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ALGORITHM TO SOLUTION OF THE OPTIMIZATION PROBLEM WITH PERIODIC CONDITION AND BOUNDARY CONTROL*

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Abstract. In the paper the optimization problem with periodic condition and boundary control is considered. It is assumed that object's motion is described by the system of ordinary nonlinear differential equations. Finding the optimal solution using the method of quasilinearization the given nonlinear problem is reduced to the linear quadratic boundary control problem. For its solution the corresponding Euler-Lagrange equation with linear boundary conditions is used. Results are illustrated on the example from oil industry which shows adequacy of the mathematical model with practice.

Keywords: periodic conditions, quasilinearization, Euler-Lagrange equation, gas-lift.

AMS Subject Classification: 49J15, 49J20, 49N05.

1. Introduction

After fountain process the main stage of the oil production is a gas-lift stage [1, 11]. It is known, that in this process the motion of the object is described by the partial differential equations of hyperbolic type [10]. The main problem of gas-lift process is that the volume of injected gas in the annular space be minimal, but extraction of the gas-liquid mixture (GLM) at the end of lift be maximal [1, 13]. In this paper the control is not included in the right hand side of the equation, but only in the initial condition. In [8] is shown that debit is 43% of GLM volume at the beginning of the lift. But the periodic condition is offered in the given problem for extracting the maximum debit with a minimum initial gas. Introducing the periodicity condition $Q(2l) = \chi Q(l+0), \quad 0 < \chi < 1$and changing χ, debit is any percent of GLM at the beginning of the lift. The optimization problem with periodic condition and boundary control [6, 9] is solved by the method of quasilinearization [4]. In this method after linearization the given equation and the condition on the well bottom the obtained equation Euler-Lagrange are linear. At the end the calculation algorithm is proposed and the efficiency of the offered algorithm is illustrated on the concrete example.

2. Problem statment

Let the object's motion be described by the system of ordinary nonlinear differential equations

$$\begin{array}{ll} \dot{y} = f_1(y(x)), & 0 \le x \le l-0 \\ \dot{y} = f_2(y(x)), & l+0 \le x \le 2l \end{array} \tag{1}$$

$$y(0) = u \tag{2}$$

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and by the following conditions on the well bottom (i.e. at $y = l$)

$$y(l+0) = f_\delta y(l-0) + \gamma(y(l-0))\bar{y}, \tag{3}$$

where is n-dimensional vector of the object coordinates, u is an unknown initial (control), $\bar{y}$ is a scalar external perturbation, f_δ is $n \times n$ dimensional matrix, $\gamma(y(l-0))$ is n-dimensional vector.

It is required to find such control u from (2), that gives minimum to the functional

$$J = \frac{1}{2}y'(2l)Ny(2l) + u'Cu + \int_0^{2l} y'(x)Q(x)y(x)dx, \tag{4}$$

where N is $n \times n$ dimensional symmetric matrix, $N < 0$ and $Q(x)$ is $n \times n$ dimensional symmetric matrix, $Q(x) \geq 0$.From the condition

$$\chi y(l+0) = y(2l), 0 < \chi < 1. \tag{5}$$

putting this condition into (4), we obtain the following functional

$$J = \frac{1}{2}\chi^2 y'(l+0)Ny(l+0) + \int_0^{2l} y'(x)Q(x)y(x)dx + u'Cu. \tag{6}$$

Euler-Lagrange equation for the extremal problem (1)-(3), (5), (6) is

$$\dot{\lambda}(x) = -\frac{8l}{\beta}y(x)Q(x) - \lambda(x)\frac{\partial f_1(y(x))}{\partial y(x)}, \quad 0 \leq x \leq l-0 \tag{7}$$

$$\dot{\lambda}(x) = -\frac{8l}{\beta}y(x)Q(x) - \lambda(x)\frac{\partial f_2(y(x))}{\partial y(x)}, \quad l+0 \leq x \leq 2l$$

with the boundary conditions

$$\begin{cases} \lambda(l+0)f_\delta + \lambda(l+0)\frac{\partial\gamma(y(l-0))}{\partial y(l-0)}\bar{y} - \frac{\beta}{4l}\lambda(l-0) = 0, \\ 2Cu + \delta + \frac{\beta}{4l}\lambda(0) = 0, \end{cases} \tag{8}$$

where $\lambda(x)$ is a Lagrange multiplier, δ is an unknown constant parameter.

