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SURVEY

AN OPERATOR THEORETIC APPROACH TO ROBUST CONTROL OF INFINITE DIMENSIONAL SYSTEMS

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Abstract. The purpose of this paper is to give an overview of the skew Toeplitz approach to $\mathcal{H}^\infty$ control of a class of infinite dimensional systems. Numerical steps involved in the computations of optimal and suboptimal controllers are demonstrated with different examples, including flexible beam models and systems with time delays.

Keywords: distributed parameter systems, time delay, robust control, skew-Toeplitz operators.

AMS Subject Classification: 93C05, 93C80, 93B28, 93B36, 93B35.

1. Introduction

This paper provides an overview of the so-called skew Toeplitz approach to $\mathcal{H}^\infty$ control of a class of infinite dimensional systems, [45]. The author has given invited presentations on this topic at various scientific meetings, such as 17th International Symposium on Mathematical Theory of Networks and Systems (MTNS 2006) Kyoto, Japan, and 2nd International Conference on Control and Optimization with Industrial Applications (COIA 2008) Baku, Azerbaijan. The present paper is a brief summary of these presentations.

It is now well-known that robust controllers, under the presence of unstructured $\mathcal{L}^\infty$-norm bounded perturbations in the plant transfer matrix, can be obtained from an $\mathcal{H}^\infty$ optimization, [168]. Many different approaches have been developed for $\mathcal{H}^\infty$ control of finite dimensional systems, and computational tools are now widely available, [32, 34, 47, 54, 56, 84], [94, 124, 137, 155, 172, 173]. Robust control under ℓ_1 optimality, and other types of uncertainty, are also studied widely, see [9, 12, 28, 148] and their references. For time delay systems (an important class of infinite dimensional systems), $\mathcal{H}^\infty$ controllers started to appear in the literature in the mid 1980s, [41, 46, 174]. Over the last 20 years there has been significant progress in the extension of these first results to larger classes of infinite dimensional systems, see e.g. [5, 24, 25, 31, 43, 53, 57, 58, 71, 75, 78, 86, 93], [107, 111, 112, 118, 119, 129, 130, 135, 140, 144, 151]. One of the methods used in the computation of $\mathcal{H}^\infty$ controllers for infinite dimensional systems is the "skew Toeplitz" approach, [45]. The importance of skew Toeplitz operators in $\mathcal{H}^\infty$ control has been noticed by Foias, Tannenbaum and their collaborators, and this term first appears in [11]. In this approach $\mathcal{H}^\infty$ optimal and suboptimal controllers are directly computed without approximating the plant.

For infinite dimensional systems robust controllers can also be obtained by approximating the plant and then using standard techniques developed for the control of finite dimensional systems,

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by keeping track of the original approximation error, see e.g. [6, 23, 76, 87, 88, 104, 105, 120, 133] and their references. See [22] for an early review of $\mathcal{H}^\infty$ control of distributed parameter systems in general, [154] for state-space approach to $\mathcal{H}^\infty$ control of such systems, and [100, 157] for reviews of state-space and operator theoretic approaches to $\mathcal{H}^\infty$ control of time delay systems. For time delay systems, and some other classes of distributed parameter systems, $\mathcal{H}^\infty$ controllers can also be derived from a game theoretic approach, [10, 139]. For the most recent results on $\mathcal{H}^\infty$ control of systems with input-output delays, see [98, 171] and their references. Repetitive control design, [68], under certain performance and robustness conditions, can be posed as a robust control problem for systems with time delays, [63, 123, 142, 158]. Sampled-data controller design, with certain types of optimality conditions result in an $\mathcal{H}^\infty$ control problem for time delay systems, see [8, 20, 67, 160] for further references. Robust stability of time delay systems (within, and outside, the framework of $\mathcal{H}^\infty$ control) is widely studied, [19, 33, 49, 50, 59, 64, 70, 74, 81, 83], [85, 89, 109, 110, 102, 121, 122, 146, 147, 153]. Several issues related to robust control of fractional delay systems has been considered in [14]. Stability robustness against small time delays have been considered for various types of plants, see e.g. [17, 90, 97, 103] and their references. Flexible structure models which include internal time delays, are considered in [65, 66]. For spatially invariant distributed parameter systems, [7], $\mathcal{H}^\infty$ optimal controllers are obtained from a parameterized family of finite dimensional problems; see also [27]. Robust control of infinite dimensional systems is also covered in the book [26].

In Section 2 some key results from operator theory are reviewed and their link with the $\mathcal{H}^\infty$ control are shown. In Section 3 numerical computations of optimal and suboptimal controllers are demonstrated based on the formulae derived in [60, 149]. Plants considered in Section 3 are systems with time delays, [39, 61], and an infinite dimensional flexible beam model, [87, 88].

