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OPTIMAL CONTROL PROBLEM DESCRIBING BY THE CAUCHY PROBLEM FOR THE FIRST ORDER LINEAR HYPERBOLIC SYSTEM WITH TWO INDEPENDENT VARIABLES

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Abstract. In the paper optimal control problems are considered, describing by the Cauchy problem for the first order linear hyperbolic system with two independent variables and integral quadratic functional. The existence and uniqueness theorems of the optimal control are proved. Necessary and sufficient conditions of optimality are derived. Finding of the optimal control and minimum of the functional is reduced to the solution of the non-linear system of integro-differential equations.

Keywords: hyperbolic system, necessary and sufficient conditions, optimal control, integro-differential equation, Cauchy problem.

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1. Introduction

Existence and uniqueness of the solution of the Cauchy problem in the class of analytical functions when the Cauchy condition is given in the part of the domain non having any characteristic direction, follows from Kovalevskiy's theorem [5]. For the system of linear equations with partial derivatives of the first order over two independent variables I.G. Petrovskiy [5] gave a definition of hyperbolic system and proved a theorem on existence and uniqueness for the Cauchy problem in the class of non-analytic functions. In this work and in [2] the solution of Cauchy problem is given by means of characteristics. Further in [4] the existence of the continuous solution of the quazilinear hyperbolic system with two independent variables is proved. A lot of work may be noted [1, 3, 6, 7], devoted to the optimization of the controlled objects with distributed parameters, described by the first order hyperbolic systems. In those works existence of the optimal control is proved and necessary conditions of optimality are derived.

2. Problem statement

Let the controlled process be described by the first order linear hyperbolic system with two independent variables

$$z_t = A\left(t,x\right) z_x + B\left(t,x\right) z + C\left(t,x\right) u + f\left(t,x\right), \tag{1}$$

where $A\left(t,x\right), B\left(t,x\right)$ are $n\times n$ dimensional matrices, $C\left(t,x\right)$ is $n\times r$ dimensional matrix, u is r dimensional control, $f\left(t,x\right)$ is n- dimensional vector function.

We assume that in the considered domain the system (1) is narrow hyperbolic in the sense of Petrovskiy, i.e.

$$A\left(t,x\right) = diag\left(\lambda_1\left(t,x\right), \lambda_2\left(t,x\right), \ldots, \lambda_n\left(t,x\right)\right),$$

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where $\lambda_i(t,x)$ $(i=1,2,\dots,n)$ are real functions of t, x.

Denote by $x=\alpha(t)$ and $x=\beta(t)$, $0\le t\le T$ $(T>0)$ solution of the problem

$$\begin{array}{l}\dot{x}=\lambda_{\min}(t,x)\,, \quad x(0)=0,\\ \dot{x}=\lambda_{\max}(t,x)\,, \quad x(0)=l \quad (l>0)\end{array}$$

correspondingly, where

$$\begin{array}{l}\lambda_{\min}(t,x)=\min\{\lambda_1(t,x)\,,\,\lambda_2(t,x)\,,\dots,\lambda_n(t,x)\}\,,\\ \lambda_{\max}(t,x)=\max\{\lambda_1(t,x)\,,\,\lambda_2(t,x)\,,\dots,\lambda_n(t,x)\}\,,\\ \alpha(t)<\beta(t)\,,\ 0\le t\le T.\end{array}$$

Let the closed domain $\bar{\Omega}=\{0\le t\le T,\ \ \alpha(t)\le x\le\beta(t)\}$ belongs to the considered domain. The Cauchy problem consists of finding the solution $z(t,x)$ of the system (1) inside of $\bar{\Omega}$, that satisfies to the condition

$$z(0,x)=\varphi(x)\,,\ \ 0\le x\le l. \tag{2}$$

Define by U_∂ the set of admissible controls $U_\partial=\{u(t):\ \ u(t)\in L_2^r\,(0,T)\}$.

On the set of solutions of the Cauchy problem (1)-(2) consider the problem of minimization of the functional

$$J(u)=\iint\limits_{\Omega} z'(t,x)\,W(t,x)\,z(t,x)\,dxdt+\int\limits_0^T u'(t)\,U(t)\,u(t)\,dt, \tag{3}$$

where $W(t,x)$ is $n\times n$ - dimensional symmetric, non-negative, continuous matrix, $U(t)$- $n\times r$ dimensional symmetric, positively-defined continuous matrix, the sign $(')$ means transpose.