Finally, solution of the optimization problem (1), (3), (6) is reduced to the solution of the Euler-Lagrange equation (1), (2), (5)-(8). For the solution of that problem the numerical method is offered below.

3. The method of quasilinearization

We use the method of quasilinearization for the nonlinear differential equation (1), where the boundary condition (2) is also nonlinear.

Let some nominal solution $y^k(x)$ of the problems (1)-(3), (5) be given. Then if the differential equation (1) is linearized with this nominal trajectory, we get the following system of linear differential equation for the (k +1)th iteration

$$\dot{y}^{k+1}(x) = A(y^k(x))y(x) + B(y^k(x)), \tag{9}$$

where

$$A(y^k(x)) = f_1'(y^k(x)),$$
$$B(y^k(x)) = f_1(y^k(x)) - f_1'(y^k(x))y^k(x).$$

The linearization form of the condition (3) with nominal trajectory $y^k(x)$ is

$$y(l+0) = \eta(y^k(l-0))y(l-0) + \mu(y^k(l-0)), \tag{10}$$

$$\mu(y^k(l-0)) = \gamma(y^k(l-0)\bar{y} - \frac{\partial\gamma(y^k(l-0)}{\partial y^k(l-0)} y^k(l-0)\bar{y},$$

$$\eta(y^k(l-0)) = \frac{\partial\gamma(y^k(l-0)}{\partial y^k(l-0)}\bar{y} + f_\delta.$$

It is required to find such control u that satisfies the conditions (9), (10) and gives the extremal value to the functional

$$J = \frac{1}{2} y'(2l)Ny(2l) + u'Cu + \int_0^{2l} y'(x)Q(x)y(x)dx, \tag{11}$$

where N is $n \times n$ dimensional symmetric matrix , $N < 0$ and $Q(x)$ is $n \times n$ dimensional matrix, $Q(x) \geq 0$. From the condition

$$\chi y(l+0) = y(2l), 0 < \chi < 1 \tag{12}$$

we have

$$J = \frac{1}{2}\chi^2 y'(l+0)Ny(l+0) + u'Cu + \int_0^{2l} y'(x)Q(x)y(x)dx. \tag{13}$$

Euler-Lagrange equation for the extremal problem (9), (10), (12), (13) is

$$\dot{\lambda}(x) = -8ly(x)Q(x) - 4lA(y^k(x))\lambda \tag{14}$$

with the boundary conditions

$$\begin{cases} (\frac{1}{4l} - 1)\lambda(l+0) + \chi Ny(2l) - \frac{\chi}{4l}\lambda(2l) = 0, \\ \frac{1}{4l}\lambda(0) + \delta + 2Cu = 0, \\ \lambda(l-0) = 4l\eta(y^k(l-0))\lambda(l+0), \end{cases} \tag{15}$$

where $\lambda(x)$ is a Lagrange multiplier, δ is a constant parameter.
Combining the equations (9), (14) we have the following linear differential equations:

$$\begin{bmatrix} \dot{y} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A(y^k(x)) & 0 \\ -8lQ(x) & -4lA(y^k(x)) \end{bmatrix} \begin{bmatrix} y \\ \lambda \end{bmatrix} + \begin{bmatrix} B(y^k(x)) \\ 0 \end{bmatrix}. \tag{16}$$