2. $\mathcal{H}^\infty$ Control problems

In this section some important results from operator theory are reviewed and a short description of the of the skew Toeplitz approach is given to illustrate the technique used in finding optimal and suboptimal $\mathcal{H}^\infty$ controllers. An excellent background material can be found in a recently published book [161].

The standard feedback system $\mathcal{F}(C, P)$ is shown in Figure 1, where C is the controller to be designed and P is the plant to be controlled. The systems P and C are assumed to be linear operators on appropriately defined function (signal) spaces. For simplicity of the presentation, all systems considered are single input single output unless otherwise stated.

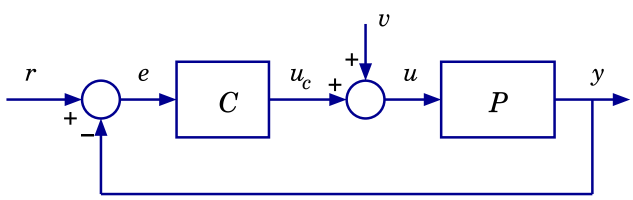

Figure 1. Feedback system $\mathcal{F}(C, P)$

We say that a linear time invariant system whose input output behavior is characterized by the transfer function $H(s)$ is said to be stable if $H \in \mathcal{H}^\infty$ (see [26, 139, 175] for a discussion on the transfer functions of infinite dimensional systems, here we assume that the transfer function is the "quotient of the Laplace transform of the output and input, with initial condition zero"). The feedback system $\mathcal{F}(C, P)$ is stable if and only if $S = (1 + PC)^{-1}$, PS and CS are in $\mathcal{H}^\infty$.

Here S is the sensitivity function. The set of all controllers stabilizing this feedback system for a given P is denoted by $\mathcal{C}(P)$.

Most important features of the $\mathcal{H}^\infty$ control problems are captured by the weighted sensitivity minimization problem, which is to find

$$\gamma_o = \inf_{C \in \mathcal{C}(P)} \|W(1+PC)^{-1}\|_\infty \tag{1}$$

and the corresponding optimal controller $C_o \in \mathcal{C}(P)$, for a given plant P and a weight W; typically $W, W^{-1} \in \mathcal{H}^\infty$. There are several approaches to this problem depending on the structure of the problem data.

First, let us assume that W is infinite dimensional (an irrational transfer function) and P is finite dimensional (a rational transfer function). We can solve this problem using Nevanlinna-Pick interpolation, as follows. For simplicity of the exposition assume that $z_1, \ldots, z_{n_z}$ are the zeros and $p_1, \ldots, p_{n_p}$ are the poles of P in $\mathbb{C}_+$, and P has no poles or zeros on the imaginary axis. In this case, $C \in \mathcal{C}(P)$ is equivalent to having the sensitivity function $S = (1+PC)^{-1}$ in $\mathcal{H}^\infty$, with $S(z_i) = 1$ and $S(p_j) = 0$. Then, γ_o is the smallest $\gamma > 0$ such that there exists a function $F \in \mathcal{H}^\infty$ such that

$$\|F\|_\infty \le 1 \quad \text{and} \quad F(z_i) = \gamma^{-1} W(z_i), \; F(p_j) = 0$$

for $i = 1, \ldots, n_z$, $j = 1, \ldots, n_p$. This is the Nevanlinna-Pick interpolation problem, and can be solved from the problem data W and P, see [44, 45, 82, 170]. Once γ_o and the corresponding optimal F is computed, the resulting controller is $C = (\gamma^{-1}W - F)/PF$. Note that in this case the problem solution depends on $W(z_i)$, and $W(s)$ can be irrational. In summary, when the plant is finite dimensional and the weight is infinite dimensional, the weighted sensitivity minimization problem can be solved using Nevanlinna-Pick interpolation.