3. Existence of the solution for the Cauchy problem

Similarly to the [1, 10] may be proved.

Theorem 3.1. *Let*

(1) *The matrix $A(t,x)=diag\,(\lambda_1(t,x)\,,\,\lambda_2(t,x)\,,\dots,\lambda_n(t,x))$ be continuous, has continuous first derivatives;*
(2) *The matrices $B(t,x)\,,\ C(t,x)$ and the vector $f(t,x)$ be continuous;*
(3) *n dimensional function $\varphi(x)$ be absolutely continuous.*

Then for the fixed function $u(t)\in U_\partial$ there exists the unique absolutely continuous solution $z(t,x)$ of the Cauchy problem (1), (2).

Theorem 3.2. *Let the conditions of the Theorem1 and condition imposed on the matrices $W(t,x)$ and $U(t)$ be fulfilled. Then there exist the only optimal control for the problem* (1)-(3).

Proof. Let the sequence $\{u_m(t)\}$ of the admissible controls be minimizing for the problem (1)-(3), i.e.

$$\lim_{m\to\infty}J(u)=\lim_{m\to\infty}\left\{\iint\limits_{\Omega} z_m'(t,x)\ W(t,x)\,z_m(t,x)\,dxdt+\int\limits_0^T u_m'(t)\ U(t)\,u_m(t)\,dt\right\}, \tag{4}$$

where $z_m(t,x)$ is a solution of the Cauchy problem (1), (2) by control $u_m(t)$.

Let's fix arbitrary point (t,x) of the domain Ω. Define by l_i the part corresponding to the characteristics $L_i$from the point (t,x) till its intersection at the some point $(0,x_i)$ with the interval $(0,l)$ of the axis $t=0$.

Let $x = x_i(\tau; t, x)$ be an equation of the characteristics of $L_i$passing trough the point (t, x). Similarly to [1] the Cauchy problem (1), (2) is equivalent to the system of integral equations, i.e.

$$\begin{aligned} z_m^i(t,x) = \varphi^i(x_i(0;t,x)) + \int_0^t \left[B^i(\tau, x_i(\tau;t,x))\, z_m(\tau, x_i(\tau;t,x)) + \right. \\ \left. + C^i(\tau, x_i(\tau;t,x))\, u_m(\tau) + f^i(\tau, x_i(\tau;t,x))\right] d\tau, \end{aligned} \tag{5}$$

where by $B^i(t,x)$ and $C^i(t,x)$ are denoted i-th line of the matrices $B(t,x)$, $C(t,x)$ correspondingly, and by $f^i(t,x)$ i-th coordinate of the function $f(t,x)$.

Since the minimizing sequence $\{u_m(t)\}$ is bounded in $L_2^r(0,T)$, then from $\{u_m(t)\}$ may be chosen a sequence (we define it also by $u_m(t)$), which weakly converges to some function $u^*(t) \in U_\partial$. Define by $z^*(t,x)$ a solution of the Cauchy problem (1), (2) by control $u^*(t)$:

$$\begin{aligned} z^{*^i}(t,x) = \varphi^i(x_i(0;t,x)) + \int_0^t \left[B^i(\tau, x_i(\tau;t,x))\, z^*(\tau, x_i(\tau;t,x)) + \right. \\ \left. + C^i(\tau, x_i(\tau;t,x))\, u^*(\tau) + f^i(\tau, x_i(\tau;t,x))\right] d\tau. \end{aligned} \tag{6}$$

Subtracting from (5) the integral identity (6) we get

$$\begin{aligned} z_m^i(t,x) - z^{*^i}(t,x) = \int_0^t B^i(\tau, x_i(\tau;t,x))\left[z_m(\tau, x_i(\tau;t,x)) - z^*(\tau, x_i(\tau;t,x))\right] d\tau + \\ + \int_0^t C^i(\tau, x_i(\tau;t,x))\left[u_m(\tau) - u^*(\tau)\right] d\tau. \end{aligned} \tag{7}$$