Combining the boundary conditions (10), (12), (15) we get

$$Kz = q, \tag{17}$$

where

$$K = \begin{bmatrix} 0 & 0 & \eta(y^k(l-0)) & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -4l\eta(y^k(l-0)) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \chi & 0 & -1 & 0 & 0 \\ 2C & \frac{1}{4l} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4l} - 1 & \chi & -\frac{\chi}{4l} & 0 \end{bmatrix}, \tag{18}$$

$$z = \begin{bmatrix} y(0) \\ \lambda(0) \\ x(l-0) \\ \lambda(l-0) \\ y(l+0) \\ \lambda(l+0) \\ y(2l) \\ \lambda(2l) \\ \delta \end{bmatrix}, q = \begin{bmatrix} -\mu(x^k(l-0)) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, let us introduce the solution of the equation (16) in the following form

$$\begin{bmatrix} y \\ \lambda \end{bmatrix} = \Phi(x,x_0)\begin{bmatrix} y(0) \\ \lambda(0) \end{bmatrix} + \int_{x_0}^{x} \Phi(x,\delta)\begin{bmatrix} B \\ 0 \end{bmatrix} d\delta, \tag{19}$$

where Φ is a fundamental matrix for (16) and the formula (19) can be written in a more convenient form for the realization at the end of the interval $0 \le x \le l-0$

$$\begin{bmatrix} y(l-0) \\ \lambda(l-0) \end{bmatrix} = \Phi(l,0)\begin{bmatrix} y(0) \\ \lambda(0) \end{bmatrix} + N(l,0) \tag{20}$$

and at the end of the interval $l+0 \le x \le 2l$

$$\begin{bmatrix} y(2l) \\ \lambda(2l) \end{bmatrix} = \Phi(2l,l)\begin{bmatrix} y(l+0) \\ \lambda(l+0) \end{bmatrix} + N(2l,l). \tag{21}$$

Note that, $N(l,0)$ $N(2l,l)$ from (20) (21) is defined by the relation

$$N(i,j) = \int_{i}^{j} \Phi(j,\delta)\begin{bmatrix} B \\ 0 \end{bmatrix} d\delta.$$

Thus, we obtain $9n$ linear algebraic equations with respect to x

$$\begin{bmatrix} & & K & & \\ \Phi(l,0) & -E & 0 & 0 & 0 \\ 0 & 0 & \Phi(2l,l) & -E & 0 \end{bmatrix} z = \begin{bmatrix} q \\ -N(l,0) \\ -N(2l,l) \end{bmatrix}. \tag{22}$$

Thus we offer the following algorithm:

(1) The initial data and parameters from (9) are introduced;
(2) Nominal trajectory $y^k(x)$ and control u^k are selected;
(3) $A(y^k(x))$, $B(y^k(x))$from (9) are calculated;
(4) The fundamental matrix Φ is determined from linear differential equations (9), (14);
(5) The system of linear algebraic equations (22) is solved with respect to x;
(6) The system of the differential equations (9), (14) is solved and $y^{k+1}(x)$ and $u^{k+1}(x)$ are found.
(7) Giving a small real number ε and the condition $\left|\frac{\partial J}{\partial u}\right| < \varepsilon$ is verified, where $\frac{\partial J}{\partial u} = \frac{1}{4l}\lambda(0) + \delta + 2Cu$, if the condition is fulfilled, then the calculation stops, otherwise passage to step 2.

4. Example

For illustration of the offered algorithm we consider the example of the gas-lift process [2, 3]. It is known that the mathematical model describing the gas-lift process is in the form of the system of partial differential equations [2, 5]:

$$-\frac{\partial P}{\partial x} = \frac{\partial(\rho\omega_c)}{\partial t} + 2a\rho\omega_c,$$
$$-\frac{\partial P}{\partial t} = c^2\frac{\partial(\rho\omega_c)}{\partial x},$$

where $P = P(x,t)$, $\omega_c = \omega_c(x,t)$ is a pressure on its fixed value and the velocity, averaged over the cross section, respectively, t,x – time and coordinate, respectively; c-sound speed in gas and gas-liquid mixture (GLM); ρ-density of mixture; $2a = \frac{g}{\omega_c} + \frac{\lambda_c\omega_c}{2D}$, g- acceleration of free fall, λ_c- coefficient of hydraulic resistance; D – effective internal diameter of the lift and the annular space; $\rho\omega_c = \frac{Q}{F}$, $Q = \rho\omega_c F$- mass flow of injected gas in annular space and GLM in lift, F- the ring space cross section area of the pump-compressor pipe.