Clearly, we cannot use this approach when the plant P is infinite dimensional (with infinitely many $\mathbb{C}_+$ zeros or poles). For such plants we will assume that W is finite dimensional and use the characterization of $\mathcal{C}(P)$ given below. First, assume that P can be written as $P(s) = N(s)/D(s)$ with $N, D \in \mathcal{H}^\infty$ such that there exist $X, Y \in \mathcal{H}^\infty$ satisfying

$$X(s)N(s) + Y(s)D(s) = 1.$$

Then the set of all controllers stabilizing $\mathcal{F}(C,P)$ is (see e.g. [3, 138, 165] and their references)

$$\mathcal{C}(P) = \left\{ \frac{X + DQ}{Y - NQ} \; : \; Q \in \mathcal{H}^\infty, \; Y - NQ \neq 0 \right\}.$$

Let us now assume that the plant has finitely many unstable modes, i.e. $D(s)$ can be taken as a rational function. In this case $X(s)$ should be chosen in such a way that

$$Y(s) = \frac{1 - X(s)N(s)}{D(s)} \in \mathcal{H}^\infty.$$

Clearly, using Lagrange interpolation one can find a rational $X(s)$ satisfying the above requirement. Note also that stabilizing controllers are in the form

$$C = \frac{D(X + DQ)}{1 - N(X + DQ)} \tag{2}$$

and they can be implemented as shown in Figure 2.

In particular, when the plant is stable we can choose $N = P$, $D = 1$ and $X = 0$. This leads to

$$\gamma_o = \inf_{Q \in \mathcal{H}^\infty} \|W(1 - PQ)\|_\infty.$$

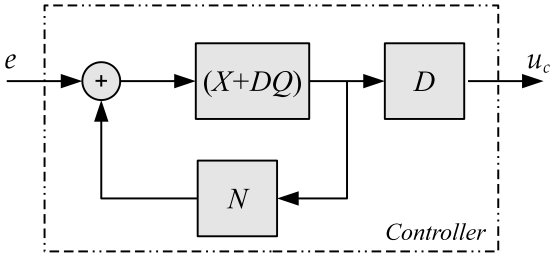

FIGURE 2. Stabilizing Controller

The next crucial step is to perform inner outer factorization of the plant:

$$P = M_n N_o \text{ where } M_n \text{ is inner and } N_o \text{ is outer.}$$

Defining $Q_1 = WN_oQ$ (we assume that WN_o is invertible in $\mathcal{H}^\infty$; otherwise, see the discussion in [40, 48] on the absorption of the outer factor), we have

$$\gamma_o = \inf_{Q_1 \in \mathcal{H}^\infty} \|W - M_n Q_1\|_\infty. \tag{3}$$

The problem (3) can be put into the framework of the Nehari problem, [108], and that gives

$$\gamma_o = \|\Gamma\|,$$

where Γ is the Hankel operator whose symbol is $M_n^* W \in \mathcal{L}^\infty$. We will assume that the norm is achieved on the discrete spectrum, i.e., $\|\Gamma\| > \|\Gamma\|_e$, the essential norm (see [45, 169] for the computation of the essential norm). In this case γ_o and the corresponding optimal $Q_1 \in \mathcal{H}^\infty$, and hence the optimal controller $C \in \mathcal{C}(P)$, can be obtained by computing the largest singular value, and the corresponding singular vector of Γ, see also [2] for the characterization of all suboptimal solutions of this problem. For more details see [45].

The problem (3) can also be solved using Sarason's Theorem, [134], or the Commutant Lifting Theorem, [44, 143], as follows. Let us map the right half plane, $\mathbb{C}_+$, to the unit disc, $\mathbb{D}$, via the conformal map $z = \varphi(s) = \frac{s-1}{s+1}$, $s = \varphi^{-1}(z) = \frac{1+z}{1-z}$. Define functions $w(z) = W(\varphi^{-1}(z))$ and $m_n(z) = M_n(\varphi^{-1}(z))$. For the inner function $m := m_n$ define $\mathcal{H}(m) = \mathcal{H}_2(\mathbb{D}) \ominus m\mathcal{H}_2(\mathbb{D})$. When m is rational (finite Blaschke product), the space $\mathcal{H}(m)$ is finite dimensional, otherwise it is infinite dimensional. Let $\mathbf{S}$ be the unit shift operator on ℓ_2, i.e. it can be seen as the multiplication by z on $\mathcal{H}_2(\mathbb{D})$. Then the compressed shift operator $\mathbf{T}$ is defined as $\mathbf{T} = \Pi_{\mathcal{H}(m)} \mathbf{S}|_{\mathcal{H}(m)}$, where $\Pi_{\mathcal{H}(m)}$ denotes the orthogonal projection onto $\mathcal{H}(m)$. Now the solution of (3) is

$$\gamma_o = \|w(\mathbf{T})\|.$$

Since $w(z)$ is rational, it can be written as $w(z) = b(z)/a(z)$. Under the assumption $\gamma_o > \|w(\mathbf{T})\|_e$, (the norm is strictly greater than the essential norm), γ_o is the largest γ for which there exists a non-zero $f \in \mathcal{H}(m)$ such that

$$0 = \left(b(\mathbf{T})^* b(\mathbf{T}) - \gamma^2 a(\mathbf{T})^* a(\mathbf{T})\right) f =: \mathbf{A}_\gamma f\ . \tag{4}$$