Let in the domain $\bar{\Omega}$ the inequality

$$\|B(t,x)\| \le M$$

be true. Then from (7) one may obtain that

$$\begin{aligned} \sum_{i=1}^n \left| z_m^i(t,x) - z^{*^i}(t,x) \right| \le M \int_0^t \sum_{i=1}^n \left| z_m^i(\tau, x_i(\tau;t,x)) - z^{*^i}(\tau, x_i(\tau;t,x)) \right| d\tau + \\ + \sum_{i=1}^n \left| \int_0^t C^i(\tau, x_i(\tau;t,x))\left[u_m(\tau) - u^*(\tau)\right] d\tau \right|. \end{aligned}$$

From this applying Gronwall's inequality we have

$$\sum_{i=1}^n \left| z_m^i(t,x) - z^{*^i}(t,x) \right| \le L \sum_{i=1}^n \left| \int_0^t C^i(\tau, x_i(\tau;t,x))\left[u_m(\tau) - u^*(\tau)\right] d\tau \right|.$$

Passing to limit by $m \to \infty$ we got that the sequence $\{z_m(t,x)\}$ uniformly converges to the function $z^*(t,x), \quad (t,x) \in \bar{\Omega}$.

Note that the set U_∂ is convex in $L_2^r(0,T)$, the functional $J(u)$ is lower semi-continuous on U_∂. From this follows that the functional $J(u)$ is weak lower semi-continuous on U_∂ (see [8], p.52).

Therefore from (4) we have

$$J(u^*) \le \lim_{m\to\infty} J(u_m) = \iint_\Omega z^{*'}(t,x)\, W(t,x)\, z^*(t,x)\, dxdt + \int_0^T u^{*'}(t)\, U(t)\, u^*(t)\, dt.$$

From this of follows that $u^*(t)$ is an optimal control and $z^*(t,x)$- optimal trajectory.

Now we prove the uniqueness of the optimal control. Let $u^*(t)$ and $\bar{u}(t)$ be two different optimal controls. Define by $z^*(t,x)$ and $\bar{z}(t,x)$ the solutions of the Cauchy problem (1), (2) by controls $u^*(t)$ and $\bar{u}(t)$ correspondingly. Take

$$u_\lambda(t) = \lambda u^*(t) + (1-\lambda)\bar{u}(t), \qquad \lambda \in (0,1), \ \ t \in [0,T].$$

Then

$$z_\lambda(t,x) = \lambda z^*(t,x) + (1-\lambda)\bar{z}(t,x), \qquad (t,x) \in \Omega,$$

where $z_\lambda(t,x)$ is a solution of the problem (1), (2) by $u_\lambda(t)$.

From the strong convexity of the functional (3) follows the validity of the inequality

$$J(u_\lambda) = J(\lambda u^* + (1-\lambda)\bar{u}) < \ \ \lambda J(u^*) + (1-\lambda) J(\bar{u}) =$$

$$= \lambda \inf_{v \in U_\partial} J(v) + (1-\lambda) \inf_{v \in U_\partial} J(v) = \inf_{v \in U_\partial} J(v).$$

From this one may derive that the functional (3) gets its minimal value by the control $u_\lambda(t)$. But this is a contradiction, consequently the optimal control is unique. □

4. Necessary and sufficient optimality conditions

The adjoint state is defined as a solution of the following problem

$$P_t = \left(A'(t,x)P\right)_x - B'(t,x)P + W(t,x)z, \tag{8}$$

$$P|_\Gamma = 0, \tag{9}$$

where by Γ is a part of the boundary of Ω consisting of the interval of the line $t = T$ and curves $x = \alpha(t)$ and $x = \beta(t)$, $\quad 0 \le t \le T$.

Further is supposed that $f(t,x) = 0$.

Theorem 4.1. *Let $u^*(t)$ be admissible control, $z^*(t,x)$- solution of the Cauchy problem* (1), (2) *by the control $u^*(t)$. The necessary and sufficient condition of optimality of the solution $u^*(t)$ is existence of the solution of the problem*

$$z_t = A(t,x) z_x + B(t,x) z + C(t,x) U^{-1}(t) \int_{\alpha(t)}^{\beta(t)} C'(t,s) P(t,s)\, ds, \tag{10}$$

$$\begin{array}{l} P_t = \left(A'(t,x)P\right)_x - B'(t,x)P + W(t,x)z, \\ z(0,x) = \varphi(x), \ \ x \in [0,l], \quad P|_\Gamma = 0. \end{array} \tag{11}$$