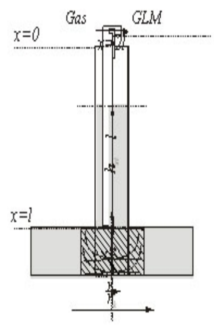

Figure 1. Gas-lift well.

The partial differential equations of flow of gas and GLM may be replaced by the ordinary differential equations

$$\begin{cases} \dot{Q} = \dfrac{2a\rho F Q^2}{c^2\rho^2F^2 - Q^2}, \ Q(0) = u, \\ \dot{P} = -\dfrac{2ac^2\rho^2 F Q}{c^2\rho^2F^2 - Q^2}, \ P(0) = P_0, \end{cases} \tag{23}$$

using time-averaging method, where $c >> \omega_c$ and except $Q = \rho\omega_c F$, all parameters are constants.

The condition on the well bottom (i.e. at $x = l$) is given as follows

$$Q(l+0) = \gamma Q(l-0) + \gamma_1(Q(l-0))\bar{Q}, \tag{24}$$

$$\gamma_1(Q(l-0)) = -\delta_3(Q(l-0) - \delta_2)^2 + \delta_1,$$

where γ and $\gamma_1(Q(l-0))$ are constants ,δ_1, δ_2, δ_3 are real numbers.
After linearization of the first equation of (23) and (24) we obtain

$$\dot{Q}(x) = A(Q^k(x))Q(x) + B(Q^k(x)), Q(0) = u, \tag{25}$$

$$Q(l+0) = \eta(Q^k(l-0))Q(l-0) + \mu(Q^k(l-0)), \tag{26}$$

where

$$A(Q^k(x)) = \frac{4c^2a\rho^3F^3Q^k(x)}{(Q^{k^2}(x) - c^2\rho^2F^2)^2},$$

$$B(Q^k(x)) = \frac{2a\rho F Q^{k^2}(x)}{c^2\rho^2F^2 - Q^{k^2}(x)} - \frac{4a\rho^3F^3Q^{k^2}(x)}{(Q^{k^2}(x) - c^2\rho^2F^2)^2},$$

$$\mu(Q^k(l-0)) = 2\delta_3\bar{Q}Q^k(l-0)[Q^k(l-0) - \delta_2] - [\delta_3[Q^k(l-0) - \delta_2]^2 - \delta_1]\bar{Q}.$$

It is required to find such control u from (25) that satisfies to the equations (25), (26) and gives minimum to the functional [7, 12]

$$J = \frac{1}{2}\alpha Q^2(2l) + \frac{1}{2}\beta u^2, \tag{27}$$

where $\alpha < 0$ and $\beta > 0$ are weight coefficients. From the condition

$$\chi y(l+0) = y(2l), 0 < \chi < 1. \tag{28}$$

putting this condition into (27), we obtain the following functional

$$J = \frac{1}{2}\alpha\chi^2 Q^2(l+0) + \frac{1}{2}\beta u^2. \tag{29}$$

It is necessary to find such minimum value of the injected gas $Q(0)$ from (25) that provides maximum value for the debit $Q(2l)$.

Euler-Lagrange equation for the extremal problem (25)-(26), (28), (29) looks like

$$\dot{\lambda} = -A(Q^k(x))\lambda \tag{30}$$

with boundary conditions

$$\begin{cases} (\frac{1}{4l} - 1)\lambda(l+0) - \frac{\chi}{4l}\lambda(2l) + \chi\alpha Q(2l) = 0 \\ \lambda(l-0) = 4l\eta(Q^k(l-0))\lambda(l+0) \\ \beta u + \delta + \frac{1}{4l}\lambda(0) = 0, \end{cases} \tag{31}$$

where $\lambda(x)$ is a Lagrange multiplier, δ is a constant parameter.