The operator $\mathbf{A}_\gamma$ is called a *skew Toeplitz operator*, and γ_o is the largest γ which makes $\mathbf{A}_\gamma$ singular; the optimal $Q_1 \in \mathcal{H}^\infty$, and hence the optimal controller $C \in \mathcal{C}(P)$, are determined from the corresponding $f \in \mathcal{H}(m)$ as

$$w - m_n q_1^{\text{opt}} = \frac{b(\mathbf{T})f}{a(\mathbf{T})f}. \tag{5}$$

A closer examination of (4) leads to a finite set of linear equations for the existence of a non-zero $f \in \mathcal{H}(m)$, even when $\mathcal{H}(m)$ is infinite dimensional. This set of linear equations determine γ_o

and the corresponding f. Then the optimal controller is obtained from (5) using this f. For complete details and further references see [45].

The weighted sensitivity minimization is known as a *one-block* $\mathcal{H}^\infty$ control problem. An extension of this problem, which also takes into account robust stability of the feedback system, [32], is the mixed sensitivity optimization: find

$$\gamma_o = \inf_{C \in \mathcal{C}(P)} \left\| \begin{bmatrix} W_1(1+PC)^{-1} \\ W_2PC(1+PC)^{-1} \end{bmatrix} \right\|_\infty \tag{6}$$

and the corresponding optimal controller. Again, using the parameterization (2), we transform the problem (6) into a problem of finding

$$\gamma_o = \inf_{Q \in \mathcal{H}^\infty} \left\| \begin{bmatrix} W_1(1-N(X+DQ)) \\ W_2N(X+DQ) \end{bmatrix} \right\|_\infty$$

and the corresponding optimal $Q \in \mathcal{H}^\infty$. After a series of inner-outer factorizations, [47], the above problem is further reduced to finding smallest γ such that there exists $Q_1 \in \mathcal{H}^\infty$ satisfying

$$\left\| \begin{bmatrix} W - MQ_1 \\ G \end{bmatrix} \right\|_\infty \leq \gamma \, label2bl_1 \tag{7}$$

where W, G, M, Q_1 are determined from the problem data W_1, W_2, N, D. In particular Q_1 is determined from Q by an invertible relation, M is inner infinite dimensional, and G is finite dimensional. If the plant is stable $N = P$ and $D = 1$, then W is finite dimensional as well. Otherwise, it has a special structure $W = W_o + M_1\widehat{W}_o$ where $W_o, \widehat{W}_o$ are finite dimensional and M_1 is the infinite dimensional part of M, i.e., $M = M_1M_2$ with M_2 being finite dimensional (assuming that the plant has finitely many unstable modes). Next step is to do a spectral factorization:

$$F_\gamma^* F_\gamma = \gamma^2 - G^*G.$$

That transforms (**??**) into a problem of finding smallest γ such that

$$\|WF_\gamma^{-1} - MQ_2\|_\infty \leq 1, \tag{8}$$

where $Q_2 = Q_1F_\gamma^{-1}$. Clearly, now the problem (8) is in the form (3), and the approach outlined earlier is applicable. The problem (**??**) is a *two-block* $\mathcal{H}^\infty$ control problem, and as we have seen above, it can be reduced to a one-block problem by a spectral factorization.

One step extension of the two-block problem is the four-block problem. That also can be reduced to a one-block problem by a series of spectral factorizations, see e.g. [47]. The key observation we make in this context is that when the weights are finite dimensional, and the inner-outer factorizations of the plant have already been made (see below for examples), we only need to do spectral factorizations for finite dimensional systems, see [45, 80] for further details. In general, there are several technical difficulties in performing spectral factorizations for infinite dimensional systems, [159].

The problem of robust stabilization in the gap metric can be posed as a special case of the mixed sensitivity minimization, (6), with special weights W_1 and W_2. This problem has been studied for various classes of infinite dimensional systems, see e.g. [35, 53, 150] and their references. The uncertainty model in this case is coprime factor perturbations. Briefly, if the uncertain plant is given as

$$P_\Delta = \frac{N_\Delta}{D_\Delta} = \frac{N + \Delta_N}{D + \Delta_D}, \quad N, D, \Delta_N, \Delta_D \in \mathcal{H}^\infty$$

with

$$N^*N + D^*D = 1 \;\| \begin{bmatrix} \Delta_N \\ \Delta_D \end{bmatrix} \|_\infty < \delta$$

then a controller $C \in \mathcal{C}(P)$, where $P = N/D$, is also in $\mathcal{C}(P_\Delta)$ if and only if it satisfies

$$\left\| D^{-1}(1+PC)^{-1} \begin{bmatrix} 1 \\ C \end{bmatrix} \right\|_\infty \leq \frac{1}{\delta}. \tag{9}$$

For certain classes of infinite dimensional systems this type of uncertainty modeling is very helpful in finding finite dimensional controllers. See for example [120] for robust controller design for a thin airfoil where the D_Δ term contains an irrational transfer function (the Theodorsen's function) which is approximated by a second order rational function, [114].