Then the unique optimal $u^*(t)$ of the problem (1)-(3) *is defined by the relation*

$$u^*(t) = U^{-1}(t) \int_{\alpha(t)}^{\beta(t)} C'(t,x) P^*(t,x)\, dx, \quad t \in [0,T]. \tag{12}$$

Proof. Let $u^*(t)$ be optimal control and $z^*(t,x)$ - solution of the Cauchy problem (1), (2) by the control $u^*(t)$. For the function $r(t) \in L_2^r(0,T)$,

$$u_\varepsilon(t) = u^*(t) + \varepsilon r(t), \quad 0 \le t \le T, \ \ \varepsilon \in R.$$

Define by $z_\varepsilon(t,x)$ the solution of the problem (1), (2) by control $u_\varepsilon(t)$. Then we get

$$z_\varepsilon(t,x) = z^*(t,x) + \varepsilon \Phi(t,x),$$

where $\Phi(t,x)$ is defined as a solution of the problem

$$\Phi_t = A(t,x)\,\Phi_x + B(t,x)\,\Phi + C(t,x)\,r(t)\,, \qquad \Phi(0,x) = 0. \tag{13}$$

Now we calculate the influence of the perturbation on the functional

$$\Delta J(u^*) = J(u_\varepsilon) - J(u^*) = \iint_\Omega z'_\varepsilon(t,x)\,W(t,x)\,z_\varepsilon(t,x)\,dxdt +$$
$$+\int_0^T u'_\varepsilon(t)\,U(t)\,u_\varepsilon(t)\,dt - \iint_\Omega z^{*'}(t,x)\,W(t,x)\,z^*(t,x)\,dxdt - \int_0^T u^{*'}(t)\,U(t)\,u^*(t)\,dt +$$
$$+2\varepsilon\iint_\Omega P'(t,x)\left[\Phi_t(t,x) - A(t,x)\,\Phi_x(t,x) - B(t,x)\,\Phi(t,x) - C(t,x)\,r(t)\right]dxdt.$$

Integrating by parts and considering the condition imposed on the function $P(t,x)$ on the boundary Γ of the domain Ω we have

$$\iint_\Omega P'(t,x)\,\Phi_t(t,x)\,dxdt = -\iint_\Omega P'_t(t,x)\,\Phi(t,x)\,dxdt,$$

$$\iint_\Omega P'(t,x)\,A(t,x)\,\Phi_x(t,x)\,dxdt = -\iint_\Omega \left(P'(t,x)\,A(t,x)\right)_x \Phi(t,x)\,dxdt.$$

Considering these identies in the increment of the functional one may write

$$\Delta J(u^*) = 2\varepsilon\iint_\Omega \Big\{ \Big[z^{*'}(t,x)\,W(t,x) - P'_t(t,x) + \left(P'(t,x)\,A(t,x)\right)_x - P'(t,x)\,B(t,x)\Big]\,\Phi(t,x) -$$
$$-P'(t,x)\,C(t,x)\,r(t)\Big\}\,dxdt + 2\varepsilon\int_0^T u^{*'}(t)\,U(t)\,r(t)\,dt + \eta(\varepsilon)\,,$$

where

$$\eta(\varepsilon) = \varepsilon^2\left\{\iint_\Omega \Phi'(t,x)\,W(t,x)\,\Phi(t,x)\,dxdt + \int_0^T r'(t)\,U(t)\,r(t)\right\}dt.$$

From this taking into account that $P = P^*(t,x)$ is a solution of the equation (8) we get

$$\Delta J(u^*) = 2\varepsilon\int_0^T\left[u^{*'}(t,x)\,U(t) - \int_{\alpha(t)}^{\beta(t)} P^{*'}(t,s)\,C(t,s)\,ds\right] r(t)\,dt + \eta(\varepsilon)\,. \tag{14}$$

Solution of the problem (13) is equivalent to the following integral equation

$$\Phi^i(t,x) = \int_0^t \left[B^i(\tau, x_i(\tau;t,x))\,\Phi(\tau, x_i(\tau;t,x)) + C^i(\tau, x_i(\tau;t,x))\,r(t)\right]d\tau.$$

From this

$$\sum_{i=1}^n \left|\Phi^i(t,x)\right| \le M\int_0^t \sum_{i=1}^n \left|\Phi^i(\tau, x_i(\tau;t,x))\right| d\tau + L\int_0^t |r(\tau)|\,d\tau. \tag{15}$$

From (15) using the Gronwall's inequality we obtain

$$|\Phi(t,x)| \le K\int_0^t |r(\tau)|\,d\tau.$$

Then from the expression for $\eta(\varepsilon)$ we obtain that $\eta(\varepsilon) = o(\varepsilon)$. For the optimal control $u^*(t)$ the relation

$$\Delta J(u^*) = 2\varepsilon \int_0^T \left\{ u^{*'}(t)\, U(t) - \int_{\alpha(t)}^{\beta(t)} P^{*'}(t,s)\, C(t,s)\, ds \right\} r(t)\, dt + o(\varepsilon) \geq 0$$

should be satisfied for any $r(t)$.

