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A SURVEY OF RESULTS IN THE THEORY OF FRACTIONAL SPACES GENERATED BY POSITIVE OPERATORS

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Abstract. This is a review paper on results for fractional spaces generated by positive operators. Its scope ranges from theory of differential and difference operators in a space to operators with local and nonlocal boundary conditions. We also discuss their applications to partial differential equations and theory of difference schemes for partial differential equations.

Keywords: fractional spaces, positive operators, differential and difference operators, Banach spaces, interpolation spaces, stability.

AMS Subject Classification: 35L40, 35L45.

1. Introduction

It is well-known that the positivity of differential and difference operators in Hilbert and Banach spaces is important in the study of various properties of boundary value problems for partial differential equations, of stability of difference schemes for partial differential equations, and of summation Fourier series converging in max norm (see, [11], [56], [12]).

An operator A densely defined in a Banach space E with domain $D(A)$ is called positive in E, if its spectrum σ_A lies in the interior of the sector of angle φ, $0<\varphi<\pi$, symmetric with respect to the real axis, and moreover on the edges of this sector $S_1(\varphi)=\{\rho e^{i\varphi}:0\leq\rho\leq\infty\}$ and $S_2(\varphi)=\{\rho e^{-i\varphi}:0\leq\rho\leq\infty\}$, and outside of the sector the resolvent $(\lambda-A)^{-1}$ is subject to the bound (see, [11])

$$\left\|(A-\lambda)^{-1}\right\|_{E\rightarrow E}\leq\frac{M}{1+|\lambda|}.$$

The infimum of all such angles φ is called the spectral angle of the positive operator A and is denoted by $\varphi(A)=\varphi(A,E)$. The operator A is said to be strongly positive in a Banach space E if $\varphi(A,E)<\frac{\pi}{2}$.

Throughout the present paper, we will indicate with M positive constants which can be different from time to time and we are not interested in precise. We will write $M(\alpha,\beta,\cdots)$ to stress the fact that the constant depends only on $\alpha,\beta,\cdots$.

Let us consider the selfadjoint positive definite operator A in a Hilbert space H with dense domain $\overline{D(A)}=H$. That means there exists $\delta>0$ such that $A=A^{*}\geq\delta I$. Then, applying the spectral representation of the selfadjoint positive definite operator, we can get

$$\left\|(A-\lambda)^{-1}\right\|_{H\rightarrow E}\leq\sup_{\delta\leq\mu<\infty}\frac{1}{|\mu-\lambda|}. \tag{1}$$

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Now, we will estimate $\frac{1}{|\mu-\lambda|}$. There are two possible case: $Re\lambda \leq \frac{\delta}{2}$ and $Re\lambda > \frac{\delta}{2}$. In the first case we have two estimates

$$|\mu - \lambda| \geq \mu - \frac{\delta}{2} \geq \frac{\delta}{2},$$

$$|\mu - \lambda| = \sqrt{(\mu - 2Re\lambda)\,\mu + |\lambda|^2} \geq |\lambda|\,.$$

Therefore, from these estimates it follows that

$$|\mu - \lambda| \geq \frac{1}{2}\left(\frac{\delta}{2} + |\lambda|\right). \tag{2}$$

In the second case we have the following estimate

$$|\mu - \lambda| = \sqrt{(\mu - |\lambda|\cos\varphi)^2 + |\lambda|^2 \sin^2\varphi} \geq |\lambda| \sin\varphi.$$

Assume that $0 < \varepsilon < \varphi$. Then

$$|\mu - \lambda| \geq \frac{\sin\varepsilon}{2}\left(\frac{\delta}{2} + |\lambda|\right) \tag{3}$$

for all $\varepsilon < \varphi$. Applying estimates (1), (2) and (3), we can write

$$\left\|(A - \lambda)^{-1}\right\|_{H \to H} \leq \frac{M(\varepsilon)}{1 + |\lambda|}.$$

So, the selfadjoint positive definite operator A in a Hilbert space H is the strongly positive operator with the spectral angle $\varphi(A, H) = 0$. Therefore, the positivity of operators in a Banach space is the generalization of the notion of selfadjoint positive definite operators in a Hilbert space.

The positivity of the wider class of differential operators in Banach spaces has been studied by K.Yosida, T. Kato, S. Agmon, A. Douglis, L. Nirenberg, A.Friedman, H.B. Stewart, M.Z. Solomyak, P.E. Sobolevskii and et al (see, [1-4, 57, 63-66]).

In [66], H.B. Stewart proved that uniformly elliptic operator of even order with general boundary conditions generates analytic semigroup in the topology of uniform convergence. As application, he gave an existence and uniqueness theorem for parabolic initial-boundary value problems, by using the Kato-Tanabe theory for temporally inhomogenous evolution equation

$$\frac{\partial u}{\partial t} + A(t)u = f.$$

M.Z. Solomyak considered in [64] the equation

$$\begin{cases} Au(x) - \lambda u(x) = f(x), x \in \Omega, \\ \frac{\partial^k u}{\partial N_k}|_\Gamma = 0, k = 0, 1, \cdots, m-1, \end{cases}$$

where $\lambda = \sigma + i\tau$, Ω is a bounded domain with sufficiently smooth boundary Γ, A is a positive and self-adjoint (for u satisfying the Dirichlet boundary conditions) with sufficiently smooth coefficients. He proved the positivity of A in $L_p(\Omega)$.

The positivity of wider class of differential and difference operators and their related applications have been investigated by many researchers (see, for example, [6-9, 13-31, 33-55, 59-62, 68-70]).

Important progress has been made in the study of positive operators from the view-point of the stability analysis of high order accuracy difference schemes for partial differential equations. It is well known that the most useful methods for stability analysis of difference schemes are difference analogue of maximum principle and energy method. The application of theory of positive difference operators allows us to investigate the stability and coercive stability properties of difference schemes in various norms for partial differential equations especially when one can not use a maximum principle and energy method. However, the positivity of difference operators is not well investigated in general. Therefore, the investigation of positivity of difference operators in Banach spaces and its applications to stability of difference schemes for partial differential equations is an important subject.

For a positive operator A in the Banach space E, let us introduce the fractional spaces $E_\alpha = E_\alpha(E, A)$, $E_{\alpha,p} = E_{\alpha,p}(E, A), (0 < \alpha < 1)$ consisting of those $v \in E$ for which norms

$$\|v\|_{E_\alpha} = \sup_{\lambda>0} \lambda^\alpha \|A(\lambda + A)^{-1}v\|_E,$$

$$\|v\|_{E_{\alpha,p}} = \left(\int_0^\infty \|\lambda^\alpha A(\lambda + A)^{-1}v\|_E^p \frac{d\lambda}{\lambda} \right)^{\frac{1}{p}}, 1 \le p < \infty$$

are finite. Clearly, the positive operator commutes A and its resolvent $(A-\lambda)^{-1}$. By the definition of the norm in the fractional space $E_\alpha = E_\alpha(E, A)$, $E_{\alpha,p} = E_{\alpha,p}(E, A), 1 \le p < \infty, (0 < \alpha < 1)$, we get

$$\|(A - \lambda)^{-1}\|_{E_\alpha \to E_\alpha}, \|(A - \lambda)^{-1}\|_{E_{\alpha,p} \to E_{\alpha,p}} \le \|(A - \lambda)^{-1}\|_{E \to E}. \tag{4}$$

Thus, from the positivity of operator A in the Banach space E it follows the positivity of this operator in fractional spaces $E_\alpha = E_\alpha(E, A)$, $E_{\alpha,p} = E_{\alpha,p}(E, A), 1 \le p < \infty, (0 < \alpha < 1)$.

This paper contains a survey of results for fractional spaces generated by positive differential and difference operators in Banach spaces. Its scope ranges from theory of differential and difference operators in a space to operators with local and nonlocal boundary conditions. We also discuss their applications to partial differential equations and theory of difference schemes for partial differential equations.

2. Fractional spaces generated by differential and difference operators in the entire space $\mathbb{R}^n$

Let us consider a differential operator with constant coefficients of the form

$$B = \sum_{|r|=2m} b_r \frac{\partial^{|r|}}{\partial_{x_1}^{r_1} ... \partial_{x_n}^{r_n}}$$

acting on functions defined on the entire space $\mathbb{R}^n$. Here $r \in \mathbb{R}^n$ is a vector with nonnegative integer components, $|r| = r_1 + ... + r_n$. If $\varphi(y)$ $(y = (y_1, ..., y_n) \in \mathbb{R}^n)$ is an infinitely differentiable function that decays at infinity together with all its derivatives, then by means of the Fourier transformation one establishes the equality

$$F(B_\varphi)(\xi) = B(\xi) F(\varphi)(\xi).$$

Here the Fourier transform operator is defined by the rule

$$F(\varphi)(\xi) = (2\pi)^{-n/2} \int_{R^n} \exp\{-i(y, \xi)\} \varphi(y)\, dy,$$

$$(y, \xi) = y_1\xi_1 + ... + y_n\xi_n.$$

The function $B(\xi)$ is called the symbol of the operator B and is given by

$$B(\xi) = \sum_{|r|=2m} b_r (i\xi_1)^{r_1} ... (i\xi_n)^{r_n}.$$

We will assume that the symbol

$$B^x(\xi) = \sum_{|r|=2m} a_r(x)\,(i\xi_1)^{r_1}\ldots(i\xi_n)^{r_n}\,, \xi = (\xi_1,\cdots,\xi_n) \in \mathbb{R}^n$$

of the differential operator of the form

$$B^x = \sum_{|r|=2m} a_r(x) \frac{\partial^{|r|}}{\partial x_1^{r_1}\ldots\partial x_n^{r_n}} \tag{5}$$

acting on functions defined on the space $\mathbb{R}^n$, satisfies the inequalities

$$0 < M_1|\xi|^{2m} \leq (-1)^m B^x(\xi) \leq M_2|\xi|^{2m} < \infty$$

for $\xi \neq 0$.

Then, for sufficiently large positive δ, an elliptic operator $A = B^x + \delta I$ is a strongly positive operator in Banach spaces $C(\mathbb{R}^n)$ and $L_p(\mathbb{R}^n), 1 \leq p < \infty$. Here $C(\mathbb{R}^n)$ is the space of all continuous functions $\varphi(x)$ defined on $\mathbb{R}^n$ with the usual norm

$$\|\varphi\|_{C(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} |\varphi(x)|,$$

$L_p(\mathbb{R}^n)$ is the space of the all integrable functions $\varphi(x)$ defined on $\mathbb{R}^n$ with the norm

$$\|\varphi\|_{L_p(\mathbb{R}^n)} = \left(\int\limits_{x\in\mathbb{R}^n} |\varphi(x)|^p\,dx\right)^{\frac{1}{p}}.$$

We will introduce the Banach space $C^\mu(\mathbb{R}^n)\,(0 < \mu < 1)$ of all continuous functions $\varphi(x)$ defined on $\mathbb{R}^n$ and satisfying a Hölder condition for which the following norm is finite:

$$\|\varphi\|_{C^\mu(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} |\varphi(x)| + \sup_{\substack{x,y\in\mathbb{R}^n\\ y\neq 0}} \frac{|\varphi(x+y)-\varphi(x)|}{|y|^\mu},$$

the Banach space $W_p^\mu(\mathbb{R}^n)\,(0 < \mu < 1)$ of all integrable functions $\varphi(x)$ defined on $\mathbb{R}^n$ and satisfying a Hölder condition for which the following norm is finite:

$$\|\varphi\|_{W_p^\mu(\mathbb{R}^n)} = \left[\int\limits_{x\in\mathbb{R}^n}\int\limits_{\substack{y\in\mathbb{R}^n\\ y\neq 0}} \frac{|\varphi(x+y)-\varphi(x)|^p}{|y|^{n+\mu p}}dydx + \|\varphi\|_{L_p(\mathbb{R}^n)}^p\right]^{\frac{1}{p}}, 1 \leq p < \infty.$$

Theorem 2.1. [69] $E_\alpha(C^\mu(\mathbb{R}^n), A) = C^{2m\alpha+\mu}(\mathbb{R}^n)$ *for all* $0 < 2m\alpha + \mu < 1, 0 \leq \mu \leq 1$.

This fact follows from the equality $D(A) = C^{2m+\mu}(\mathbb{R}^n)$ for an $2m$-th order elliptic operator A in $C^\mu(\mathbb{R}^n), 0 < \mu < 1$, via the real interpolation method.