It turns out that, see e.g. [51, 55, 156], minimizing the left hand side of (9) over all $C \in \mathcal{C}(P)$ (for the largest allowable δ), is equivalent to finding the norm of a Hankel operator whose symbol is $[D^* \; N^*]$. Clearly, (9) is a special case of the two block problem, (6), (with $W_1 = D_o^{-1}$, the inverse of the outer part of D, and $W_2 = N_o^{-1}$), and it too reduces to a one block problem.

In summary, several different $\mathcal{H}^\infty$ control problems can be reduced to the generic form (3), which is solved via (5). Then the optimal controller is computed using the optimal $Q \in \mathcal{H}^\infty$ in (2).

3. Optimal solution of the mixed sensitivity minimization problem

In this section we present a closed form solution for the mixed sensitivity problem (6), taken from [149].

3.1. **Examples of Plants Considered.**

The plants considered in [149] have the coprime factorization in the form

$$P(s) = \frac{M_n(s)N_o(s)}{M_d(s)}, \tag{10}$$

where M_n is inner, N_o is outer and M_d is finite dimensional and inner. The formula given here is also valid for certain different classes of infinite dimensional plants (including plants with infinitely many $\mathbb{C}_+$ poles) with some modifications, see [60, 61].

Examples of infinite dimensional plants in the form (10) are as follows.

1. A stable first order system with transport delay:

$$P_1(s) = \frac{e^{-hs}}{\tau_p s + 1}, \;\; h > 0, \;\; \tau_p > 0.$$

$$M_d(s) = 1, \;\; M_n(s) = e^{-hs}, \;\; N_o(s) = \frac{1}{\tau_p s + 1}.$$

2. An unstable system with transport delay:

$$P_2(s) = e^{-hs}\frac{1}{s-a}, \;\; h > 0, \;\; a > 0,$$

$$M_d(s) = \frac{s-a}{s+a}, \;\; M_n(s) = e^{-hs}, \;\; N_o(s) = \frac{1}{s+a}.$$

3. An unstable system with internal time delays:

$$P_3(s) = \frac{s+3+2(s-1)e^{-0.4s}}{s^2 + se^{-0.2s} + 5e^{-0.5s}},$$

which can be re-written as

$$P_3(s) = \frac{P_N(s)}{P_D(s)},$$

where

$$P_N(s) = \frac{s+3+2(s-1)e^{-0.4s}}{(s+1)^2},$$

$$P_D(s) = \frac{s^2+se^{-0.2s}+5e^{-0.5s}}{(s+1)^2}.$$

It can be shown that $P_D(s)$ has only two zeros in $\mathbb{C}_+$, these are the unstable poles of the plant, p_1 and p_2. On the other hand, $P_N(s)$ has infinitely many zeros in $\mathbb{C}_+$. Inner-outer factorization of this plant can be done, [61], by finding p_1, p_2, and the single $\mathbb{C}_+$ zero, z_1, of

$$\overline{P_N}(s) = \frac{2(s+1)+(s-3)e^{-0.4s}}{(s+1)^2}.$$

At this point we should mention that there are several tools for finding the zeros of a quasi-polynomial, see e.g. [36, 37, 38, 131] for the Matlab-based program DDE-BIFTOOL, and also [29, 91, 113] for other techniques and further references on this topic. Now, the zeros of $P_D(s)$ in $\mathbb{C}_+$ are computed as $p_{1,2} \approx 0.467 \pm j1.889$, and the zero of $\overline{P_N}(s)$ in $\mathbb{C}_+$ is $z_1 \approx 0.247$. We define

$$M_d(s) = \frac{(s-p_1)(s-p_2)}{(s+p_1)(s+p_2)}, \quad M_1(s) = \frac{s-z_1}{s+z_1}.$$

Then, the plant P_3 can be written in the form (10) with M_d as above and

$$M_n(s) = M_1(s)\ \frac{s+3+2(s-1)e^{-0.4s}}{2(s+1)+(s-3)e^{-0.4s}},$$

$$N_o(s) = \frac{\overline{P_N}(s)}{M_1(s)}\ \frac{M_d(s)}{P_D(s)}.$$

4. A flexible beam: Consider an Euler-Bernoulli beam having free ends with Kelvin-Voigt damping. Assume that the beam length and all other parameters are normalized to unity, except the damping constant $\varepsilon > 0$, [88]. Denote the deflection of the beam at time t and location x along the elastic axis of the beam by $w(x,t)$. Suppose that a transverse force, $-u(t)$, is applied at one end of the beam, e.g. at $x = 1$ (see Figure 3).