Therefore the term $o(\varepsilon)$ may be neglected. Since ε may be positive and also negative it is true

$$\int_0^T \left\{ u^{*'}(t)\, U(t) - \int_{\alpha(t)}^{\beta(t)} P^{*'}(t,s)\, C(t,s)\, ds \right\} r(t)\, dt = 0.$$

As the function $r(t)$ is arbitrary one from $L_2^r(0,T)$ we have

$$u^{*'}(t)\, U(t) - \int_{\alpha(t)}^{\beta(t)} P^{*'}(t,s)\, C(t,s)\, ds = 0, \quad [0,T].$$

From this we obtain the equality (12). Substituting (12) into (1) we obtain (10). Thus the necessary condition is proved.

Now let $z^*(t,x)$, $P^*(t,x)$ be solutions of the problem (10), (11). Let's prove that the function $u^*(t)$ defined by the formula (12) is an optimal control for the problem (1)-(3). Let $u(t)$ be any admissible control, $z(t,x)$- solution of the problem (1)-(2) by this control.

Take

$$u(t) = u^*(t) + r(t), \quad z(t,x) = z^*(t,x) + \Phi(t,x),$$

where $\Phi(t,x)$ is a solution of the problem (13).

Similarly to the obtaining of (14) we have

$$\Delta J(u^*) = 2 \int_0^T \left\{ u^{*'}(t)\, U(t) - \int_{\alpha(t)}^{\beta(t)} P^{*'}(t,s)\, C(t,s)\, ds \right\} r(t)\, dt + \eta,$$

where

$$\eta = \iint_\Omega \Phi'(t,x)\, W(t,x)\, \Phi(t,x)\, dx dt + \int_0^T r'(t)\, U(t)\, r(t)\, dt.$$

Considering equality (12) and $\eta \geq 0$, we obtain

$$\Delta J(u^*) \geq 0.$$

It follows from the last that the control $u^*(t)$ defined (11) is optimal. □

The theorem is proved.

5. Non-linear integro-differential systems

Let $n \times n$-matrix $G(t,x,s)$ be in the domain a solution of the integro-differential equation

$$\begin{aligned} \frac{\partial G}{\partial t} &= \frac{\partial}{\partial x}\left(A'(t,x)\, G\right) + \frac{\partial}{\partial s}\left(G A(t,s)\right) - B'(t,x)\, G - G B(t,s) - \\ &- \int_{\alpha(t)}^{\beta(t)} G(t,x,\sigma)\, Q(t,\sigma)\, G(t,\sigma,s)\, d\sigma + \tilde{\Phi}(x,s)\, W(t,s) \end{aligned} \tag{16}$$

$\Pi = \{0 < t < T,\ \alpha(t) < x < \beta(t)\ ,\ \alpha(t) < s < \beta(t)\}$ satisfying to the condition

$$G|_S = 0, \tag{17}$$

where

$$Q\left(t,\sigma\right) = C\left(t,\sigma\right) U^{-1}\left(t\right) d\sigma \int\limits_{\alpha(t)}^{\beta(t)} C'\left(t,\xi\right) d\xi, \tag{18}$$

$$\int\limits_{\alpha(t)}^{\beta(t)} \tilde{\Phi}\left(t,s\right) W\left(t,s\right) z\left(t,s\right) ds = W\left(t,x\right) z\left(t,x\right), \tag{19}$$

S- surface of the domain Π excepting $\{t = 0,\ 0 \le x \le l,\ \ 0 \le s \le l\}$.

