TWMS J. Pure Appl. Math., V.6, N.2, 2015, pp.233-237

CONSTRAINED QUADRATIC STOCHASTIC OPERATORS

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Abstract. In this paper we consider a class of constrained quadratic stochastic operators defined on the simplex S^m and show that such operators have a unique fixed point and the trajectory of the operators starting from any initial point converges to the unique fixed point, i.e. each operator has the regular property.

Keywords: simplex, trajectory, Volterra quadratic stochastic operator, non-Volterra operators.

AMS Subject Classification: Primary 37N25, Secondary 92D10.

1. Introduction

The quadratic stochastic operator (qso) is a mapping of the simplex

$$S^{m-1} = \{x = (x_1, ..., x_m) \in R^m : x_i \geq 0, \sum_{i=1}^{m} x_i = 1\} \tag{1}$$

into itself and has the following form

$$V : x'_k = \sum_{i,j=1}^{m} p_{ij,k} x_i x_j, \quad (k = 1, ..., m), \tag{2}$$

where $p_{ij,k}$ are the coefficients of heredity and satisfy the following conditions

$$p_{ij,k} \geq 0, \quad \sum_{k=1}^{m} p_{ij,k} = 1, \quad (i, j, k = 1, ..., m). \tag{3}$$

Note that each element $x \in S^{m-1}$ is a probability distribution on $E = \{1, ..., m\}$.

For a given initial point $x^{(0)} \in S^{m-1}$ the trajectory $\{x^{(n)}\}, n = 0, 1, 2, ...$ under qso (2) is defined by $x^{(n+1)} = V(x^{(n)})$, where $n = 0, 1, 2, ...$.

In the mathematical biology the main problem for a given qso is to study the asymptotical behaviour of the trajectory. This problem was more completely solved for the class of Volterra qso ([4]-[6]), which are defined by relations (2), (3) and by the additional assumption

$$p_{ij,k} = 0, \qquad k \notin \{i, j\}. \tag{4}$$

The biological interpretation of the relation (4) is clear: an individual repeats the genotype of one of its parents. In [4]-[6] the general form of the Volterra qso V is given, i.e. $V : \mathbf{x} = (x_1, ..., x_m) \in S^{m-1} \to V(\mathbf{x}) = \mathbf{x}' = (x'_1, ..., x'_m) \in S^{m-1}$:

$$x'_k = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i\right), \tag{5}$$

where $a_{ki} = 2p_{ik,k} - 1$ for $i \neq k$ and $a_{kk} = 0$. In addition $a_{ki} = -a_{ik}$ and $|a_{ki}| \leq 1$.

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Manuscript received January 2015.

In [4]-[6] the theory of Volterra qso (5) was developed using the theory of Lyapunov functions and tournaments. However, in the non-Volterra case (i.e., where condition (4) is violated), many questions remain open and there seems to be no general theory available (e.g. see [8]-[12]). See [7] for a recent review of qso.

In [11] the class of $F-$ quadratic stochastic operators is considered. It is shown that such operator has unique fixed point and all trajectories converges to this point exponential rapidly. In [9] the results of paper [11] was generalized for the class conditional quadratic stochastic operators. In [8] similar results have proved for quadratic stochastic operators corresponding to graphs.

In this paper we consider another class of non-Volterra operators. These operators we call constrained quadratic stochastic operators. For any such operator we show uniqueness of the fixed point and prove that any trajectory converges to this fixed point exponentially fast. Thus each constrained operator has the regular property.

2. Definition of cqso

Motivated by [9] and [11] we consider the set $E_0 = \{0, 1, 2, ..., m\} = \{0\} \cup E$. Let us consider partition of the set E_0 as the following way: let $\{F_i \subset E, i = 1, ..., s\}$ be some sets of females and let $\{M_j \subset E, j = 1, .., t\}$ be sets of males such that

$$\bigcup_{i=1}^{s} F_i \cup \bigcup_{j=1}^{t} M_j = E, \quad F_i \cap F_j = \emptyset, \quad M_i \cap M_j = \emptyset, i \neq j.$$

The element 0 plays the role of "empty body".