Combining the equations (25), (30) we have the following linear differential equations:

$$\begin{bmatrix} \dot{Q} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A(Q^k(x)) & 0 \\ 0 & -A(y^k(x)) \end{bmatrix} \begin{bmatrix} Q(x) \\ \lambda(x) \end{bmatrix} + \begin{bmatrix} B(y^k(x)) \\ 0 \end{bmatrix}. \tag{32}$$

Combining the boundary conditions (26), (28), (31) we get

$$Kz = q, \tag{33}$$

$$K = \begin{bmatrix} 0 & 0 & \eta(Q^k(l-0)) & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -4l\eta(Q^k(l-0)) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \chi & 0 & -1 & 0 & 0 \\ \beta & \frac{1}{4l} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4l} - 1 & \chi\alpha & -\frac{\chi}{4l} & 0 \end{bmatrix}, \tag{34}$$

$$z = \begin{bmatrix} Q(0) \\ \lambda(0) \\ Q(l-0) \\ \lambda(l-0) \\ Q(l+0) \\ \lambda(l+0) \\ Q(2l) \\ \lambda(2l) \\ \delta \end{bmatrix}, \ q = \begin{bmatrix} -\mu(Q^k(l-0)) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, let us search the solution of the equation (32) in the form

$$\begin{bmatrix} Q \\ \lambda \end{bmatrix} = \Phi(x, x_0) \begin{bmatrix} Q(0) \\ \lambda(0) \end{bmatrix} + \int_{x_0}^{x} \Phi(x, \delta) \begin{bmatrix} B \\ 0 \end{bmatrix} d\delta, \tag{35}$$

where Φ is a fundamental matrix for (32) and the formula (35) can be written in a more convenient form for the realization at the end of the interval $0 \le x < l - 0$

$$\begin{bmatrix} Q(l-0) \\ \lambda(l-0) \end{bmatrix} = \Phi(l, 0) \begin{bmatrix} Q(0) \\ \lambda(0) \end{bmatrix} + N(l, 0), \tag{36}$$

Table 1

χ	0.1	0.3	0.4	0.6	0.999
$\frac{Q(2l)}{Q(l+0)} \cdot 100\%$	10%	30%	40%	60%	99.99%

and at the end of the interval $l+0 < x \leq 2l$

$$\begin{bmatrix} Q(2l) \\ \lambda(2l) \end{bmatrix} = \Phi(2l,l) \begin{bmatrix} Q(l+0) \\ \lambda(l+0) \end{bmatrix} + N(2l,l). \tag{37}$$

Note that, $N(l,0)$ $N(2l,l)$ from (36) (37) are defined by the following relation

$$N(i,j) = \int\limits_i^j \Phi(j,\delta) \begin{bmatrix} B \\ 0 \end{bmatrix} d\delta.$$

Thus, we obtain $9n$ linear algebraic equations with respect to x

$$\begin{bmatrix} & & K & & \\ \Phi(l,0) & -E & 0 & 0 & 0 \\ 0 & 0 & \Phi(2l,l) & -E & 0 \end{bmatrix} z = \begin{bmatrix} q \\ -N(l,0) \\ -N(2l,l) \end{bmatrix}. \tag{38}$$

Let us illustrate the application of this method on the following example.
Example 1. Let the parameters of the equation (25) be in the following form: $0 \leq x \leq l$: l=1485 m, c=331 m/s, $\rho = 0.717 \frac{kg}{m^3}$, d=$\sqrt{114^2 - 73^2} \cdot 10^{-3}$ m , λ=0.01; $l \leq x \leq 2l$: c=850 m/s, $\rho = 700 \frac{kg}{m^3}$, d=0.073m, λ=0.23.
By using algorithm described above, we see that to achieve the accuracy 10^{-16} is needed 13 iterations and finally the following result is obtained

$$Q(0) = 0.0015291, \quad Q(l+0) = 0.051239, \quad Q(2l) = 0.05124.$$

It is shown that, changing χ debit ($Q(2l)$) is any percent of GLM ($Q(l{+}0)$) at the beginning of the lift in table 1:

5. Conclusion

In the paper the results are given, which allows one to define the efficiency of the quasilinearization method. The problem has been reduced to the linear quadratic control problem for which adequacy of the proposed mathematical model is shown.

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