Theorem 5.1. *Assume that* $G\left(t,x,s\right)$ *is a solution of the problem* (16), (17) *and* $\left(u^*\left(t\right),\ z^*\left(t,x\right)\right)$ *is an optimal process for the problem* (1)-(3). *Then, the functional*

$$P^*\left(t,x\right) = \int\limits_{\alpha(t)}^{\beta(t)} G\left(t,x,s\right) z^*\left(t,s\right) ds \tag{20}$$

is a solution of the adjoint problem (8), (9), *optimal control* $u^*\left(t\right)$ *is defined by* (12) *and minimal value of the functional* (3) *is*

$$J\left(u^*\right) = -\int\limits_0^l \int\limits_0^l \varphi'\left(x\right) G\left(0,x,s\right) \varphi\left(s\right) dxds. \tag{21}$$

Proof. First we prove that the function $P^*\left(t,x\right)$ defined by (20) is a solution of the adjoint problem (8), (9). From (20) we obtain

$$P_t^*\left(t,x\right) = \int\limits_{\alpha(t)}^{\beta(t)} \left[G_t\left(t,x,s\right) z^*\left(t,s\right) + G\left(t,x,s\right) z_t^*\left(t,s\right)\right] ds, \tag{22}$$

$$\left(A'\left(t,x\right) P^*\left(t,x\right)\right)_x = \int\limits_{\alpha(t)}^{\beta(t)} \left(A'\left(t,x\right) G\left(t,x,s\right)\right)_x z^*\left(t,s\right) ds. \tag{23}$$

Substituting (20), (22), (23) into the equation (8) and considering that $z^*\left(t,x\right)$ satisfies to the equation (10) we have

$$\begin{aligned} &\int\limits_{\alpha(t)}^{\beta(t)} \Big\{G_t\left(t,x,s\right) z^*\left(t,s\right) + G\left(t,x,s\right) \Big[A\left(t,s\right) z_s^*\left(t,s\right) + B\left(t,s\right) z^*\left(t,s\right) + \\ &+ C\left(t,s\right) U^{-1}\left(t\right) \int\limits_{\alpha(t)}^{\beta(t)} C'\left(t,\xi\right) \left(\int\limits_{\alpha(t)}^{\beta(t)} G\left(t,\xi,\sigma\right) z^*\left(t,\sigma\right) d\sigma \right) d\xi \Big] - \\ &- \left(A'\left(t,x\right) G\left(t,x,s\right)\right)_x z^*\left(t,s\right) + B'\left(t,x\right) G\left(t,x,s\right) z^*\left(t,s\right) - \\ &- \tilde{\Phi}\left(x,s\right) W\left(t,s\right) z^*\left(t,s\right)\Big\} ds = 0. \end{aligned} \tag{24}$$

From this integrating by parts and considering the conditions (17) we obtain

$$\int\limits_{\alpha(t)}^{\beta(t)} G\left(t,x,s\right) A\left(t,s\right) z_s^*\left(t,s\right) ds = -\int\limits_{\alpha(t)}^{\beta(t)} \left(G\left(t,x,s\right) A\left(t,s\right)\right)_s z^*\left(t,s\right) ds.$$

Changing the integrating turn, we have

$$\int\limits_{\alpha(t)}^{\beta(t)} G(t,x,s)\left[\int\limits_{\alpha(t)}^{\beta(t)} C(t,s)\,U^{-1}(t)\,C'(t,\xi)\left(\int\limits_{\alpha(t)}^{\beta(t)} G(t,\xi,\sigma)\,z^*(t,\sigma)\,d\sigma\right)d\xi\right]ds =$$

$$= \int\limits_{\alpha(t)}^{\beta(t)}\left\{\int\limits_{\alpha(t)}^{\beta(t)} G(t,x,\sigma)\,Q(t,\sigma)\,G(t,\sigma,s)\,d\sigma\right\} z^*(t,s)\,ds.$$

Taking into account these equations in (24) we get

$$\int\limits_{\alpha(t)}^{\beta(t)} \{G_t(t,x,s) - (A'(t,x)\,G(t,x,s))_x - (G(t,x,s)\,A(t,s))_s + B'(t,x)\,G(t,x,s) +$$
$$+G(t,x,s)\,B(t,s) + \int\limits_{\alpha(t)}^{\beta(t)} G(t,x,\sigma)\,Q(t,\sigma)\,G(t,\sigma,s)\,d\sigma - \tilde{\Phi}(x,s)\,W(t,s)\Bigg\}\, z^*(t,s)\,ds = 0.$$

It is easy to get from this that the function $P^*(t,x)$ defined by relation (20) is a solution of the adjoint problem (8), (9).

