely proportional to $I_k^{(0)}$. Then

$$\sum_{i=1}^{s} \frac{\partial I_k}{\partial \sigma_i} \cdot \frac{1}{I_k^{(0)}} \cdot \Delta\sigma_i = \ln \tilde{I}_k - \ln I_k^{(0)}, \quad k = 1, 2, \dots, n_\theta. \tag{21}$$

It might be noticed that system (21) is a result of linearization of equations $\ln I_k(\sigma_1, \dots, \sigma_n) = \ln \tilde{I}_k$, $k = 1, 2, \dots, n_\theta$, in $\big(\sigma_1^{(0)}, \dots, \sigma_n^{(0)}\big)$. System (21) is an overspecified problem, therefore we apply least squares method.

Estimation of full section $\sigma(\vec{r}, \lambda) = \sigma_a(\vec{r}, \lambda) + \sigma_m(\vec{r}, \lambda) + \sigma_c(\vec{r}, \lambda)$ is done within two steps: first, estimate $\sigma_a(\vec{r}, \lambda)$ with indicatrix $g_a(\mu, \vec{r}, \lambda)$, later, due to the symmetry, we can estimate $\sigma_m(\vec{r}, \lambda)$ with indicatrix $g_m(\mu, \vec{r}, \lambda)$. As $\sigma_c(\vec{r}, \lambda)$ is known, then

$$\sigma(\vec{r}, \lambda) = \sigma_a(\vec{r}, \lambda) + \sigma_m(\vec{r}, \lambda) + \sigma_c(\vec{r}, \lambda). \tag{22}$$

Remark 1. Using the methods above $\sigma_c(\vec{r}, \lambda)$ also can be estimated (if $\sigma_c(\vec{r}, \lambda)$ is an unknown parameter). Having estimations for $(\sigma_a(\vec{r}, \lambda),\ \sigma_m(\vec{r}, \lambda),\ \sigma_c(\vec{r}, \lambda))$ we can estimate full section by formula (22).

Remark 2. Estimation of derivatives of I_k by Monte Carlo methods with respect to parameters τ (is optical depth) and P_a (is albedo) allows one to estimate these parameters with the methods mentioned in the article.

5. Computing Experiment

Optical depth of the extinction τ of cloudless sky was measured near Astrophysical Institute named after V.G. Fesenkov (Almaty, Kazakhstan, 1350 meters above sea level) during a period from 1996 to 2004 in summer and autumn months in spectral range of 0.42 – 1.28 micrometers. Full-sized measurements for 2003 – 2004 are given in the table. They are provided by V.N. Glushko – research worker of Astrophysical Institute and Institute of Space Researches named after U.M. Sultangazin. I express him my acknowledgement of thanks.

Optical depth of the extinction over a period 2003 – 2004 years (antemeridians values)

Yars	λ micrometer	0.421	0.478	0.540	0.667	0.796	1.28
2003	τ	0.364	0.259	0.193	0.113	0.078	0.060
2003	σ	0.031	0.027	0.023	0.014	0.013	0.008
2003	Δ	8.64	10.5	12.0	12.7	17.3	13.2
2003	N	13	13	13	13	13	13
2004	τ	0.360	0.258	0.199	0.122	0.083	0.060
2004	σ	0.047	0.039	0.042	0.036	0.041	0.030
2004	Δ	13.0	15.2	21.3	29.1	49.0	52.2
2004	N	7	7	7	7	7	7
mean	τ	0.364	0.256	0.194	0.115	0.081	0.065
mean	σ	0.042	0.035	0.031	0.023	0.022	0.020
mean	Δ	11.5	13.4	16.1	20.4	27.6	30.7
mean	N	65	66	66	66	63	65

The numerical experiment was implemented in Institute of Space Researches. Optical depth of the extinction τ was estimated based on the data of 2003-2004 (August and September). Comparison with full-sized measurements shows that an error (difference) is no more than 5-10 percent, which is quite satisfactory for Monte Carlo methods.

Here to bring the means for 1996 – 2004 years, λ is wave-length, τ is optical depth of the extinction, σ is mean square difference (standard deviate), Δ is variation coefficient (in percentage terms) for every years of measuring, N is number days of observations.

Many questions of the given article are considered in [4] - [11].

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