The coefficients $p_{ij,k}$ of the matrix $\mathbf{P}$ we define as

$$p_{ij,k} = \begin{cases} 1, & \text{if } \ k = 0, \ \ i, j \in F_u \ \text{ or } \ i, j \in M_v \ \text{ or } \ i \in F_u, j \in M_v, u = v; \\ 0, & \text{if } \ k \neq 0, \ \ i, j \in F_u \ \text{ or } \ i, j \in M_v \ \text{ or } \ i \in F_u, j \in M_v, u = v; \\ \geq 0, & \text{if } \ i \in F_u, j \in M_v, u \neq v. \end{cases} \tag{6}$$

The biological interpretation of the coefficients (6): each individual i should interbreed with individual j from other class and other gender, to have offspring k.

Definition 2.1. *For any subsets $\{F_1, F_2, ..., F_s, M_1, M_2, ..., M_t\} \subset E$, the qso with (2),(3) and (6) is called a constrained quadratic stochastic operator (cqso).*

Remark 2.1. *Since $p_{ii,0} = 1$ for each $i \neq 0$, any cqso is a non-Volterra qso.*

Remark 2.2. *For $s = t = 1$ cqso coincides with the $F-$qso defined in [12].*

3. The main result

The canonical form of an arbitrary cqso is

$$V: \begin{cases} x_0' = 1 - 2 \sum\limits_{u=1}^{s} \sum\limits_{\substack{v=1: \\ v \neq u}}^{t} \sum\limits_{i \in F_u} \sum\limits_{j \in M_v} (1 - p_{ij,0}) x_i x_j; \\ x_k' = 2 \sum\limits_{u=1}^{s} \sum\limits_{\substack{v=1: \\ v \neq u}}^{t} \sum\limits_{i \in F_u} \sum\limits_{j \in M_v} p_{ij,k} x_i x_j, \quad k = 1, 2, ..., m, \end{cases} \tag{7}$$

where

$$p_{ij,k} = p_{ji,k} \geq 0, \quad k \in E_0; \quad \sum_{k=0}^{m} p_{ij,k} = 1. \tag{8}$$

Theorem 3.1. *For any values of coefficients $p_{ij,k}$ with conditions (8) the cqso (7) has a unique fixed point $(1,0,0,...,0)$ (with m zeros). Moreover for any initial point $x^{(0)} \in S^m$ the trajectory $\{x^{(n)}\}$ converges to this fixed point exponentially fast.*

Proof. For a point $x \in S^m$ we consider the function

$$\varphi(x) = \sum_{u=1}^{s} \sum_{\substack{v=1:\\ v\neq u}}^{t} \sum_{i\in F_u} \sum_{j\in M_v} x_i x_j. \tag{9}$$

Using (7),(8) and (9) we obtain

$$x_k^{(n+1)} = 2\sum_{u=1}^{s} \sum_{\substack{v=1:\\ v\neq u}}^{t} \sum_{i\in F_u} \sum_{j\in M_v} p_{ij,k} x_i^{(n)} x_j^{(n)} \leq 2\sum_{u=1}^{s} \sum_{\substack{v=1:\\ v\neq u}}^{t} \sum_{i\in F_u} \sum_{j\in M_v} x_i^{(n)} x_j^{(n)} = 2\varphi(x^{(n)}), \tag{10}$$

where $k = 1,2,...,m$ and $n = 0,1,2,...$.

We estimate $\varphi(x^{(n+1)})$:

$$\varphi(x^{(n+1)}) = \sum_{u=1}^{s} \sum_{\substack{v=1:\\ v\neq u}}^{t} \sum_{i\in F_u} \sum_{j\in M_v} x_i^{(n+1)} x_j^{(n+1)} \leq \left(\sum_{u=1}^{s} \sum_{i\in F_u} x_i^{(n+1)}\right)\left(\sum_{v=1}^{t} \sum_{j\in M_v} x_j^{(n+1)}\right) \leq$$