Theorem 2.2. [69] $E_{\alpha,p}(L_p(\mathbb{R}^n), A) = W_p^{2m\alpha}(\mathbb{R}^n)$ *for all* $0 < 2m\alpha < 1, 1 \leq p < \infty$.

This fact follows from the equality $D(A) = W_p^{2m}(\mathbb{R}^n)$ for an $2m$-th order elliptic operator A in $L_p(\mathbb{R}^n)$, $1 < p < \infty$, via the real interpolation method. The alternative method of investigation adopted in [11],[12], based on estimates of fundamental solution of the resolvent equation for the operator A^x, allows us to consider also the cases $p = 1$ and $p = \infty$.

From the strong positivity of an elliptic operator $A = B^x + \delta I$ in Banach spaces $C(\mathbb{R}^n)$ and $L_p(\mathbb{R}^n), 1 \leq p < \infty$ and estimate (4) it follows the strong positivity of this operator in Banach spaces $C^\mu(\mathbb{R}^n)$ and $W_p^{2m\alpha}(\mathbb{R}^n)$.

In applications, we consider the Cauchy problem for the $2m$-th order multidimensional parabolic equation

$$\begin{cases} \frac{\partial u}{\partial t} + \sum\limits_{|r|=2m} a_r(x) \frac{\partial^{|r|} u}{\partial x_1^{r_1} ... \partial x_n^{r_n}} + \delta u(t,x) = f(t,x), \\ \\ 0 < t < T, \ x, r \in \mathbb{R}^n, |r| = r_1 + \cdots + r_n, \\ \\ u(0,x) = \varphi(x), x \in \mathbb{R}^n, \end{cases} \tag{6}$$

where $a_r(x)$ and $f(t,x)$, $\varphi(x)$ are given sufficiently smooth functions and $\delta > 0$ is the sufficiently large number.The problem (6) has a unique smooth solution. This allows us to reduce the problem (6) to the abstract Cauchy problem

$$\frac{du(t)}{dt} + Au(t) = f(t), 0 < t < T, u(0) = \varphi \tag{7}$$

in a Banach space $E = C^{\mu}(\mathbb{R}^n)$ with a strongly positive operator $A = B^x + \delta I$ defined by (5).

Theorem 2.3. [11],[55] *Let* $0 < 2m\mu < 1$. *Then, for the solution of the Cauchy problem (6) the following coercivity inequalities are satisfied:*

$$\max_{0 \le t \le T} \left\| \frac{\partial u}{\partial t} \right\|_{C^{2m\mu}(\mathbb{R}^n)} + \sum_{|r|=2m} \max_{0 \le t \le T} \left\| \frac{\partial^{|r|} u}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} \right\|_{C^{2m\mu}(\mathbb{R}^n)}$$

$$\le M(\mu) \left[\max_{0 \le t \le T} \|f\|_{C^{2m\mu}(\mathbb{R}^n)} + \sum_{|r|=2m} \left\| \frac{\partial^{|r|} \varphi}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} \right\|_{C^{2m\mu}(\mathbb{R}^n)} \right],$$

$$\left(\int_0^T \left\| \frac{\partial u}{\partial t} \right\|^p_{W_p^{2m\mu}(\mathbb{R}^n)} dt \right)^{\frac{1}{p}} + \sum_{|r|=2m} \left(\int_0^T \left\| \frac{\partial^{|r|} u}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} \right\|^p_{W_p^{2m\mu}(\mathbb{R}^n)} dt \right)^{\frac{1}{p}}$$

$$\le M(\mu) \left[\left(\int_0^T \|f\|^p_{W_p^{2m\mu}(\mathbb{R}^n)} dt \right)^{\frac{1}{p}} + \sum_{|r|=2m} \left(\int_0^T \left\| \frac{\partial^{|r|} \varphi}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} \right\|^p_{W_p^{2m\mu}(\mathbb{R}^n)} dt \right)^{\frac{1}{p}} \right], 1 \le p < \infty.$$

The proof of Theorem 2.3 is based on Theorem 2.1 and Theorem 2.2 on the structure of fractional spaces $E_{\alpha}(C^{\mu}(\mathbb{R}^n), A)$ and $E_{\alpha,p}(L_p(\mathbb{R}^n), A)$, on the strongly positivity of the operator A in $C^{\mu}(\mathbb{R}^n)$ and $W_p^{\mu}(\mathbb{R}^n)$, on following theorems on coercive stability of elliptic problem and the abstract Cauchy problem for the abstract parabolic equation (7).

Theorem 2.4. [11],[55] *Let* $0 < 2m\mu < 1$. *Then, for the solution of elliptic problem*

$$\sum_{|r|=2m} a_r(x) \frac{\partial^{|r|} v}{\partial x_1^{r_1} ... \partial x_n^{r_n}} + \delta v(x) = g(x), \ x \in \mathbb{R}^n \tag{8}$$

the following coercive inequalities hold:

$$\sum_{|r|=2m} \left\| \frac{\partial^{|r|} v}{\partial x_1^{r_1} ... \partial x_n^{r_n}} \right\|_{C^{2m\mu}(\mathbb{R}^n)} \le M(\mu) \|g\|_{C^{2m\mu}(\mathbb{R}^n)},$$

$$\sum_{|r|=2m} \left\| \frac{\partial^{|r|} v}{\partial x_1^{r_1} ... \partial x_n^{r_n}} \right\|_{W_p^{2m\mu}(\mathbb{R}^n)} \le M(\mu) \|g\|_{W_p^{2m\mu}(\mathbb{R}^n)}, 1 \le p < \infty.$$

Theorem 2.5. [58], [48], [49] *Let A be a strongly positive operator in a Banach space E. Then, for the solution of the abstract Cauchy problem (7) the following coercive inequalities hold:*

$$\max_{0\le t\le T}\|u'(t)\|_{E_\alpha}+\max_{0\le t\le T}\|Au(t)\|_{E_\alpha}\le M\left[\|A\varphi\|_{E_\alpha}+\frac{M}{\alpha(1-\alpha)}\max_{0\le t\le T}\|f(t)\|_{E_\alpha}\right],$$

$$\left(\int_0^T\left\|u'(t)\right\|^p_{E_{\alpha,p}}dt\right)^{\frac{1}{p}}+\left(\int_0^T\|Au(t)\|^p_{E_{\alpha,p}}dt\right)^{\frac{1}{p}}$$

$$\le M\left[\|A\varphi\|_{E_{\alpha,p}}+\frac{M}{\alpha(1-\alpha)}\left(\int_0^T\|f(t)\|^p_{E_{\alpha,p}}dt\right)^{\frac{1}{p}}\right].$$

In this paper we do not discuss results on the well-posedness in Holder spaces in t of the local and nonlocal boundary-value problems for parabolic equations, for which the reader is referred to the papers [10-12, 15].

In [50]-[54], Yu. A. Simirnitskii and P.E. Sobolevskii considered the difference operator A_h which is an elliptic difference operator of an arbitrary high order of accuracy approximating the multidimensional elliptic operator $A = B^x + \delta I$. Let us define the grid space $\mathbb{R}^n_h$ $(0 < h \le h_0)$ as the set of all points of the Euclidean space $\mathbb{R}^n$ whose coordinates are given by

$$x_k = s_k h, \qquad s_k = 0, \pm1, \pm2, \cdots,\ k = 1, \cdots, n.$$

The number h is called the step of the grid space. A function defined on $\mathbb{R}^n_h$ will be called a grid function. To the differential operator B with constant coefficients of the form

$$B=\sum_{|r|=2m} b_r\frac{\partial^{r_1+\cdots+r_n}}{\partial_{x_1^{r_1}}\cdots\partial_{x_n^{r_n}}},$$

we assign the difference operator

$$B^x_h = h^{-m}\sum_{2m\le|s|\le S} d_s\Delta^{s_1}_{1-}\Delta^{s_2}_{1+}\cdots\Delta^{s_{2n-1}}_{n-}\Delta^{s_{2n}}_{n+}, \tag{9}$$

which acts on functions defined on the entire space $\mathbb{R}^n_h$. Here $s \in \mathbb{R}^{2n}$ is a vector with nonnegative integer coordinates,

$$\Delta_{k\pm}f^h(x) = \pm\left(f^h(x\pm e_k h) - f^h(x)\right),$$

and e_k is the unit vector of the axis x_k.

An infinitely differentiable function of the continuous argument $y \in \mathbb{R}^n$ that is continuous and bounded together with all its derivatives is said to be smooth. Let $\varphi(y)$ be a smooth function on $\mathbb{R}^n$. Using the Taylor expansion of $\varphi(y)$, one can show that

$$\sup_{x\in\mathbb{R}^n_h}\left|h^{-1}\Delta_{k\pm}\varphi(x)-\frac{\partial}{\partial y_k}\varphi(x)\right|\le M(\varphi)\,h.$$

Here the grid function $\varphi(x)$ and $\frac{\partial}{\partial y_k}\varphi(x)$ are the traces of the functions $\varphi(y)$ and $\frac{\partial}{\partial_{yk}}\varphi(y)$, respectively. The last inequality means that the difference operator $h^{-1}\Delta_{k\pm}$ is a first-order approximation for the differential operator $\frac{\partial}{\partial y_k}$.

We say that the difference operator B^x_h is a λ-th order $(\lambda > 0)$ approximation of the differential operator B^x if the inequality

$$\sup_{x\in R^n_h}|B^x_h\varphi(x)-B^x\varphi(x)|\le M(\varphi)\,h^\lambda$$

holds for any smooth function $\varphi(y)$. We shall assume that the operator B_h^x approximates the differential operator B^x with any prescribed order.

A function of a continuous [resp., discrete] argument that decays at infinity faster than any negative power of $|y|$ [resp., $|x|$] is said to be rapidly decreasing. Let us define the Fourier transform of a grid function $f^h(x)$ by the formula

$$\tilde{f}(\xi) = (2\pi)^{-n} \sum_{x \in R_h^n} \exp\{-i(x,\xi)\} f^h(x) h^n, \xi \in \mathbb{R}^n. \tag{10}$$

This formula defines a $2\pi h^{-1}$-periodic smooth function of the continuous argument ξ whenever $f^h(x)$ is a rapidly decreasing grid function. In this last case (10) is just Fourier series expansion of the function $\tilde{f}(\xi)$ and the numbers $f^h(x)$ are the Fourier coefficients, given by the formula

$$f^h(x) = \int_{|\xi_1| \le \pi h^{-1}} \cdots \int_{|\xi_n| \le \pi h^{-1}} \exp\{i(x,\xi)\} \tilde{f}(\xi) d\xi_1 \cdots d\xi_n. \tag{11}$$

The inverse Fourier transform of a $2\pi h^{-1}$ periodic function $\varphi(\xi)$ is defined to be the grid function $\tilde{\varphi}^h(x)$ given by the formula

$$\tilde{\varphi}^h(x) = \int_{|\xi_1| \le \pi h^{-1}} \cdots \int_{|\xi_n| \le \pi h^{-1}} \exp\{i(x,\xi)\} \varphi(\xi) d\xi_1 \cdots d\xi_n. \tag{12}$$

Formulas (11) and (12) establish a one-to-one correspondence between rapidly decreasing grid functions of a continuous argument. In particular, if $f^h(x)$ is a rapidly decreasing grid function, then

$$\widehat{\left[\tilde{f}\right]}^h(x) = f^h(x).$$

If $f^h(x)$ is a rapidly decreasing grid function, then the grid function $B_h^x f^h(x)$ exists and is given by (10) and we have the equality

$$\widetilde{B^x}_h f(\xi) = B(\xi h, h) \tilde{f}(\xi).$$

The function $B(\xi h, h)$ is obtained by replacing the operator $\Delta_{k\pm}$ in the right-hand side of equality (9) with the expression $\pm(\exp\{\pm i\xi_k h\} - 1)$, respectively and is called the symbol of the difference operator. Since $\exp\{\pm i\xi_k h\}$ is bounded analytic $2\pi h^{-1}$periodic function, the symbol $B(\xi h, h)$ is a bounded analytic $2\pi h^{-1}$-periodic function. It follows that for large $|\xi|$ one has the estimate

$$|B(\xi h, h)| \le M(h) |\xi|^m, |\xi|^2 = |\xi_1|^2 + ... + |\xi_n|^2.$$

Let us give the difference operator A_h^x by the formula

$$A_h^x u^h(x) = \sum_{2m \le |r| \le S} a_r^x D_h^r u^h(x) + \delta u^h(x). \tag{13}$$