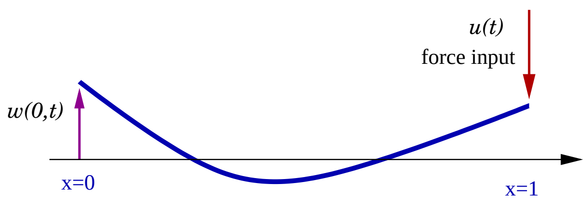

FIGURE 3. Beam with free ends.

The beam dynamics are given by the following PDE, [16, 87],

$$\frac{\partial^2 w}{\partial t^2} + \varepsilon \frac{\partial^5 w}{\partial x^4 \partial t} + \frac{\partial^4 w}{\partial x^4} = 0 \tag{11}$$

with boundary conditions

$$\frac{\partial^2 w}{\partial x^2}(0,t) + \varepsilon \frac{\partial^3 w}{\partial x^2 \partial t}(0,t) = 0,$$

$$\frac{\partial^2 w}{\partial x^2}(1,t) + \varepsilon \frac{\partial^3 w}{\partial x^2 \partial t}(1,t) = 0,$$

$$\frac{\partial^3 w}{\partial x^3}(0,t) + \varepsilon \frac{\partial^4 w}{\partial x^3 \partial t}(0,t) = 0,$$

$$\frac{\partial^3 w}{\partial x^3}(1,t) + \varepsilon \frac{\partial^4 w}{\partial x^3 \partial t}(1,t) = u(t).$$

Let the output of the system be the acceleration at $x = 0$, i.e. $y_o(t) := \frac{\partial^2}{\partial t^2} w(0,t)$ and consider a low-pass characteristics for the sensor dynamics: $H(s) = e^{-hs}/(1+\tau s)$, with $h \geq 0$ and $\tau > 0$. Define the output available for feedback as $Y(s) = H(s)Y_o(s)$. Note that $w(0,t)$ is the deflection at the opposite end of the beam as the applied force $-u(t)$, i.e., even if the sensing delay is zero the plant is non-minimum phase due to non-collocated actuator and sensor. We will ignore the actuator dynamics here. Transfer function of the plant (including the sensor dynamics) can be derived as in [87]: $P_4(s) := \frac{Y(s)}{U(s)}$,

$$P_4(s) = \frac{s^2(\sinh\beta - \sin\beta)\ e^{-hs}}{\beta^3(\cos\beta\cosh\beta - 1)(1+\varepsilon s)(1+\tau s)}, \tag{12}$$

where $\beta^4 = \frac{-s^2}{(1+\varepsilon s)}$.

One can show that $P_4(s)$ can be expressed as infinite products of second order terms. These product representations display its poles and zeros. It also facilitate inner/outer factorizations which are essential for solving the $\mathcal{H}^\infty$ optimization problems.

$$P_4(s) = \frac{2e^{-hs}}{\tau s + 1} \prod_{n=1}^{\infty} g_n(s), \tag{13}$$

where

$$g_n(s) = \frac{\left(1 + \varepsilon s - \frac{s^2}{4\alpha_n^4}\right)}{\left(1 + \varepsilon s + \frac{s^2}{\phi_n^4}\right)}$$

for values of s where this infinite product converges. In [87] it is shown that (13) converge everywhere in the closed right half plane and can be written as quotients of $\mathcal{H}^\infty$ functions. We factor $P_4 = M_n N_o$, where $M_n(s) = e^{-hs}B(s)$,

$$N_o(s) = \frac{2}{(\tau s + 1)} \prod_{n=1}^{\infty} \frac{\left(1 + s\sqrt{\varepsilon^2 + \frac{1}{\alpha_n^4}} + \frac{s^2}{4\alpha_n^4}\right)}{\left(1 + \varepsilon s + \frac{s^2}{\phi_n^4}\right)}, \tag{14}$$

$$B(s) = \prod_{n=1}^{\infty} \left(\frac{2\alpha_n^4\left(\varepsilon + \sqrt{\varepsilon^2 + \frac{1}{\alpha_n^4}}\right) - s}{2\alpha_n^4\left(\varepsilon + \sqrt{\varepsilon^2 + \frac{1}{\alpha_n^4}}\right) + s} \right). \tag{15}$$