Now we prove that the minimal value of the functional (3) is defined by (21).

For this purpose consider the function

$$g(t) = \int\limits_{\alpha(t)}^{\beta(t)}\int\limits_{\alpha(t)}^{\beta(t)} z^{*'}(t,x)\,G(t,x,s)\,z^*(t,s)\,dxds. \qquad (25)$$

Differenting this with respect to t and considering the condition (11) one may get

$$g'(t) = \int\limits_{\alpha(t)}^{\beta(t)}\int\limits_{\alpha(t)}^{\beta(t)} \Big[z_t^{*'}(t,x)\,G(t,x,s)\,z^*(t,s) + z^{*'}(t,x)\,G_t(t,x,s)\,z^*(t,s) +$$
$$+z^{*'}(t,x)\,G(t,x,s)\,z_t^*(t,s)\Big]\,dxds.$$

Replacing here $z^*(t,x)$ and $z^*(t,s)$ by their expressions from the equations (1) and (10) respectively, we obtain

$$g'(t) = \int\limits_{\alpha(t)}^{\beta(t)}\int\limits_{\alpha(t)}^{\beta(t)} \Big\{[A(t,x)\,z_x^*(t,x) + B(t,x)\,z^*(t,x) + C(t,x)\,u^*(t)]'\,G(t,x,s)\,z^*(t,s) +$$
$$+z^{*'}(t,x)\,G_t(t,x,s)\,z^*(t,s) + z^{*'}(t,x)\,G(t,x,s)\,[A(t,s)\,z_s^*(t,s) + B(t,s)\,z^*(t,s) + \qquad (26)$$
$$+C(t,s)\,U^{-1}(t)\int\limits_{\alpha(t)}^{\beta(t)} C'(t,\xi)\,P^*(t,\xi)\,d\xi\Big\}\,dxds.$$

Integrating the last by parts and considering (17) we have

$$\int\limits_{\alpha(t)}^{\beta(t)} (A(t,x)\,z_x^*(t,x))'\,G(t,x,s)\,z^*(t,s)\,dx = -\int\limits_{\alpha(t)}^{\beta(t)} z^{*'}(t,x)\,\big(A'(t,x)\,G(t,x,s)\big)_x\,z^*(t,s)\,dx,$$

$$\int\limits_{\alpha(t)}^{\beta(t)} z^{*'}(t,x)\,G(t,x,s)\,A(t,s)\,z_s^*(t,s)\,ds = -\int\limits_{\alpha(t)}^{\beta(t)} z^{*'}(t,x)\,(G(t,x,s)\,A(t,s))_s\,z^*(t,s)\,ds.$$

Use of (18) and changing the integrating turn gives

$$\int\limits_{\alpha(t)}^{\beta(t)}\int\limits_{\alpha(t)}^{\beta(t)} z^{*'}(t,x)\,G(t,x,s)\,C(t,s)\,U^{-1}(t)\int\limits_{\alpha(t)}^{\beta(t)} C'(t,\xi)\left(\int\limits_{\alpha(t)}^{\beta(t)} G(t,\xi,\sigma)\,z^*(t,\sigma)\,d\sigma\right)d\xi dxds =$$
$$= \int\limits_{\alpha(t)}^{\beta(t)}\int\limits_{\alpha(t)}^{\beta(t)} z^{*'}(t,x)\left(\int\limits_{\alpha(t)}^{\beta(t)} G(t,x,\sigma)\,Q(t,\sigma)\,G(t,\sigma,s)\,d\sigma\right)z^*(t,s)\,dxds.$$

The obtained results we put into (26) and considering that $G(t,x,s)$ is a solution of the problem (16), (17) get

$$\begin{aligned} g'(t) = \int\limits_{\alpha(t)}^{\beta(t)}\int\limits_{\alpha(t)}^{\beta(t)} \Big[& z^{*'}(t,x)\,\tilde{\Phi}(x,s)\,W(t,s)\,z^*(t,s) \\ &(C(t,x)\,u^*(t))'\,G(t,x,s)\,z^*(t,s)\Big]\,dxds. \end{aligned} \tag{27}$$