The coefficients are chosen in such a way that the operator A_h^x approximates in a specified way the operator

$$\sum_{|r|=2m} a_r(x) \frac{\partial^{|r|}}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}} + \delta.$$

We shall assume that for $|\xi_k h| \le \pi$ and fixed x the symbol $A^x(\xi h, h)$ of the operator $A_h^x - \delta$ satisfies the inequalities

$$(-1)^m A^x(\xi h, h) \ge M_1 |\xi|^{2m}, |\arg A^x(\xi h, h)| \le \phi < \phi_0 \le \frac{\pi}{2}.$$

We will introduce the space $C_h^\beta = C^\beta(\mathbb{R}_h^n)$, $0 \leq \beta \leq 1$ of all bounded grid functions $u^h(x)$ defined on $\mathbb{R}_h^n$, equipped with the norm

$$||u^h||_{C_h^\beta} = ||u^h||_{C_h} + \sup_{x,y\varepsilon\mathbb{R}_h^n, x\neq y} \frac{|u^h(x) - u^h(y)|}{|x-y|^\beta},$$

where $C_h = C(\mathbb{R}_h^n)$ denotes the Banach space of bounded grid functions $u^h(x)$ defined on $\mathbb{R}_h^n$, equipped with the norm

$$||u^h||_{C_h} = \sup_{x\varepsilon\mathbb{R}_h^n} |u^h(x)|.$$

Next, we will introduce the space $W_{p,h}^\beta = W_p^\beta(\mathbb{R}_h^n)$, $0 \leq \beta \leq 1$, $1 \leq p < \infty$ of all bounded grid functions $u^h(x)$ defined on $\mathbb{R}_h^n$, equipped with the norm

$$||u^h||_{W_{p,h}^\beta} = \left[\sum_{x\varepsilon R_h^n} \sum_{y\varepsilon R_h^n, y\neq 0} \frac{|u^h(x) - u^h(x+y)|^p}{|y|^{n+\beta p}} h^{2n} + ||u^h||_{L_{p,h}}^p \right]^{\frac{1}{p}}.$$

Here $L_{p,h} = L_p(\mathbb{R}_h^n)$ denotes the Banach space of bounded grid functions $u^h(x)$ defined on $\mathbb{R}_h^n$, equipped with the norm

$$||u^h||_{L_{p,h}} = \left[\sum_{x\varepsilon R_h^n} |u^h(x)|^p h^n \right]^{\frac{1}{p}}.$$

Theorem 2.6. [54] *An elliptic difference operator* $A_h^x = B_h^x + \delta I_h$ *is the strongly positive operator in Banach spaces* C_h *and* $L_{p,h}, 1 \leq p < \infty$.

Theorem 2.7. [11], [34] $E_\alpha(C_h, A_h^x) = C_h^{2m\alpha}$ *for all* $0 < 2m\alpha < 1$.

This fact follows from the equality $D(A_h^x) = C_h^{2m+\mu}$ *for an* $2m$*-th order elliptic difference operator* A_h^x *in* $C_h^\mu, 0 < \mu < 1$, *via the real interpolation method.*

Theorem 2.8. [11], [34] $E_{\alpha,p}(L_{p,h}, A_h^x) = W_{p,h}^{2m\alpha}$ *for all* $0 < 2m\alpha < 1, 1 \leq p < \infty$.

This fact follows from the equality $D(A_h^x) = W_{p,h}^{2m}$ for an $2m$-th order elliptic difference operator A_h^x in $L_{p,h}$, $1 < p < \infty$, via the real interpolation method. The alternative method of investigation adopted in [11], [12], based on estimates of fundamental solution of the resolvent equation for the elliptic difference operator A_h^x, allows us to consider also the cases $p = 1$ and $p = \infty$.

From the strong positivity of an elliptic operator $A_h^x = B_h^x + \delta I_h$ in Banach spaces C_h and $L_{p,h}, 1 \leq p < \infty$ and estimate (4) it follows the strong positivity of this operator in Banach spaces C_h^μ and $W_{p,h}^{2m\alpha}$.

In applications, we consider the implicit Rothe difference scheme for the approximate solution of Cauchy problem (6). The discretization of problem (6) is carried out in two steps. In the first step let us give the difference operator A_h^x by the formula (13). With the help of A_h^x we arrive at the initial value problem

$$\frac{du^h(t,x)}{dt} + A_h^x u^h(t,x) = f^h(t,x), 0 < t < T, u^h(0,x) = \varphi^h(x), x \in \mathbb{R}_h^n, \tag{14}$$

for an infinite system of ordinary differential equations.

In the second step we replace problem (14) by the implicit Rothe difference scheme

$$\begin{cases} \frac{1}{\tau}(u_k^h(x) - u_{k-1}^h(x)) + A_h^x u_k^h(x) = f_k^h(x), f_k^h(x) = f^h(t_k, x), t_k = k\tau, 1 \leq k \leq N, \\ N\tau = T, u_0^h(x) = \varphi^h(x), \ x \in \mathbb{R}_h^n. \end{cases} \tag{15}$$

Theorem 2.9. [34] *The solution of difference schemes (15) satisfies the following stability estimates:*

$$\max_{1\le k\le N}\left\|u_k^h\right\|_{C_h}\le M\left[||\varphi^h||_{C_h}+\max_{1\le k\le N}\left\|f_k^h\right\|_{C_h}\right],$$

$$\max_{1\le k\le N}\left\|u_k^h\right\|_{L_{p,h}}\le M\left[||\varphi^h||_{L_{p,h}}+\max_{1\le k\le N}\left\|f_k^h\right\|_{L_{p,h}}\right],1\le p<\infty.$$

The proof of Theorem 2.9 is based on Theorem 2.6 on a strong positivity of an elliptic difference operator $A_h^x=B_h^x+\delta I_h$ in Banach spaces C_h and $L_{p,h}, 1\le p<\infty$ and on the following abstract theorem on the stability of the difference scheme

$$\frac{1}{\tau}\left(u_k-u_{k-1}\right)+Au_k=f_k, f_k=f(t_k), t_k=k\tau, 1\le k\le N, N\tau=T, u_0=\varphi \tag{16}$$

for the approximate solution of the abstract Cauchy problem (7).

Theorem 2.10. [34] *Let A be a strongly positive operator in a Banach space E. Then, for the solution of difference scheme (16) the following stability inequality holds:*

$$\max_{1\le k\le N}\|u_k\|_E\le M\left[||\varphi||_E+\max_{1\le k\le N}\|f_k\|_E\right].$$

Theorem 2.11. [34] *The solution of difference scheme (15) satisfies the following almost coercive stability estimates:*

$$\max_{1\le k\le N}\left\|\frac{1}{\tau}(u_k^h-u_{k-1}^h)\right\|_{C_h}+\max_{1\le k\le N}\sum_{|r|=2m}\left\|D_h^r u_k^h\right\|_{C_h}$$

$$\le M\left[\ln\frac{1}{h}\sum_{|r|=2m}||D_h^r\varphi^h||_{C_h}+\ln\frac{1}{\tau+h}\max_{1\le k\le N}\left\|f_k^h\right\|_{C_h}\right],$$

$$\max_{1\le k\le N}\left\|\frac{1}{\tau}(u_k^h-u_{k-1}^h)\right\|_{L_{p,h}}+\max_{1\le k\le N}\sum_{|r|=2m}\left\|D_h^r u_k^h\right\|_{L_{p,h}}$$

$$\le M\left[\ln\frac{1}{h}\sum_{|r|=2m}||D_h^r\varphi^h||_{L_{p,h}}+\ln\frac{1}{\tau+h}\max_{1\le k\le N}\left\|f_k^h\right\|_{L_{p,h}}\right],1\le p<\infty.$$

The proof of Theorem 2.11 is based on Theorem 2.6 on a strong positivity of an elliptic difference operator $A_h^x=B_h^x+\delta I_h$ in Banach spaces C_h and $L_{p,h}, 1\le p<\infty$ and on the estimate

$$\min\left\{\ln\frac{1}{\tau},1+\left|\ln\|A_h^x\|_{C_h\to C_h}\right|\right\}\le M\ln\frac{1}{\tau+h}$$

and on the following theorems on almost coercive stability of the elliptic difference equation and on almost coercive stability of difference scheme (16).

Theorem 2.12. [35] *For the solution of elliptic difference problem*

$$\sum_{2m\le|r|\le S}a_r^x D_h^r u^h(x)+\delta u^h(x)=g^h(x),\ x\in\mathbb{R}_h^n \tag{17}$$

the following almost coercive stability inequalities hold:

$$\sum_{2m\le|r|\le S}\left\|D_h^r u^h\right\|_{C_h}\le M\ln\frac{1}{h}\|g^h\|_{C_h},$$

$$\sum_{2m\leq|r|\leq S}\left\|D_h^r u^h\right\|_{L_{p,h}} \leq M\ln\frac{1}{h}\|g^h\|_{L_{p,h}}, 1\leq p<\infty.$$

Theorem 2.13. [60], *Let A be a strongly positive operator in a Banach space E. Then, for the solution of difference scheme (16) the following almost coercive stability inequality holds:*

$$\max_{1\leq k\leq N}\left\|\frac{1}{\tau}(u_k-u_{k-1})\right\|_E+\max_{1\leq k\leq N}\|Au_k\|_E$$

$$\leq M\left[||A\varphi||_E+\min\left\{\ln\frac{1}{\tau}, 1+|\ln\|A\|_{E\to E}|\right\}\max_{1\leq k\leq N}\|f_k\|_E\right].$$

Theorem 2.14. [60] *Let $0<2m\mu<1$. Then, the solution of difference scheme (15) satisfies the following coercive stability estimates:*

$$\max_{1\leq k\leq N}\left\|\frac{1}{\tau}(u_k^h-u_{k-1}^h)\right\|_{C_h^{2m\mu}}+\max_{1\leq k\leq N}\sum_{|r|=2m}\left\|D_h^r u_k^h\right\|_{C_h^{2m\mu}}$$

$$\leq M(\mu)\left[\sum_{|r|=2m}||D_h^r\varphi^h||_{C_h^{2m\mu}}+\max_{1\leq k\leq N}\left\|f_k^h\right\|_{C_h^{2m\mu}}\right],$$

$$\left[\sum_{k=1}^N\left\|\frac{1}{\tau}(u_k^h-u_{k-1}^h)\right\|^p_{W_{p,h}^{2m\mu}}\tau\right]^{\frac{1}{p}}+\left[\sum_{|r|=2m}\sum_{k=1}^N\left\|D_h^r u_k^h\right\|^p_{W_{p,h}^{2m\mu}}\tau\right]^{\frac{1}{p}}$$

$$\leq M(\mu)\left[\sum_{|r|=2m}||D_h^r\varphi^h||_{W_{p,h}^{2m\mu}}+\left[\sum_{k=1}^N\left\|f_k^h\right\|^p_{W_{p,h}^{2m\mu}}\tau\right]^{\frac{1}{p}}\right], 1\leq p<\infty.$$

The proof of Theorem 2.14 is based on Theorem 2.6 on a strong positivity of an elliptic difference operator $A_h^x = B_h^x+\delta I_h$ in Banach spaces C_h and $L_{p,h}, 1\leq p<\infty$ and on the following theorems on coercive stability of the elliptic difference equation (17) and on coercive stability of difference scheme (16).