It has been shown that [87, 88], $N_o(s) \in \mathcal{H}^\infty$ and $B(s) \in \mathcal{H}^\infty$ converge in the closed right half-plane. The zeros of P_4 are at

$$s = 2\alpha_n^4 \left(\varepsilon \pm \sqrt{\varepsilon^2 + \frac{1}{\alpha_n^4}}\right) \text{ for } n = 1, 2, ..., \tag{16}$$

where

$$\cos(\alpha_n)\sinh(\alpha_n) = \sin(\alpha_n)\cosh(\alpha_n), \text{ for } \alpha_n > 0.$$

Also, P_4 has a singularity at $-1/\varepsilon$, and poles at

$$s = \frac{-\phi_n^4}{2}\left(\varepsilon \pm \sqrt{\varepsilon^2 - \frac{4}{\phi_n^4}}\right) \text{ for } n = 1, 2, ...,$$

where $\cos(\phi_n)\cosh(\phi_n) = 1$, for $\phi_n > 0$.

3.2. $\mathcal{H}^\infty$ **Optimal Controller.** Assume that the weights W_1 and W_2 are rational and $(W_2N_o), (W_2N_o)^{-1} \in \mathcal{H}^\infty$, then optimal $\mathcal{H}^\infty$ controller for plant (10) can be written as, [149],

$$C_{\text{opt}} = E_{\gamma_o}(s)M_d(s)\frac{N_o^{-1}(s)F_{\gamma_o}(s)L(s)}{1+M_n(s)F_{\gamma_o}(s)L(s)}, \tag{17}$$

where $E_\gamma(s) = \left(\frac{W_1(-s)W_1(s)}{\gamma^2} - 1\right)$, and for the definition of the other terms, let the right half plane zeros of $E_\gamma(s)$ be β_i, $i = 1, \ldots, n_1$, the right half plane poles of $P(s)$ be α_k, $k = 1, \ldots, \ell$ and that of $W_1(-s)$ be η_i $i = 1, \ldots, n_1$. Then,

$$F_\gamma(s) = G_\gamma(s)\prod_{i=1}^{n_1}\frac{s-\eta_i}{s+\eta_i},$$

where

$$G_\gamma^* G_\gamma = \left(1 - \left(\frac{W_2^*W_2}{\gamma^2} - 1\right)E_\gamma\right)^{-1} \tag{18}$$

and $G_\gamma, G_\gamma^{-1} \in \mathcal{H}^\infty$, and $L(s) = \frac{L_2(s)}{L_1(s)}$, L_1 and L_2 are polynomials with degrees $\leq (n_1 + \ell - 1)$ and they are determined by the following interpolation conditions,

$$\begin{aligned} 0 &= L_1(\beta_i) + M_n(\beta_i)F_\gamma(\beta_i)L_2(\beta_i), \\ 0 &= L_2(-\beta_i) + M_n(\beta_i)F_\gamma(\beta_i)L_1(-\beta_i), \\ 0 &= L_1(\alpha_k) + M_n(\alpha_k)F_\gamma(\alpha_k)L_2(\alpha_k), \\ 0 &= L_2(-\alpha_k) + M_n(\alpha_k)F_\gamma(\alpha_k)L_1(-\alpha_k) \end{aligned} \tag{19}$$

for $i = 1, \ldots, n_1$ and $k = 1, \ldots, \ell$. The optimal performance level, γ_o, is the largest γ value such that the spectral factorization (18) can be done and the interpolation conditions (19) are satisfied for some non-zero L_1, L_2. A Matlab-based computer program is available for computing γ_o and all the functions appearing in (17); it can be downloaded from the author's web site: `http://www.ee.bilkent.edu.tr/~ozbay/HINFCON.rar`