$$\leq \frac{1}{4}\left(\sum_{i\in F} x_i^{(n+1)} + \sum_{j\in M} x_j^{(n+1)}\right)^2 \leq \frac{\left(1 - x_0^{(n+1)}\right)^2}{4}, \tag{11}$$

where $F = \bigcup_{u=1}^{s} F_u$ and $M = \bigcup_{v=1}^{t} M_v$

Using the expression $x_0^{(n+1)}$, from (11) we get

$$\varphi(x^{(n+1)}) \leq \frac{1}{4} \cdot \left[2\sum_{u=1}^{s} \sum_{\substack{v=1:\\ v\neq u}}^{t} \sum_{i\in F_u} \sum_{j\in M_v} \left(1 - p_{ij,0}\right) x_i^{(n)} x_j^{(n)}\right]^2 \leq$$

$$\leq \left(\sum_{u=1}^{s} \sum_{\substack{v=1:\\ v\neq u}}^{t} \sum_{i\in F_u} \sum_{j\in M_v} x_i^{(n)} x_j^{(n)}\right)^2 = \left(\varphi(x^{(n)})\right)^2, \quad n \geq 0. \tag{12}$$

Iterating (12) we obtain

$$\varphi(x^{(n)}) \leq \left(\varphi(x^{(0)})\right)^{2^n}, \quad n \geq 1. \tag{13}$$

We claim that $\varphi(x^{(0)}) \leq 1 - \frac{1}{2(s+t+1)}$. Indeed, using the notation

$$f_u = \sum_{i\in F_u} x_i, \quad u = 1,...,s, \quad \text{and} \quad \tilde{m}_v = \sum_{j\in M_v} x_j, \quad v = 1,...,t,$$

we obtain

$$\varphi(x) = \sum_{u=1}^{s} \sum_{\substack{v=1:\\ v\neq u}}^{t} \sum_{i\in F_u} \sum_{j\in M_v} x_i x_j \leq \frac{(1-x_0)^2 - \sum\limits_{i=1}^{s} f_i^2 - \sum\limits_{j=1}^{t} \tilde{m}_j^2 - \sum\limits_{k=1}^{\min(s,t)} f_k \tilde{m}_k}{2} \leq$$

$$\leq \frac{1 - \sum_{i=1}^{s} f_i^2 - \sum_{j=1}^{t} \tilde{m}_j^2}{2}.$$

It is easy to check that the function $\psi(x) = x_0^2 + x_1^2 + ... + x_m^2$ with condition $x_0 + x_1 + ... + x_m = 1$ has minimum $\min\limits_{x \in S^m} \psi(x) = \frac{1}{m+1}$ and using this fact we get

$$\varphi(x) \leq \frac{2 - 1/(s+t+1)}{2} \leq 1 - \frac{1}{2(s+t+1)}. \tag{14}$$

Thus by (13) from (14) it follows that

$$\varphi(x^{(n)}) \leq \Big(1 - \frac{1}{2(s+t+1)}\Big)^{2^n}. \tag{15}$$

From the relations (10) and (15) it follows that

$$\lim_{n\to\infty} x_k^{(n)} = 0, \quad \text{for} \quad \text{any} \quad k = 1, 2, ..., m, \quad \text{i.e.}$$

$$\lim_{n\to\infty} x^{(n)} = (1, 0, ..., 0), \quad \text{for} \quad \text{all} \quad x^{(0)} \in S^m. \tag{16}$$

It is evident that the point $(1, 0, ..., 0)$ is unique fixed point since the relations (16) holds for any $x^{(0)} \in S^m$. Theorem is proved. □

Remark 3.1. *The qso V is called regular if the limit $\lim\limits_{n\to\infty} x^{(n)}$ exists for all $x^{(0)}$ from the simplex. The qso V satisfies the ergodic theorem if the limit $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=0}^{n-1} x^{(k)}$ exists for any $x^{(0)} \in S^{m-1}$. On the basis of numerical calculations, Ulam conjectured* [13] *that the ergodic theorem is valid for any qso. In* [14]*, it was shown that this conjecture is, in general, false. It follows from Theorem 3.1 that the cqso has regular property and the ergodic theorem holds for any cqso.*

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