Theorem 2.15. [35] *Let $0<2m\mu<1$. Then, for the solution of elliptic difference equation (17) the following coercive stability inequalities hold:*

$$\sum_{2m\leq|r|\leq S}\left\|D_h^r u^h\right\|_{C_h^{2m\mu}}\leq M(\mu)\|g^h\|_{C_h^{2m\mu}},$$

$$\sum_{2m\leq|r|\leq S}\left\|D_h^r u^h\right\|_{W_{p,h}^{2m\mu}}\leq M(\mu)\|g^h\|_{W_{p,h}^{2m\mu}}, 1\leq p<\infty.$$

Theorem 2.16. [60] *Let A be a strongly positive operator in a Banach space E. Then, for the solution of difference scheme (16) the following coercive stability inequalities hold:*

$$\max_{1\leq k\leq N}\left\|\frac{1}{\tau}(u_k-u_{k-1})\right\|_{E_\mu}+\max_{1\leq k\leq N}\|Au_k\|_{E\mu}$$

$$\leq M(\mu)\left[||A\varphi||_{E_\mu}+\max_{1\leq k\leq N}\|f_k\|_{E_\mu}\right],$$

$$\left[\sum_{k=1}^N\left\|\frac{1}{\tau}(u_k-u_{k-1})\right\|^p_{E_{\mu,p}}\tau\right]^{\frac{1}{p}}+\left[\sum_{k=1}^N\|Au_k\|^p_{E_{\mu,p}}\tau\right]^{\frac{1}{p}}$$

$$\leq M(\mu)\left[||A\varphi||_{E_{\mu,p}}+\left[\sum_{k=1}^{N}\|f_k\|_{E_{\mu,p}}^{p}\tau\right]^{\frac{1}{p}}\right].$$

Note that the positivity of differential and difference operators in the space and structure of fractional spaces generated by these positive operators were well investigated. We have given only simple applications of these results to well-posedness of partial differential and difference equations. For more details see [11] and [12].

3. Positive operators in the half-space. Fractional spaces generated by differential and difference operators in the half-space

In [41]-[44], S.I. Danelich considered the difference elliptic operator A_h^x which is an arbitrary high order of accuracy approximating the multi dimensional elliptic operator A^x defined by

$$A^x=(-1)^m a(x)\frac{\partial^{2m}}{\partial x_{n+1}^{2m}}+\sum_{|r|=2m}a_r(x)\frac{\partial^{|r|}}{\partial x_1^{r_1}\cdots\partial x_n^{r_n}}+\delta I$$

with the domain

$$D(A^x)=\left\{u:\frac{\partial^{2m}u(x_{n+1},x)}{\partial x_{n+1}^{2m}},\frac{\partial^{|r|}u(x_{n+1},x)}{\partial x_1^{r_1}\cdots\partial x_n^{r_n}}\in C(\mathbb{R}^+\times\mathbb{R}^n),|r|=r_1+...+r_n=2m,\right.$$

$$\left.u(0,x)=0,\frac{\partial u(0,x)}{\partial x_{n+1}}=0,\cdots,\frac{\partial^{m-1}u(0,x)}{\partial x_{n+1}^{m-1}}=0,x\in\mathbb{R}^n,\mathbb{R}^+=[0,\infty).\right\}$$

Here, $a(x)$ is a smooth function defined on $\mathbb{R}^n$ with $a(x)\geq a>0$. She proved the strong positivity of A_h^x in the Banach space $C_h=C(\mathbb{R}_h^+\times\mathbb{R}_h^n)$ (difference analogue of $C(\mathbb{R}^+\times\mathbb{R}^n)$) for sufficiently large positive δ. Passing to limit when $h\to 0$, we can get the strong positivity of differential operator A^x in the Banach space $C(\mathbb{R}^+\times\mathbb{R}^n)$.

In [22], the two-dimensional elliptic differential operator $A^x$with dependent coefficients on the half-space $\mathbb{R}^+\times\mathbb{R}^1$

$$A^x u(x)=-a_{11}(x)u_{x_1x_1}(x)-a_{22}(x)u_{x_2x_2}(x)+\sigma u(x),x=(x_1,x_2)\in\mathbb{R}^+\times\mathbb{R}^1 \tag{18}$$

with the domain

$$D(A^x)=\left\{u:\frac{\partial^2 u(x)}{\partial x_1^2},\frac{\partial^2 u(x)}{\partial x_1^2}\in C(\mathbb{R}^+\times\mathbb{R}^1),u(0,x_2)=0,x_2\in\mathbb{R}^1.\right\}$$

Here, the coefficients $a_{ii}(x)$, $i=1,2$ are continuously differentiable and satisfy the uniform ellipticity

$$a_{11}^2(x)+a_{22}^2(x)\geq\delta>0, \tag{19}$$

and $\sigma>0$.

The Green function of differential operator A^x defined by (18) was constructed. Moreover, applying Green's function of A^x the following results were proved.

Theorem 3.1. [22] A^x *is the positive operator in the Banach space* $C\left(\mathbb{R}^+\times\mathbb{R}\right)$ *of all continuous bounded functions* $\varphi(x)$ *defined on* $\mathbb{R}^+\times\mathbb{R}$ *with the norm*

$$\|\varphi\|_{C(\mathbb{R}^+\times\mathbb{R})}=\sup_{x\in\mathbb{R}^+\times\mathbb{R}}|\varphi(x)|.$$

Theorem 3.2. [22] A^x *is the strongly positive operator in the space* $C^{\beta}(\mathbb{R}^+\times\mathbb{R})$.*Here* $C^{\beta}(\mathbb{R}^+\times\mathbb{R})$ *be the Hölder space of all continuous bounded functions* φ *defined on* $\mathbb{R}^+\times\mathbb{R}$ *satisfying a Hölder condition with the indicator* $\beta\in(0,1)$ *with the norm*

$$\|f\|_{C^{\beta}(\mathbb{R}^+\times\mathbb{R})}=\|f\|_{C(\mathbb{R}^+\times\mathbb{R})}+\sup_{\substack{x,x'\in\mathbb{R}^+\times\mathbb{R},\\ x\neq x'}}\frac{|f(x)-f(x')|}{|x-x'|^{\beta}}.$$

Theorem 3.3. [22] *Suppose* $\beta, 2\alpha+\beta\in(0,1)$. *Then, the norms of the spaces* $E_\alpha(A, C^{\beta}(\mathbb{R}^+\times\mathbb{R}))$ *and* $C^{2\alpha+\beta}(\mathbb{R}^+\times\mathbb{R})$ *are equivalent.*

In applications, we will consider the boundary value problem for the elliptic equation

$$\begin{cases} -\frac{\partial^2u(t,x)}{\partial t^2}-a_{11}(x)\frac{\partial^2u(t,x)}{\partial x_1^2}-a_{22}(x)\frac{\partial^2u(t,x)}{\partial x_2^2}+\sigma u(t,x)\\ =f(t,x),\ 0<t<T,\ x\in\mathbb{R}^+\times\mathbb{R}^1,\\ u(0,x)=\varphi(x),\ u(T,x)=\psi(x),\ \ x\in\mathbb{R}^+\times\mathbb{R}^1,\\ u(t,0,x_2)=0,\ 0\le t\le T,\ x_2\in\mathbb{R}^1. \end{cases}\tag{20}$$

Here, $a_{11}(x)$, $a_{22}(x)$, $\varphi(x)$, *and* $f(t,x)$ *are sufficiently smooth functions. Assume that the assumption of the uniform ellipticity holds.*

Theorem 3.4. [22] *For the solution of boundary value problem (20), we have the following estimate*

$$\|u_{tt}\|_{C(C^{2\alpha+\beta}(\mathbb{R}^+\times\mathbb{R}))}+\|u_{x_1x_1}\|_{C(C^{2\alpha+\beta}(\mathbb{R}^+\times\mathbb{R}))}+\|u_{x_2x_2}\|_{C(C^{2\alpha+\beta}(\mathbb{R}^+\times\mathbb{R}))}$$

$$\le\ M(\alpha,\beta)\Big[\|\varphi_{x_1x_1}\|_{C^{2\alpha+\beta}(\mathbb{R}^+\times\mathbb{R})}+\|\varphi_{x_2x_2}\|_{C^{2\alpha+\beta}(\mathbb{R}^+\times\mathbb{R})}+\|\psi_{x_1x_1}\|_{C^{2\alpha+\beta}(\mathbb{R}^+\times\mathbb{R})}$$

$$+\|\psi_{x_2x_2}\|_{C^{2\alpha+\beta}(\mathbb{R}^+\times\mathbb{R})}+\|f\|_{C(C^{2\alpha+\beta}(\mathbb{R}^+\times\mathbb{R}))}\Big].$$

The proof of Theorem 3.4 is based on Theorem 3.3 on the structure of the fractional spaces $E_\alpha(A^x, C^{\beta}(\mathbb{R}^+\times\mathbb{R}))$, Theorem 3.2 on the positivity of the operator A^x, on the following theorems on coercive stability of elliptic problems, nonlocal boundary value for the abstract elliptic equation and on the structure of the fractional space $E'_\alpha=E_\alpha(A^{1/2},E)$ which is the Banach space consists of those $v\in E$ for which the norm

$$||v||_{E'_\alpha}=\sup_{\lambda>0}\lambda^{\alpha}\left\|A^{1/2}\left(\lambda+A^{1/2}\right)^{-1}v\right\|_E+||v||_E$$

is finite.

Theorem 3.5. [22] *Under assumption (19) for the solution of elliptic problem*

$$\begin{cases} a_{11}(x)\frac{\partial^2u(t,x)}{\partial x_1^2}+a_{22}(x)\frac{\partial^2u(t,x)}{\partial x_2^2}-\sigma u(x)=g(x),\ x\in\mathbb{R}^+\times\mathbb{R}^1,\\ u(0,x_2)=0,\ \ x_2\in\mathbb{R}^1 \end{cases}$$

the following coercive inequality holds

$$\left\|\frac{\partial^2u}{\partial x_1^2}\right\|_{C^{\mu}(\mathbb{R}^+\times\mathbb{R})}+\left\|\frac{\partial^2u}{\partial x_2^2}\right\|_{C^{\mu}(\mathbb{R}^+\times\mathbb{R})}\le M(\mu)\|g\|_{C^{\mu}(\mathbb{R}^+\times\mathbb{R})}.$$

The proof of Theorem 3.5 uses the techniques introduced in [12] and it is based on estimates for the Green's function of operator A^x defined by (18).

Theorem 3.6. [33] *The spaces* $E_\alpha(A,E)$ *and* $E'_{2\alpha}(A^{1/2},E)$ *coincide for any* $0<\alpha<\frac{1}{2}$, *and their norms are equivalent.*

Theorem 3.7. [38] *Let* A *be positive operator in a Banach space* E *and* $f\in C([0,T],E'_\alpha)$ $(0<\alpha<1)$. *Then, for the solution of the nonlocal boundary value problem*

$$\begin{cases} -u''(t)+Au(t)=f(t),\ 0<t<T, \\ \\ u(0)=\varphi,\ u(T)=\psi \end{cases}$$

in a Banach space E *with positive operator* A *the coercive inequality*

$$\|u''\|_{C([0,T],E'_\alpha)}+\|Au\|_{C([0,T],E'_\alpha)}$$
$$\leq M\left[\|A\varphi\|_{E'_\alpha}+\|A\psi\|_{E'_\alpha}+\frac{M}{\alpha(1-\alpha)}\|f\|_{C([0,T],E'_\alpha)}\right]$$

holds.

Note that the positivity of differential and difference operators in the half-space and the structure of fractional spaces generated by these positive operators were well investigated. For more details see [5], [23] and [43].

4. Positive differential and difference operators with local boundary conditions

In [6-8], Kh. A. Alibekov and P.E. Sobolevskii considered the simple difference operator A_h^x which is an elliptic difference operator of second order of accuracy approximating the simple multidimensional elliptic differential operator A defined by

$$A^x u=-\sum_{r=1}^{n}\alpha_r(x)\frac{\partial^2 u(x)}{\partial x^2} \tag{21}$$

acting on functions Ω satisfying the condition $u=0$ on S, where $\Omega\subset\mathbb{R}^n$ is the open unit cube with boundary S. They proved the strong positivity of A_h^x in the Banach spaces $L_p(\overline{\Omega}_h)$ and $C(\overline{\Omega}_h)$ (difference analogue of $L_p(\overline{\Omega})$ and $C(\overline{\Omega})$). Passing to limit when $h\to 0$, we can get the strong positivity of differential operator A^x in Banach spaces $L_p(\overline{\Omega})$ and $C(\overline{\Omega})$. At first time in [6], P.E. Sobolevskii proved the strong positivity of A_h^x in difference analogue of Hölder spaces $C_{01}^{\beta}(\overline{\Omega})$ with weight on boundary.

We consider the differential operator A^x defined by the formula

$$A^x u=-a(x)\frac{d^2u}{dx^2}+\delta u \tag{22}$$

with domain $D(A^x)=\left\{u\in C^{(2)}[0,l]:u(0)=u(l)=0\right\}$. Let $a(x)$ be the smooth function defined on the segment $[0,l]$ and $a(x)\geq a>0$.The pointwise estimates for the Green function of differential operator A^x defined by (22) were obtained. Moreover, applying these estimates of Green's function of A^x the following results were proved.