According to (19) is obtained

$$\int\limits_{\alpha(t)}^{\beta(t)}\int\limits_{\alpha(t)}^{\beta(t)} z^{*'}(t,x)\,\tilde{\Phi}(x,s)\,W(t,s)\,z^*(t,s)\,dxds = \int\limits_{\alpha(t)}^{\beta(t)} z^{*'}(t,x)\,W(t,x)\,z^*(t,x)\,dx. \tag{28}$$

In (12) putting (20) instead of $P^*(t,x)$ we have

$$\begin{aligned} U(t)\,u^*(t) &= \int\limits_{\alpha(t)}^{\beta(t)} C'(t,x)\left(\int\limits_{\alpha(t)}^{\beta(t)} G(t,x,s)\,z^*(t,s)\,ds\right)dx = \\ &= \int\limits_{\alpha(t)}^{\beta(t)}\int\limits_{\alpha(t)}^{\beta(t)} C'(t,x)\,G(t,x,s)\,z^*(t,s)\,dxds. \end{aligned} \tag{29}$$

Consideration of (28), (29) in (27) gives

$$g'(t) = \int\limits_{\alpha(t)}^{\beta(t)} z^{*'}(t,x)\,W(t,x)\,z^*(t,x)\,dx + u^{*'}(t)\,U(t)\,u^*(t).$$

From this integrating with respect to t over $[0,T]$ we have

$$g(T) - g(0) = \int\limits_0^T\int\limits_{\alpha(t)}^{\beta(t)} z^{*'}(t,x)\,W(t,x)\,z^*(t,x)\,dxdt + \int\limits_0^T u^{*'}(t)\,U(t)\,u^*(t)\,dt = J(u^*).$$

Then as follows from (17) $G(T,x,s) = 0,\ 0 \le x,\ s \le l$. Therefore from (25) we obtain that $g(T) = 0$ and

$$J(u^*) = -g(0) = -\int\limits_0^l\int\limits_0^l \varphi'(x)\,G(0,x,s)\,\varphi(s)\,dxds.$$

□

Example. Distribution of plane waves is described by the system

$$\frac{\partial y}{\partial t} + \frac{1}{\rho_0}\frac{\partial p}{\partial x} = u_1,$$
$$\frac{\partial p}{\partial t}\rho_0 c_0^2\frac{\partial y}{\partial x} = u_2,$$

with conditions

$$y(x,0)=\varphi_1(x)\ ,\quad p(x,0)=\varphi_2(x),$$

where y- speed of the disturbed environment, p- pressure environment, ρ_0 and c_0- density and compressibility of environment correspondingly, u_1, u_2 - perturbation.

After some transformations the system assumes the form:

$$\frac{\partial y_1}{\partial t}+c_0\frac{\partial y_1}{\partial x}=u_1\ +\frac{1}{\rho_0 c_0}u_2,$$

$$\frac{\partial y_2}{\partial t}\ -c_0\frac{\partial y_2}{\partial x}=u_1\ -\frac{1}{\rho_0 c_0}u_2\,,$$

with conditions

$$y_1(x,0)=\varphi_1(x)\ +\frac{1}{\rho_0 c_0}\,\varphi_2(x)$$

$$y_2(x,0)=\varphi_1(x)\ -\frac{1}{\rho_0 c_0}\,\varphi_2(x).$$

Let Ω is triangle with the basis$[0,l]$, which is determined by the inequality $t\geq 0,\ \ x-c_0t\geq 0,\ \ x+c_0t\leq l$

On a set of solutions of the system we consider the minimization of the functional

$$J\left(u\right)=\iint\limits_{\Omega}\left(y_1^2+y_2^2\right)dxdt+\int\limits_{0}^{\frac{l}{2c_0}}\left(u_1^2+u_2^2\right)dt.$$

The adjoint problem has the following form:

$$\frac{\partial z_1}{\partial t}+c_0\frac{\partial z_1}{\partial x}=y_1,$$

$$\frac{\partial z_2}{\partial t}\ -c_0\frac{\partial x_2}{\partial x}=y_2$$

with conditions

$$z_1|_\Gamma=0,\ \ z_2|_\Gamma=0,$$

where Γ is side of triangle of Ω.

Optimal control is defined by the following formula

$$u_1\left(t\right)\ =\int_{c_0t}^{l-c_0t}\left(z_1+z_2\right)\,dx,$$

$$u_2\left(t\right)\ =\frac{1}{\rho_0 c_0}\int_{c_0t}^{l-c_0t}\left(z_1-z_2\right)\,dx.$$

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