With respect to the above controller formula, first point to note is that the spectral factorization in question is finite dimensional and the order of F_γ is at most the order of W_1 plus the order of W_2. Secondly, $L(s)$ is characterized by its $2(n_1 + \ell)$ coefficients, and the interpolation conditions (19) can be re-written as

$$\mathsf{R}_\gamma \ \Phi = 0, \tag{20}$$

where the entries of the $2(n_1 + \ell) \times 1$ vector Φ contain the coefficients of L_1 and L_2, and the $2(n_1+\ell)\times 2(n_1+\ell)$ matrix R_γ is constructed from $M_n(s)$, $F(s)$ and β_i, α_k, for $i = 1, \ldots, n_1$ and $k = 1, \ldots, \ell$. Clearly, explicit computation of $M_n(s)$ is not necessary for the construction of R_γ; all we need is its values at β_i's and α_k's. See [42] for more detailed discussion of this point and further references. Due to symmetry in the interpolation conditions, $L(s)$ term in the optimal controller satisfies $|L(j\omega)| = 1$, but it may or may not be stable. The final observation we make, probably the most important one from the point of view of "controller implementation," is that the controller has internal unstable pole-zero cancelations: the $\mathbb{C}_+$ zeros of E_{γ_o} and M_d are canceled by the zeros of $1 + M_n(s)F_{\gamma_o}(s)L(s)$. Since M_n is infinite dimensional, e.g. time delay, exact cancelation is not always possible. For this reason, these cancelations should be studied in more detail and a new equivalent structure, which can be implemented in a stable manner, should be investigated, see [61, 62] for a discussion of this problem within the framework of general time delay systems.

Here we illustrate the optimal $\mathcal{H}^\infty$ controller for P_1, with the mixed sensitivity problem weights

$$W_1(s) = \frac{1+\epsilon s}{s+\epsilon}, \quad W_2(s) = k(1+\alpha\tau_p s).$$

In this case the plant is stable, so $\ell = 0$, and W_1 is first order, so $n_1 = 1$, and thus $n_1+\ell-1 = 0$, which means that $L(s)$ is just a constant, $+1$ or -1. We have only one linear equation to form, and that gives γ_o. When we let $\epsilon \to 0$, we obtain γ_o as the largest root of the equation

$$\sqrt{1-(k/\gamma)^2} - k\alpha\tau_p/\gamma^2 - \sin(h/\gamma) = 0$$

in the interval $\frac{2h}{\pi} \le \gamma < \infty$. It can be shown that, [116, 141], the optimal $\mathcal{H}^\infty$ controller has an "internal model controller" structure

$$C_{\text{opt}} = \frac{Q_{\text{opt}}}{1-P(s)Q_{\text{opt}}}, \quad Q_{\text{opt}} = \frac{Q_o}{1+F_oQ_o},$$

where

$$Q_o(s) = k\frac{1+\alpha\tau_p s}{\gamma_o s},$$

$$F_o(s) = \frac{1}{1+\alpha\tau_p s}\ F_1(s) \quad \text{where } F_1(s) = \frac{\gamma_o s(\sin(h/\gamma_o)+\gamma_o s\cos(h/\gamma_o)) + e^{-hs}}{1+(\gamma_o s)^2}. \tag{21}$$

It should be noted that $F_1(s)$ in (21) is the transfer function of a linear time invariant system whose impulse response is of finite duration, (it is non-zero only on the interval $[0\ ,h]$). Thus F_1 is a stable system. On the other hand, if $F_1(s)$ is implemented as in (21) by using γ_o and h with a slight uncertainty in these parameters, then stability of F_1 can be lost. So, one must be careful about the "fragility" of a particular controller implementation in this framework; this is a topic discussed in detail in the literature [79]. If $F_1(s)$ is implemented by taking a stable approximation (for example, approximation of its impulse response by another finite impulse response filter), then this will lead to a small stable perturbation in the numerator and denominator coprime factors of the controller, which will not cause any fragility problems as shown in [52, 95, 125].

4. Conclusions

The purpose of this paper was to introduce the readership of this mathematics journal to a specific *control engineering* problem.

There are also very interesting applications of the theory given here to data flow control in computer communication networks. Interested readers are referred to [106] [21, 69, 96], [13, 15, 136], [126], [127], [152], for related issues and technical details related to this particular application.

Another useful application of the theory presented here is in the suppression of cavity flow oscillations in aerodynamics. Typically, this is an application involving nonlinear infinite dimensional models, [1, 4, 18, 73, 77, 145]. Nevertheless, using simplified linear models $\mathcal{H}^\infty$ controllers can be designed for performance improvement, see e.g. [162, 163, 166, 167] and their references. In particular [166] shows that in this application the plant and the weights are infinite dimensional, but by exploiting the special structure of the plant and how the weights are defined, it is possible to find a suboptimal $\mathcal{H}^\infty$ controller suppressing cavity flow oscillations.

Recently, in [117] it has been shown that the set of equations (20) can be further simplified and the size of the γ-dependent matrix whose singularity gives the optimum performance level can be reduced to $n_1 \times n_1$, where n_1 is the order of the weight $W_1(s)$. For detailed discussion the simplified formulae see [117].

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