Theorem 4.1. [12] A^x *is the positive operator in the Banach space* $C[0,l]$ *of all continuous functions* $\varphi(x)$ *defined on* $[0,l]$ *with the norm*

$$\|\varphi\|_{C[0,l]}=\max_{x\in[0,l]}|\varphi(x)|.$$

Let $C^{\beta}[0,l]$ be the Hölder space of all continuous functions $\varphi(x)$ defined on $[0,l]$ satisfying a Hölder condition with the indicator $\beta \in (0,1)$ with the norm

$$\|f\|_{C^{\beta}[0,l]} = \|f\|_{C[0,l]} + \sup_{\substack{x, x' \in [0,l], \\ x \neq x'}} \frac{|f(x) - f(x')|}{|x - x'|^{\beta}}.$$

Theorem 4.2. [12] *For $\mu \in (0, \frac{1}{2})$, the norms of the space $E_{\mu}(C[0,l], A^x)$ and the Hölder space$\overset{\cdot}{C}^{2\mu}[0,l]$ are equivalent. Here $\overset{\cdot}{C}^{2\mu}[0,l] = \left\{\varphi(x) : \varphi(x) \in C^{2\mu}[0,l], \varphi(0) = \varphi(l) = 0\right\}$.*

Theorem 4.3. [12] *A^x is the strongly positive operator in $\overset{\cdot}{C}^{2\mu}[0,l]$ for $\mu \in (0, \frac{1}{2})$.*

In applications, we consider the initial-boundary value problem for the parabolic equation

$$\begin{cases} \frac{\partial u}{\partial t} - a(x)\frac{\partial^2 u}{\partial x^2} + \delta u(t,x) = f(t,x), 0 < t < T, \ x \in (0,l), \\ u(t,0) = u(t,l) = 0, 0 \leq t \leq T, \\ u(0,x) = \varphi(x), x \in [0,l], \end{cases} \tag{23}$$

where $a(x)$ and $f(t,x)$, $\varphi(x)$ are given sufficiently smooth functions and $\delta > 0$ is the sufficiently large number.The problem (23) has a unique smooth solution. This allows us to reduce the problem (23) to the abstract Cauchy problem (6) in a Banach space $E = C^{\mu}[0,l]$ with a strongly positive operator A^x defined by (22).

Theorem 4.4. [12] *Let $0 < 2m\mu < 1$. Then, for the solution of problem (23) the following coercivity inequality is satisfied:*

$$\max_{0\leq t\leq T} \|u_t(t)\|_{\overset{\cdot}{C}^{2m\mu}[0,l]} + \max_{0\leq t\leq T} \|u(t)\|_{\overset{\cdot}{C}^{2+2m\mu}[0,l]}$$
$$\leq M(\mu)\left[\max_{0\leq t\leq T} \|f\|_{\overset{\cdot}{C}^{2m\mu}[0,l]} + \|\varphi\|_{\overset{\cdot}{C}^{2+2m\mu}[0,l]}\right].$$

The proof of Theorem 4.4 is based on Theorem 4.2 on the structure of the fractional spaces $E_{\alpha}(C^{\mu}[0,l], A^x)$, on Theorem 4.3 on the strongly positivity of the operator A^x in $\overset{\cdot}{C}^{\mu}[0,l]$ and on Theorem 2.5 on coercive stability of the abstract Cauchy problem for the abstract parabolic equation (7).

In [39], M. A. Bazarov considered a second order of approximation of the differential operator A^x defined by formula (22) difference operator A_h^x defined by formula

$$A_h^x u^h = \left\{-a(x_k)\frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} + \delta u_k\right\}_1^{M-1}, \ u^h = \{u_k\}_0^M, Mh = l$$

with $u_0 = u_M = 0$. He proved the following results.

Theorem 4.5. [39] *A_h^x is the strongly positive operator in the space $C_h = C[0,l]_h$ of all mesh functions $\varphi^h(x) = \{\varphi_k\}_0^M$ defined on $[0,l]_h$ with the norm*

$$\left\|\varphi^h\right\|_{C_h} = \max_{0\leq k\leq M} |\varphi_k|.$$

Let $C_h^{\beta} = C^{\beta}[0,l]_h$ be the Hölder space of all mesh functions $\varphi^h(x) = \{\varphi_k\}_0^M$ defined on $[0,l]_h$ satisfying a Hölder condition with the indicator $\beta \in (0,1)$ with the norm

$$\left\|\varphi^h\right\|_{C_h^{\beta}} = \left\|\varphi^h\right\|_{C_h} + \sup_{0\leq k<k+n\leq M} \frac{|\varphi_{k+n} - \varphi_k|}{(nh)^{\beta}}.$$

Theorem 4.6. [39] *For* $\mu \in (0, \frac{1}{2})$, *the norms of the space* $E_\mu(C_h, A_h^x)$ *and the Hölder space* $\overset{\cdot}{C_h}{}^{2\mu}$ *are equivalent uniformly in h. Here* $\overset{\cdot}{C_h}{}^{2\mu} = \left\{ \varphi^h(x) : \varphi^h(x) \in C_h^{2\mu}, \varphi_0 = \varphi_M = 0 \right\}$.

Theorem 4.7. [39] A^x *is the strongly positive operator in* $\overset{\cdot}{C_h}{}^{2\mu}$ *for* $\mu \in (0, \frac{1}{2})$.

In applications, we consider the Crank-Nicholson difference scheme for the approximate solution of problem (23). The discretization of problem (23) is carried out in two steps. In the first step let us give the difference operator A_h^x by formula (22). With the help of A_h^x we arrive at the initial value problem

$$\frac{du^h(t,x)}{dt} + A_h^x u^h(t,x) = f^h(t,x), 0 < t < T, u^h(0,x) = \varphi^h(x), x \in [0,l]_h \,, \tag{24}$$

for an infinite system of ordinary differential equations.

In the second step we replace problem (24) by the Crank-Nicholson difference scheme

$$\begin{cases} \frac{1}{\tau}(u_k^h(x) - u_{k-1}^h(x)) + \frac{1}{2}A_h^x \left[u_k^h(x) + u_{k-1}^h(x)\right] = f_k^h(x), \\ f_k^h(x) = f^h(t_k - \frac{\tau}{2}, x), t_k = k\tau, 1 \le k \le N, \\ N\tau = T, u_0^h(x) = \varphi^h(x), \ x \in [0,l]_h \,. \end{cases} \tag{25}$$

Theorem 4.8. [11] *The solution of difference scheme (25) satisfies the following stability estimate:*

$$\max_{1\le k\le N} \left\| u_k^h \right\|_{C_h^\mu} \le M(\mu) \left[||\varphi^h||_{C_h^\mu} + \max_{1\le k\le N} \left\| f_k^h \right\|_{C_h^\mu} \right].$$

The proof of Theorem 4.8 is based on Theorem 4.5 on a strong positivity of difference operator A_h^x in the Banach space C_h^μ and on the following abstract theorem on stability of the difference scheme

$$\begin{cases} \frac{1}{\tau}\left(u_k - u_{k-1}\right) + \frac{1}{2}A\left[u_k + u_{k-1}\right] = f_k, \\ f_k = f(t_k - \frac{\tau}{2}), t_k = k\tau, 1 \le k \le N, N\tau = T, u_0 = \varphi \end{cases} \tag{26}$$

for the approximate solution of the abstract Cauchy problem (7).

Theorem 4.9. [11] *Let A be a strongly positive operator in a Banach space E. Then, for the solution of difference scheme (26) the following stability inequality holds:*

$$\max_{1\le k\le N} \|u_k\|_{E_\mu} \le M(\mu) \left[||\varphi||_{E_\mu} + \max_{1\le k\le N} \|f_k\|_{E_\mu} \right].$$

Theorem 4.10. [11] *The solution of difference scheme (26) satisfies the following almost coercive stability estimate:*

$$\max_{1\le k\le N} \left\| \frac{1}{\tau}(u_k^h - u_{k-1}^h) \right\|_{C_h} + \max_{1\le k\le N} \left\| \frac{D_h^2\left(u_k^h + u_{k-1}^h\right)}{2} \right\|_{C_h}$$
$$\le M \left[||D_h^2\varphi^h||_{C_h} + \ln\frac{1}{\tau+h} \max_{1\le k\le N} \left\| f_k^h \right\|_{C_h} \right].$$

The proof of Theorem 4.10 is based on Theorem 4.5 on a strong positivity of an elliptic difference operator A_h^x in the Banach space $C\left[0,l\right]_h$ and on the estimate

$$\min\left\{ \ln\frac{1}{\tau}, 1 + \left| \ln \| \ A_h^x \ \|_{C_h \to C_h} \right| \right\} \le M \ln\frac{1}{\tau+h} \tag{27}$$

and on the following theorem on almost coercive stability of difference scheme (26).

Theorem 4.11. [11] *Let A be a strongly positive operator in a Banach space E. Then, for the solution of difference scheme (26) the following almost coercive stability inequality holds:*

$$\max_{1\leq k\leq N}\left\|\frac{1}{\tau}(u_k-u_{k-1})\right\|_E+\max_{1\leq k\leq N}\left\|A\frac{u_k+u_{k-1}}{2}\right\|_E$$
$$\leq M\left[||A\varphi||_E+\min\left\{\ln\frac{1}{\tau},1+|\ln\|A\|_{E\to E}|\right\}\max_{1\leq k\leq N}\|f_k\|_E\right].$$

Theorem 4.12. [11] *Let $0<2m\mu<1$. Then, the solution of difference scheme (26) satisfies the following coercive stability estimate:*

$$\max_{1\leq k\leq N}\left\|\frac{1}{\tau}(u_k^h-u_{k-1}^h)\right\|_{C_h^{2m\mu}}+\max_{1\leq k\leq N}\left\|\frac{D_h^2\left(u_k^h+u_{k-1}^h\right)}{2}\right\|_{C_h^{2m\mu}}$$
$$\leq M(\mu)\left[||D_h^2\varphi^h||_{C_h^{2m\mu}}+\max_{1\leq k\leq N}\left\|f_k^h\right\|_{C_h^{2m\mu}}\right].$$

The proof of Theorem 4.12 is based on Theorem 4.5 on a strong positivity of an elliptic difference operator A_h^x in the Banach space C_h and on the following theorem on coercive stability of difference scheme (26).

Theorem 4.13. [11] *Let A be a strongly positive operator in a Banach space E. Then, for the solution of difference scheme (26) the following coercive stability inequality holds:*

$$\max_{1\leq k\leq N}\left\|\frac{1}{\tau}(u_k-u_{k-1})\right\|_{E_\mu}+\max_{1\leq k\leq N}\left\|A\frac{u_k+u_{k-1}}{2}\right\|_{E\mu}$$
$$\leq M(\mu)\left[||A\varphi||_{E_\mu}+\max_{1\leq k\leq N}\|f_k\|_{E_\mu}\right].$$

Note that the positivity of difference operators which are a high order of approximation of the operator defined by formula (21) is not studied. Nevertheless structure of fractional spaces generated by these positive operators is not well-investigated.

5. Positive differential and difference operators with nonlocal boundary conditions

Finally, we should mention that the positivity of difference operators with nonlocal conditions is investigated only in one-dimensional case. In [19], A. Ashyralyev, I. Karakaya considered the differential operator A^x defined by the formula

$$A^xu=-a(x)\frac{d^2u}{dx^2}+\delta u \tag{28}$$

with domain $D(A^x)=\left\{u\in C^{(2)}[0,l]:u(0)=u(l),u'(0)=u'(l)\right\}$. Let $a(x)$ be the smooth function defined on the segment $[0,l]$ and $a(x)\geq a>0$.

Theorem 5.1. *A^x is the strongly positive operator in $C[0,l]$.*

Theorem 5.2. *For $\alpha\in(0,\frac{1}{2})$, the norms of the space $E_\alpha(C[0,l],A^x)$ and the Hölder space $C^{2\alpha}[0,l]$ are equivalent.*

Theorem 5.3. *A^x is the strongly positive operator in $C^{2\alpha}[0,l]$.*

In [13]-[14], A. Ashyralyev and B. Kendirli considered the difference operator A_h^x defined by formula

$$A_h^x u^h = \left\{ -a(x_k) \frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} + \delta u_k \right\}_1^{M-1}, \ u^h = \{u_k\}_0^M, Mh = l \tag{29}$$

with $u_0 = u_M$, $u_1 - u_0 = u_M - u_{M-1}$. This operator is a first order of approximation of the differential operator A^x defined by formula (28). They proved the following results.

Theorem 5.4. [13] A_h^x *is the strongly positive operator in* C_h.

Theorem 5.5. [14] *For* $\alpha \in (0, \frac{1}{2})$, *the norms of the space* $E_\alpha(C_h, A_h^x)$ *and the Hölder space* $C_h^{2\alpha}$ *are equivalent.*

Theorem 5.6. [14] A_h^x *is the strongly positive operator in* $C_h^{2\alpha}$.

A. Ashyralyev and N. Yenial-Altay considered in [16] the difference operator defined by formula

$$A_h^x u^h = \left\{ -a(x_k) \frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} + \delta u_k \right\}_1^{M-1}, \ u^h = \{u_k\}_0^M, Mh = l \tag{30}$$

with $u_0 = u_M$, $-u_2 + 4u_1 - 3u_0 = u_{M-2} - 4u_{M-1} + 3u_M$. This operator is a second order of approximation of the differential operator A^x defined by formula (28). They proved the following results.

Theorem 5.7. [16] A_h^x *is the strongly positive operator in* C_h.

Theorem 5.8. [16] *For* $\alpha \in (0, \frac{1}{2})$, *the norms of the space* $E_\alpha(C_h, A_h^x)$ *and the Hölder space* $C_h^{2\alpha}$ *are equivalent.*

Theorem 5.9. [16] A_h^x *is the strongly positive operator in* $C_h^{2\alpha}$.

A. Ashyralyev considered in [18] the differential operator defined by (28) and difference operator A_h^x which is a second order approximation of A^x and defined by formula (30). He proved the following results.

Theorem 5.10. [18] A^x *is the strongly positive operator in the space* $L_p[0,l]$, $1 \le p < \infty$ *of the all integrable functions* $\varphi(x)$ *defined on* $[0,l]$ *with the norm*

$$\|\varphi\|_{L_p[0,l]} = \left(\int_0^l |\varphi(x)|^p \, dx \right)^{\frac{1}{p}}.$$

Theorem 5.11. [18] $E_{\alpha,p}(L_p[0,l], A^x) = W_p^{2\alpha}[0,l]$ *for all* $0 < 2\alpha < 1, 1 \le p < \infty$. *Here,* $W_p^\mu[0,l]$ $(0 < \mu < 1)$ *is the Banach space of all integrable functions* $\varphi(x)$ *defined on* $[0,l]$ *and satisfying a Hölder condition for which the following norm is finite:*

$$\|\varphi\|_{W_p^\mu[0,l]} = \left[\int_0^l \int_0^l \frac{|\varphi(x+y) - \varphi(x)|^p}{|y|^{1+\mu p}} dy dx + \|\varphi\|_{L_p[0,l]}^p \right]^{\frac{1}{p}}, 1 \le p < \infty.$$

This fact follows from the equality $D(A^x) = W_p^2[0,l]$ for a second order differential operator A^x in $L_p[0,l]$, $1 < p < \infty$, via the real interpolation method. The alternative method of investigation adopted in [11], [12], based on estimates of fundamental solution of the resolvent equation for the operator A^x, allows us to consider also the cases $p = 1$ and $p = \infty$.

Theorem 5.12. [18] A^x *is the strongly positive operator in the space* $W_p^{2\alpha}[0,l]$ *for all* $0 < 2\alpha < 1, 1 \leq p < \infty$.

In applications, we consider the initial-boundary value problem for the parabolic equation

$$\begin{cases} \frac{\partial u}{\partial t} - a(x)\frac{\partial^2 u}{\partial x^2} + \delta u(t,x) = f(t,x), 0 < t < T,\ x \in (0,l), \\ u(t,0) = u(t,l), u_x(t,0) = u_x(t,l), 0 \leq t \leq T, \\ u(0,x) = \varphi(x), x \in [0,l], \end{cases} \tag{31}$$

where $a(x)$ and $f(t,x)$, $\varphi(x)$ are given sufficiently smooth functions and $\delta > 0$ is the sufficiently large number.The problem (31) has a unique smooth solution. This allows us to reduce the problem (31) to the abstract Cauchy problem (6) in Banach spaces $E = C^\mu[0,l]$ and $L_p[0,l]$, $1 \leq p < \infty$ with a strongly positive operator A^x defined by (28).

Theorem 5.13. [12] *Let* $0 < 2m\mu < 1$. *Then, for the solution of problem (31) the following coercivity inequalities are satisfied:*

$$\max_{0\leq t\leq T}\|u_t(t)\|_{C^{2m\mu}[0,l]} + \max_{0\leq t\leq T}\|u(t)\|_{C^{2+2m\mu}[0,l]}$$

$$\leq M(\mu)\left[\max_{0\leq t\leq T}\|f\|_{C^{2m\mu}[0,l]} + \|\varphi\|_{C^{2+2m\mu}[0,l]}\right],$$

$$\left(\int_0^T \|u_t\|^p_{W_p^{2m\mu}[0,l]}dt\right)^{\frac{1}{p}} + \left(\int_0^T \|u(t)\|^p_{W_p^{2+2m\mu}[0,l]}dt\right)^{\frac{1}{p}}$$

$$\leq M(\mu)\left[\left(\int_0^T \|f\|^p_{W_p^{2m\mu}[0,l]}dt\right)^{\frac{1}{p}} + \left(\int_0^T \|\varphi\|^p_{W_p^{2m\mu}[0,l]}dt\right)^{\frac{1}{p}}\right], 1 \leq p < \infty.$$

The proof of Theorem 5.13 is based on Theorems 5.2 and 5.11 on the structure of the fractional spaces $E_\alpha(C^\mu[0,l], A^x)$ and $E_{\alpha,p}(L_p[0,l], A^x)$, on Theorems 5.3 and 5.12 on the strongly positivity of the operator A^x in $C^\mu[0,l]$ and $W_p^{2m\mu}[0,l]$, on Theorem 2.5 on coercive stability of the abstract Cauchy problem for the abstract parabolic equation (7).

Theorem 5.14. [18] A_h^x *is the strongly positive operator in the space* $L_p = L_{p,h}$, $1 \leq p < \infty$ *of mesh functions* $\varphi^h(x)$ *defined on* $[0,l]_h$ *with the norm*

$$\left\|\varphi^h\right\|_{L_{p,h}} = \left(\sum_{x\in[0,l]_h}\left|\varphi^h(x)\right|^p h\right)^{\frac{1}{p}}.$$

Theorem 5.15. [18] $E_{\alpha,p}(L_{p,h}, A_h^x) = W_{p,h}^{2\alpha}$ *for all* $0 < 2\alpha < 1, 1 \leq p < \infty$. *Here,* $W_{p,h}^\mu = W_p^\mu[0,l]_h$ $(0 < \mu < 1)$ *is the Banach space of all mesh functions* $\varphi^h(x)$ *defined on* $[0,l]_h$ *with the norm:*

$$\left\|\varphi^h\right\|_{W_{p,h}^\mu} = \left[\sum_{x\in[0,l]_h}\sum_{\substack{y\in[0,l]_h\\ y\neq 0}}\frac{\left|\varphi^h(x+y) - \varphi^h(x)\right|^p}{|y|^{1+\mu p}}h^2 + \left\|\varphi^h\right\|^p_{L_{p,h}}\right]^{\frac{1}{p}}, 1 \leq p < \infty.$$

This fact follows from the equality $D(A_h^x) = W_{p,h}^2$ *for a second order differential operator* A_h^x *in* $L_{p,h}$, $1 < p < \infty$, *via the real interpolation method. The alternative method of investigation adopted in* [11], [12], *based on estimates of fundamental solution of the resolvent equation for the operator* A_h^x, *allows us to consider also the cases* $p = 1$ *and* $p = \infty$.

Theorem 5.16. [18] A_h^x *is the strongly positive operator in the space* $W_{p,h}^{2\alpha}$ *for all* $0 < 2\alpha < 1, 1 \leq p < \infty$.

In applications, we consider the Crank-Nicholson difference scheme for the approximate solution of problem (28). The discretization of problem (28) is carried out in two steps. In the first step let us give the difference operator A_h^x by formula (30). With the help of A_h^x we arrive at the initial value problem (24). In the second step we replace problem (24) by the Crank-Nicholson difference scheme (26).

Theorem 5.17. [11] *The solution of difference scheme (25) satisfies the following stability estimates:*

$$\max_{1\leq k\leq N} \left\| u_k^h \right\|_{C_h^\mu} \leq M(\mu) \left[||\varphi^h||_{C_h^\mu} + \max_{1\leq k\leq N} \left\| f_k^h \right\|_{C_h^\mu} \right],$$

$$\max_{1\leq k\leq N} \left\| u_k^h \right\|_{W_{p,h}^\mu} \leq M(\mu) \left[||\varphi^h||_{W_{p,h}^\mu} + \max_{1\leq k\leq N} \left\| f_k^h \right\|_{W_{p,h}^\mu} \right], 1 \leq p < \infty.$$

The proof of Theorem 5.17 is based on Theorems 5.6 and 5.16 on a strong positivity of difference operator A_h^x in Banach space $C^\mu [0,l]_h$ and $W_{p,h}^\mu$ for all $0 < \mu < 1, 1 \leq p < \infty$, on Theorem 4.9 on stability of the difference scheme (26).

Theorem 5.18. [11] *The solution of difference scheme (25) satisfies the following almost coercive stability estimates:*

$$\max_{1\leq k\leq N} \left\| \frac{1}{\tau}(u_k^h - u_{k-1}^h) \right\|_{C_h} + \max_{1\leq k\leq N} \left\| \frac{D_h^2 \left(u_k^h + u_{k-1}^h \right)}{2} \right\|_{C_h}$$

$$\leq M \left[||D_h^2 \varphi^h||_{C_h} + \ln \frac{1}{\tau + h} \max_{1\leq k\leq N} \left\| f_k^h \right\|_{C_h} \right],$$

$$\max_{1\leq k\leq N} \left\| \frac{1}{\tau}(u_k^h - u_{k-1}^h) \right\|_{L_{p,h}} + \max_{1\leq k\leq N} \left\| \frac{D_h^2 \left(u_k^h + u_{k-1}^h \right)}{2} \right\|_{L_{p,h}}$$

$$\leq M \left[||D_h^2 \varphi^h||_{L_{p,h}} + \ln \frac{1}{\tau + h} \max_{1\leq k\leq N} \left\| f_k^h \right\|_{L_{p,h}} \right], 1 \leq p < \infty.$$

The proof of Theorem 5.18 is based on Theorems 5.4 and 5.14 on a strong positivity of an elliptic difference operator A_h^x in Banach space $C[0,l]_h$ and $L_{p,h}, 1 \leq p < \infty$, on estimate (27) and on Theorem 4.11 on almost coercive stability of difference scheme (26).

Theorem 5.19. [11] *Let* $0 < 2m\mu < 1$. *Then, the solution of difference scheme (25) satisfies the following coercive stability estimates:*

$$\max_{1\leq k\leq N} \left\| \frac{1}{\tau}(u_k^h - u_{k-1}^h) \right\|_{C_h^{2m\mu}} + \max_{1\leq k\leq N} \left\| \frac{D_h^2 \left(u_k^h + u_{k-1}^h \right)}{2} \right\|_{C_h^{2m\mu}}$$

$$\leq M(\mu) \left[||D_h^2 \varphi^h||_{C_h^{2m\mu}} + \max_{1\leq k\leq N} \left\| f_k^h \right\|_{C_h^{2m\mu}} \right],$$

$$\left[\sum_{k=1}^N \left\| \frac{1}{\tau}(u_k^h - u_{k-1}^h) \right\|_{W_{p,h}^{2m\mu}}^p \tau \right]^{\frac{1}{p}} + \left[\sum_{k=1}^N \left\| \frac{D_h^2 \left(u_k^h + u_{k-1}^h \right)}{2} \right\|_{W_{p,h}^{2m\mu}}^p \tau \right]^{\frac{1}{p}}$$

$$\leq M(\mu) \left[||D_h^2 \varphi^h||_{W_{p,h}^{2m\mu}} + \left[\sum_{k=1}^N \left\| f_k^h \right\|_{W_{p,h}^{2m\mu}}^p \tau \right]^{\frac{1}{p}} \right], 1 \leq p < \infty.$$

The proof of Theorem 5.19 is based on Theorems 5.4 and 5.14 on a strong positivity of an elliptic difference operator A_h^x in Banach space $C\left[0,l\right]_h$ and $L_{p,h}, 1\leq p<\infty$, and on Theorem 4.13 on coercive stability of difference scheme (26) and on the following theorem on coercive stability of difference scheme (26).

Theorem 5.20. [11] *Let A be a strongly positive operator in a Banach space E. Then, for the solution of difference scheme (26) the following coercive stability inequality holds:*

$$\left[\sum_{k=1}^{N}\left\|\frac{1}{\tau}(u_k-u_{k-1})\right\|_{E_{\mu,p}}^{p}\tau\right]^{\frac{1}{p}}+\left[\sum_{k=1}^{N}\left\|Au_k\right\|_{E_{\mu,p}}^{p}\tau\right]^{\frac{1}{p}}$$
$$\leq M(\mu)\left[||A\varphi||_{E_{\mu,p}}+\left[\sum_{k=1}^{N}\left\|f_k\right\|_{E_{\mu,p}}^{p}\tau\right]^{\frac{1}{p}}\right].$$

In [17], A. Ashyralyev and N. Yaz investigated the differential operator A^x defined by the formula

$$A^x u=-a(x)\frac{d^2u}{dx^2}+\delta u \tag{32}$$

with domain

$$D(A^x)=\{u\in C^{(2)}[0,l]:u(0)=u(\mu),u'(0)=u'(l),l/2\leq\mu\leq l\}. \tag{33}$$

Here $a(x)$ is the smooth function defined on the segment $[0,l]$ and $a(x)\geq a>0$.They proved the following results.

Theorem 5.21. [17] A^x *is the strongly positive operator in $C\left[0,l\right]$.*

Theorem 5.22. [17] *For $\alpha\in(0,\frac{1}{2})$, the norms of the space $E_\alpha(C\left[0,l\right],A^x)$ and the Hölder space $C^{2\alpha}\left[0,l\right]$ are equivalent.*

Theorem 5.23. [17] A^x *is the strongly positive operator in $C^{2\alpha}\left[0,l\right]$.*

Ashyralyev A., Nalbant N. and Sozen Y. considered in [31] the difference operator defined by formula

$$A_h^x u^h=\left\{-a(x_k)\frac{u_{k+1}-2u_k+u_{k-1}}{h^2}+\delta u_k\right\}_1^{M-1},\ u^h=\{u_k\}_0^M, Mh=l \tag{34}$$

with $u_0=u_\ell$, $u_1-u_0=u_N-u_{N-1}$, where $\ell=\left[\frac{\mu}{h}\right]$, $[\cdot]$ is the greatest integer function. This operator is a first order of approximation of the differential operator A^x defined by formula (32) with domain $D(A^x)=\{u\in C^{(2)}[0,l]:u(0)=u(\mu),u'(0)=u'(l),l/2\leq\mu\leq l\}$. They proved the following results.

Theorem 5.24. [31] A_h^x *is the strongly positive operator in C_h.*

Theorem 5.25. [31] *For $\alpha\in(0,\frac{1}{2})$, the norms of the space $E_\alpha(C_h,A_h^x)$ and the Hölder space $C_h^{2\alpha}$ are equivalent uniformly in h.*

Theorem 5.26. [31] A_h^x *is the strongly positive operator in $C_h^{2\alpha}$.*

In the paper [24], the operator defined by formula

$$Au=\begin{pmatrix} a(x)\frac{du_1(x)}{dx}+\delta u_1(x) & -\delta u_2(x)\\ 0 & -a(x)\frac{du_2(x)}{dx}+\delta u_2(x)\end{pmatrix} \tag{35}$$

with domain

$$D(A) = \left\{ \left(\begin{array}{c} u_1(x) \\ u_2(x) \end{array} \right) : u_m(x), \frac{du_m(x)}{dx} \in C[0,l], m = 1,2; \right.$$
$$\left. u_1(0) = \gamma u_1(l), \beta u_2(0) = u_2(l) \right\}$$

generated by the hyperbolic system of equations with nonlocal boundary conditions was considered. Let us introduce the Banach space $\mathbb{C}^\alpha[0,l] = C^\alpha[0,l] \times C^\alpha[0,l]$ $(0 \leq \alpha \leq 1)$ of all continuous vector functions $u = \left(\begin{array}{c} u_1(x) \\ u_2(x) \end{array} \right)$ defined on $[0,l]$ and satisfying a Hölder condition for which the following norm is finite

$$\|u\|_{\mathbb{C}^\alpha[0,l]} = \|u\|_{\mathbb{C}[0,l]}$$
$$+ \sup_{\substack{x,x+\tau\in[0,l] \\ \tau\neq 0}} \frac{|u_1(x+\tau) - u_1(x)|}{|\tau|^\alpha} + \sup_{\substack{x,x+\tau\in[0,l] \\ \tau\neq 0}} \frac{|u_2(x+\tau) - u_2(x)|}{|\tau|^\alpha}.$$

Here $\mathbb{C}[0,l] = C[0,l] \times C[0,l]$ is the Banach space of all continuous vector functions $u = \left(\begin{array}{c} u_1(x) \\ u_2(x) \end{array} \right)$ defined on $[0,l]$ with norm

$$\|u\|_{\mathbb{C}[0,l]} = \max_{x\in[0,l]} |u_1(x)| + \max_{x\in[0,l]} |u_2(x)|.$$

The Green's matrix function of A was constructed. Moreover, applying Green's matrix function of A the following results were proved.

Theorem 5.27. [24] A *is the positive operator in* $\mathbb{C}[0,l]$.

Theorem 5.28. [24] *For* $\alpha \in (0, \frac{1}{2})$, *the norms of the space* $E_\alpha(\mathbb{C}[0,l], A)$ *and the Hölder space* $\overset{\circ}{\mathbb{C}}{}^\alpha[0,l]$ *are equivalent. Here*

$$\overset{\circ}{\mathbb{C}}{}^\alpha[0,l] = \left\{ \left(\begin{array}{c} \varphi(x) \\ \psi(x) \end{array} \right) \in \mathbb{C}^\alpha[0,l] : \right.$$
$$\left. \varphi(0) = \gamma\varphi(l), 0 \leq \gamma \leq 1, \beta\psi(0) = \psi(l), 0 \leq \beta \leq 1 \right\}.$$

Theorem 5.29. [24] A *is the strongly positive operator in* $\overset{\circ}{\mathbb{C}}{}^\alpha[0,l]$.

In applications, we consider the initial-boundary value problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + a(x)\frac{\partial u(t,x)}{\partial x} + \delta(u(t,x) - v(t,x)) = f_1(t,x), \\ 0 < x < l, 0 < t < T, \\ \frac{\partial v(t,x)}{\partial t} - a(x)\frac{\partial v(t,x)}{\partial x} + \delta v(t,x) = f_2(t,x), \\ 0 < x < l, 0 < t < T, \\ u(t,0) = \gamma u(t,l), 0 \leq \gamma \leq 1, \beta v(t,0) = v(t,l), \\ 0 \leq \beta \leq 1, 0 \leq t \leq T, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad 0 \leq x \leq l \end{cases} \tag{36}$$

for the hyperbolic system of equations with nonlocal boundary conditions was obtained. Here

$$a(x) \geq a > 0, \tag{37}$$

$u_0(x)$, $v_0(x)$, $(x \in [0,l])$, $f_1(t,x)$, $f_2(t,x)$, $((t,x) \in [0,T] \times [0,l])$ are given smooth functions and they satisfy every compatibility conditions which guarantees the problem (36) has a smooth solution $u(t,x)$ and $v(t,x)$.

For A a positive operator in E the following result was established in papers [45]-[40].

Theorem 5.30. [24] *Let A be a positive operator in E. Then the following estimate*

$$\|\mathbf{R}^k_{q,q-1}(\tau A)\|_{E\to E} \leq M, 1 \leq k \leq N, N\tau = T \tag{38}$$

is satisfied, where M does not depend on τ and k. Here $\mathbf{R}^k_{q,q-1}(z)$ is the Pade approximation of $exp(-z)$ near $z = 0$.

Putting $t_k = k\tau$ and passing to limit when $\tau \to 0$, we get $t_k \to t$ and

$$\|\exp\{-tA\}\|_{E\to E} \leq M, 0 \leq t \leq T. \tag{39}$$

We introduce the Banach space $\mathbb{C}([0,T],E)$ of all continuous abstract vector functions $u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$ defined on $[0,T]$ with values in E, equipped with the norm

$$\|u\|_{\mathbb{C}([0,T],E)} = \max_{0\leq t\leq T} \|u_1(t)\|_E + \max_{0\leq t\leq T} \|u_2(t)\|_E \,.$$

Note that the problem (36) can be written in the form as the abstract Cauchy problem

$$\frac{d}{dt}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + A\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}, \tag{40}$$
$$0 < t < T, \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

in a Banach space $E = \mathbb{C}[0,l]$ with a positive operator A defined by (35). Here $\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = \begin{pmatrix} f_1(t,x) \\ f_2(t,x) \end{pmatrix}$ is the given abstract vector function defined on $[0,T]$ with values in E, $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} u_0(x) \\ v_0(x) \end{pmatrix}$ is the element of $D(A)$.

It is well known that (see, for example [46]) the following formula

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \exp\{-tA\}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \int_0^t \exp\{-(t-s)A\}\begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix} ds \tag{41}$$

gives a solution of problem (40) in $\mathbb{C}([0,T],E)$ for continuously differentiable on $[0,T]$ vector function $\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$ and smooth given element $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$.

Theorem 5.31. [24] *For the solution of problem* (40) *the stability inequality holds:*

$$\left\|\begin{pmatrix} u \\ v \end{pmatrix}\right\|_{\mathbb{C}([0,T],E)} \leq M\left[\left\|\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}\right\|_E + \left\|\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right\|_{\mathbb{C}([0,T],E)}\right].$$

The proof of Theorem 5.31 is based on Theorem 5.27 on the positivity of operator A in $\mathbb{C}[0,l]$, on formula (41) and estimate (39).

Applying results of Theorem 5.30 and Theorem 5.31, we get the following theorem.

Theorem 5.32. [24] *The solution of problem*(36) *satisfies the following estimate*

$$\max_{t\in[0,T]}\max_{x\in[0,l]}|u(t,x)|+\max_{t\in[0,T]}\max_{x\in[0,l]}|v(t,x)|$$

$$\leq M\left[\max_{x\in[0,l]}|u_0(x)|+\max_{x\in[0,l]}|v_0(x)|+\max_{t\in[0,T]}\max_{x\in[0,l]}|f_1(t,x)|+\max_{t\in[0,T]}\max_{x\in[0,l]}|f_2(t,x)|\right].$$

Applying results of Theorems 5.29, 5.30 and 5.31, we get the following theorem.

Theorem 5.33. [24] *Assume that*

$$f_1(t,0)=\gamma f_1(t,l),0\leq\gamma\leq 1,\quad \beta f_2(t,0)=f_2(t,l),0\leq\beta\leq 1,t\in[0,T].$$

Then the solution of problem (5) *satisfies the following estimate*

$$\max_{t\in[0,T]}\left(\max_{x\in[0,l]}|u(t,x)|+\max_{\substack{x,x+\tau\in[0,l]\\ \tau\neq 0}}\frac{|u(t,x+\tau)-u(t,x)|}{|\tau|^{\alpha}}\right)$$

$$+\max_{t\in[0,T]}\left(\max_{x\in[0,l]}|v(t,x)|+\max_{\substack{x,x+\tau\in[0,l]\\ \tau\neq 0}}\frac{|v(t,x+\tau)-v(t,x)|}{|\tau|^{\alpha}}\right)$$

$$\leq M\left[\max_{x\in[0,l]}|u_0(x)|+\max_{\substack{x,x+\tau\in[0,l]\\ \tau\neq 0}}\frac{|u_0(x+\tau)-u_0(x)|}{|\tau|^{\alpha}}\right.$$

$$+\max_{x\in[0,l]}|v_0(x)|+\max_{\substack{x,x+\tau\in[0,l]\\ \tau\neq 0}}\frac{|v_0(x+\tau)-v_0(x)|}{|\tau|^{\alpha}}$$

$$+\max_{t\in[0,T]}\left(\max_{x\in[0,l]}|f_1(t,x)|+\max_{\substack{x,x+\tau\in[0,l]\\ \tau\neq 0}}\frac{|f_1(t,x+\tau)-f_1(t,x)|}{|\tau|^{\alpha}}\right)$$

$$\left.+\max_{t\in[0,T]}\left(\max_{x\in[0,l]}|f_2(t,x)|+\max_{\substack{x,x+\tau\in[0,l]\\ \tau\neq 0}}\frac{|f_2(t,x+\tau)-f_2(t,x)|}{|\tau|^{\alpha}}\right)\right].$$

In the paper [25], the difference space operator A_h^x defined by the formula

$$A_h^x u^h=\begin{pmatrix} a(x_n)\frac{u_{1,n}-u_{1,n-1}}{h}+\delta u_{1,n} & -\delta u_{2,n} \\ 0 & -a(x_n)\frac{u_{1,n+1}-u_{1,n}}{h}+\delta u_{2,n}\end{pmatrix} \tag{42}$$

acting on the space of mesh vector functions $u^h=\left\{\begin{pmatrix} u_{1,n} \\ u_{2,n-1}\end{pmatrix}\right\}_{n=1}^{M}$ defined on $[0,l]_h$ satisfying conditions

$$u_{1,0}=\gamma u_{1,M},\beta u_{2,0}=u_{2,M}$$

was investigated. Let us introduce the Banach spaces $\mathbb{C}_h^{\alpha}=C_h^{\alpha}\times C_h^{\alpha}$ $(0\leq\alpha\leq 1)$ and $\mathbb{C}_h=C_h\times C_h$ of all mesh vector functions $u^h=\left\{\begin{pmatrix} u_{1,n} \\ u_{2,n-1}\end{pmatrix}\right\}_{n=1}^{M}$ defined on

$$[0,l]_h=\{x_n=nh,0\leq n\leq M,Mh=l\}$$

with following norms

$$\left\|u^h\right\|_{\mathbb{C}_h^{\alpha}}=\left\|u^h\right\|_{\mathbb{C}_h}$$

$$+\sup_{1\le n<n+m\le M}\frac{|u_{1,n+m}-u_{1,n}|}{(mh)^{\alpha}}+\sup_{1\le k<k+m\le M-1}\frac{|u_{2,n+m}-u_{2,n}|}{(mh)^{\alpha}},$$

$$\left\|u^h\right\|_{\mathbb{C}_h}=\max_{1\le n\le M}|u_{1,n}|+\max_{0\le n\le M-1}|u_{2,n}|.$$

The Green's matrix function of A_h^x was constructed. Moreover, applying Green's matrix function of A_h^x the following results were proved.

Theorem 5.34. [25] A_h^x *is the positive operator in* $\mathbb{C}_h$.

Theorem 5.35. [25] *For* $\alpha\in(0,\frac{1}{2})$, *the norms of the space* $E_\alpha(\mathbb{C}_h,A_h^x)$ *and the Hölder space* $\overset{\circ}{\mathbb{C}}_h^{\ \alpha}$ *are equivalent. Here*

$$\overset{\circ}{\mathbb{C}}_h^{\ \alpha}=\left\{\left\{\begin{pmatrix}\varphi_n\\ \psi_{n-1}\end{pmatrix}\right\}_{n=1}^{M}\in\mathbb{C}_h^{\alpha}:\right.$$
$$\left.\varphi_0=\gamma\varphi_M,0\le\gamma\le1,\beta\psi_0=\psi_M,0\le\beta\le1\right\}.$$

Theorem 5.36. [25] *A is the strongly positive operator in* $\overset{\circ}{\mathbb{C}}_h^{\ \alpha}$.

In applications, for numerical solution of an initial-boundary value problem (36) the following difference scheme is presented:

$$\begin{cases}\frac{u_n^k-u_n^{k-1}}{\tau}+a(x_n)\frac{u_n^k-u_{n-1}^k}{h}+\delta(u_n^k-v_n^k)=f_{1,n}^k,f_{1,n}^k=f_1(t_k,x_n),\\ t_k=k\tau,x_n=nh,1\le k\le N,N\tau=T,1\le n\le M,Mh=l,\\ \frac{v_n^k-v_n^{k-1}}{\tau}-a(x_{n+1})\frac{v_{n+1}^k-v_n^k}{h}+\delta v_n^k=f_{2,n}^k,f_{2,n}^k=f_2(t_k,x_n),\\ t_k=k\tau,x_n=nh,1\le k\le N,N\tau=T,0\le n\le M-1,Mh=l,\\ u_0^k=\gamma u_M^k,0\le\gamma\le1,\quad\beta v_0^k=v_M^k,0\le\beta\le1,0\le k\le N,\\ u_n^0=u_0(x_n),\quad v_n^0=v_0(x_n),\quad x_n=nh,0\le n\le M,Mh=l.\end{cases}\tag{43}$$

We introduce the Banach space $\mathbb{C}([0,T]_\tau,E)$ of all continuous abstract mesh vector functions $u^\tau=\left\{u^k\right\}_{k=1}^N=\left\{\begin{pmatrix}u_{1,n}^k\\ u_{2,n}^k\end{pmatrix}^h\right\}_{k=1}^N$ defined on

$$[0,T]_\tau=\{t_k=k\tau,1\le k\le N,N\tau=T\}$$

with values in E, equipped with the norm

$$\|u^\tau\|_{\mathbb{C}([0,T]_\tau,E)}=\max_{1\le k\le N}\left\|\left\{u_{1,n}^k\right\}_{n=1}^M\right\|_E+\max_{1\le k\le N}\left\|\left\{u_{2,n}^k\right\}_{n=1}^M\right\|_E.$$

Note that the problem (43) can be written in the form as the abstract Cauchy problem

$$\left\{\begin{pmatrix}\frac{u^k-u^{k-1}}{\tau}\\ \frac{v^k-v^{k-1}}{\tau}\end{pmatrix}^h\right\}_{k=1}^N+A_h^x\left\{\begin{pmatrix}u^k\\ v^k\end{pmatrix}^h\right\}_{k=1}^N=\left\{\begin{pmatrix}f_1^k\\ f_2\end{pmatrix}^h\right\}_{k=1}^N,\tag{44}$$

$$1\le k\le N,\begin{pmatrix}u_0\\ v_0\end{pmatrix}^h=\begin{pmatrix}u_n^0\\ v_{n-1}^0\end{pmatrix}_{n=1}^M$$

in a Banach space $E=\mathbb{C}_h$ with a positive operator A_h^x defined by (42). Here $\left\{\left(\begin{array}{c} f_1^k \\ f_2^k \end{array}\right)^h\right\}_{k=1}^N = \left\{\left(\begin{array}{c} f_{1,n}^k \\ f_{2,n-1}^k \end{array}\right)_{n=1}^M\right\}_{k=1}^N$ is the given abstract vector function defined on $[0,T]_\tau$ with values in E, $\left(\begin{array}{c} u_0 \\ v_0 \end{array}\right) = \left(\begin{array}{c} u_n^0 \\ v_{n-1}^0 \end{array}\right)_{n=1}^M$ is the element of $D(A_h^x)$. It is well known that the following formula

$$\left(\begin{array}{c} u^k \\ v^k \end{array}\right)^h = (I+\tau A_h^x)^{-k}\left(\begin{array}{c} u_0 \\ v_0 \end{array}\right)^h + \sum_{j=1}^k (I+\tau A_h^x)^{-k+j-1}\left(\begin{array}{c} f_1^j \\ f_2^j \end{array}\right)^h \tau \tag{45}$$

gives a solution of problem (44) in $\mathbb{C}\left([0,T]_\tau, E\right)$.

Theorem 5.37. [25] *For the solution of problem* (44) *the stability inequality holds:*

$$\left\|\left\{\left(\begin{array}{c} u^k \\ v^k \end{array}\right)\right\}_{k=1}^N\right\|_{\mathbb{C}([0,T]_\tau,E)} \leq M(a,\delta)\left[\left\|\left(\begin{array}{c} u_0 \\ v_0 \end{array}\right)^h\right\|_E + \left\|\left\{\left(\begin{array}{c} f_1^k \\ f_2^k \end{array}\right)^h\right\}_{k=1}^N\right\|_{\mathbb{C}([0,T]_\tau,E)}\right].$$

The proof of Theorem 5.37 is based on the positivity of operator A_h^x, formula (45) and estimate (38). Applying results of Theorem 5.37 and Theorem 5.34 on the positivity of operator A_h^x in $\mathbb{C}_h$, we get the following theorem

Theorem 5.38. [25] *The solution of problem*(43) *satisfy the following estimate*

$$\max_{1\leq k\leq N}\max_{1\leq n\leq M}|u_n^k| + \max_{1\leq k\leq N}\max_{0\leq n\leq M-1}|v_n^k|$$

$$\leq M(a,\delta)\left[\max_{1\leq n\leq M}|u_n^0| + \max_{0\leq n\leq M-1}|v_n^0| + \max_{1\leq k\leq N}\max_{1\leq n\leq M}\left|f_{1,n}^k\right| + \max_{1\leq k\leq N}\max_{0\leq n\leq M-1}\left|f_{2,n}^k\right|\right].$$

Applying results of Theorem 5.37 and Theorem 5.36 on the positivity of operator A_h^x in $\overset{\circ}{\mathbb{C}}{}_h^{\alpha}$, we get the following theorem

Theorem 5.39. [25] *Assume that*

$$f_{1,0}^k = \gamma f_{1,M}^k, 0\leq\gamma\leq 1, \quad \beta f_{2,0}^k = f_{2,M}^k, 0\leq\beta\leq 1, 1\leq k\leq N.$$

Then the solution of problem (43) *satisfies the following estimate*

$$\max_{1\leq k\leq N}\left(\max_{1\leq n\leq M}|u_n^k| + \sup_{1\leq n<n+m\leq N}\frac{|u_{n+m}^k-u_n^k|}{(m\tau)^\alpha}\right)$$

$$+\max_{1\leq k\leq N}\left(\max_{0\leq n\leq M-1}|v_n^k| + \sup_{0\leq n<n+m\leq M-1}\frac{|v_{n+m}^k-v_n^k|}{(m\tau)^\alpha}\right)$$

$$\leq M(a,\delta,\alpha)\left[\max_{1\leq n\leq M}|u_n^0| + \sup_{1\leq n<n+m\leq N}\frac{|u_{n+m}^0-u_n^0|}{(m\tau)^\alpha}\right.$$

$$+\max_{0\leq n\leq M-1}|v_n^0| + \sup_{0\leq n<n+m\leq M-1}\frac{|v_{n+m}^0-v_n^0|}{(m\tau)^\alpha}$$

$$+\max_{1\leq k\leq N}\left(\max_{1\leq n\leq M}|f_{1,n}^k| + \sup_{1\leq n<n+m\leq N}\frac{|f_{1,n+m}^k-f_{1,n}^k|}{(m\tau)^\alpha}\right)$$

$$+ \max_{1\le k\le N}\left(\max_{0\le n\le M-1}|f_{2,n}^{k}| + \sup_{0\le n<n+m\le M-1}\frac{|f_{2,n+m}^{k}-f_{2,n}^{k}|}{(m\tau)^{\alpha}}\right)\Bigg].$$

Note that the positivity of difference operators which are a high order of approximation of the operator with nonlocal boundary conditions is not studied. Nevertheless structure of fractional spaces generated by these positive operators is not well-investigated.

6. Conclusions

In this study, a survey of results in the theory of fractional spaces generated by positive differential and difference operators is given. Its scope ranges from theory of differential and difference operators in a space to operators with local and nonlocal boundary conditions. We also discuss their applications to partial differential equations and theory of difference schemes for partial differential equations. This paper does not touch upon the results of papers [20] and [21] on the structure of fractional spaces generated by the neutron transport differential and difference operators. In this paper we do not discuss results of papers [26]- [30] on the structure of fractional spaces generated by the second order positive differential operator with periodic and Neumann conditions and papers [32], [67] structure of fractional spaces generated by the differential operator of the first order with the nonlocal boundary condition.

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