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EQUI-AFFINE DIFFERENTIAL INVARIANTS OF A PAIR OF CURVES

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Abstract. Let $G = SAff(n, R)$ be the group of all transformations in R^n as $F(x) = gx + b$ such that $g \in SL(n, R)$ and $b \in R^n$. The system of generators for the differential algebra of all G-invariant differential polynomials of a pair of curves is found for the group $SAff(n, R)$. The conditions for G-equivalence of a pair of curves is obtained.

Keywords: differential invariant, equi-affine group, equivalence of curves.

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1. Introduction

The concept of affine geometry was introduced by Felix Klein in Erlangen Programme in 1872. According to this programme, affine geometry deals with the properties of curves and surfaces which are invariant under affine maps. Since that time, affine differential invariants of curves and surfaces have been investigated.

The theory of differential invariants consists of three fundamental theorems. The first of these is finding the generators for invariant functions. The second is finding the conditions of equivalence for curves (similarly for surfaces) and the third one is finding the relations (if they exist) between of these generators.

The fundamental theorems of curves and hypersurfaces in centro-affine is investigated in [4]. The complete system of global differential and integral invariants for one curves is found in [8] for equi-affine geometry and in [14] for centro-equiaffine curves. In [17], it is obtained differential invariants for some groups in according to one curve. For two curves, it is investigated the differential invariants and its applications to ruled surfaces for the group $SL(n, R)$ in [13].

In [3], it was provided a rigorous theoretical justification of Cartan's method of moving frames for arbitrary finite-dimensional Lie group actions on manifolds. The method given there also leads to complete classification of generating systems of differential invariants. This paper provides a new approach to the construction of differential invariants and equivalence of curves for the group $SAff(n, R)$.

In this paper, we investigate the differential invariants of a pair of curves for the group $SAff(n, R)$. In section 1, we give some introductory definitions. In section 2, the generator system of differential invariants is found for the polynomials of a pair of curves. Then the conditions of equivalence for two pairs of curves is given by the differential invariants. Also it is shown that the set of generator invariants is minimal.

Let R be the field of real numbers and R^n be n-dimensional Euclidean space. The set $SAff(n, R)$ which defined by

$$\{F : F(x) = Ax + b \ , \ A \ is \ a \ real \ nxn \ matrix \ which \ detA = 1 \, and \, b \in R^n\}$$

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is a group in according to composition of transformations.

Definition 1.1. *A $C^{\infty}-$function $x : I \to R^n$ will be called a parametric curve or briefly a curve in R^n.*

This paper treats the case of parametrized curves. The unparametrized case is more challenging. Olver, completely classified equi-affine joint differential invariants for unparametrized curves in $2-$ and $3-$dimensions.([11])

Definition 1.2. *Let $\{x_1, x_2\}$ and $\{y_1, y_2\}$ be two pairs of curves. If $y_i = Ax_i + b$, $i = 1, 2$ for some $A \in SL(n, R)$, $b \in R^n$, then these curve families will be called $SAff(n, R)-$equivalent and denoted by $\{x_1, x_2\} \overset{G}{\approx} \{y_1, y_2\}$ for the group $G = SAff(n, R)$.*

Definition 1.3. *Let x_1 and x_2 be two curve in R^n. The polynomial*

$$P\{x_1, x_2\} = P(x_1, x_2, x_1', x_2', \dots, x_1^{(m)}, x_2^{(m)})$$

for some finite natural number m will be called a differential polynomial of x_1 and x_2.

The derivation of $P\{x_1, x_2\}$ will be denoted by P' and this derivation is obtained as follows: Since x_1, x_2 are variables, then the derivative of P is taken in according to the functional variables x_1, x_2 in the polynomial.

Definition 1.4. *If $P\{Ax_1 + b, Ax_2 + b\} = P\{x_1, x_2\}$ for some $A \in SL(n, R)$, $b \in R^n$, the differential polynomial P is called an equi-affine invariant differential polynomial.*

The set of all differential polynomials will be denoted by $R\{x_1, x_2\}$. It is a differential R-algebra. Let G be the group $SAff(n, R)$. The set of all equi-affine invariant differential polynomials will be denoted by $R\{x_1, x_2\}^G$. $R\{x_1, x_2\}^G$ is a differential subalgebra of $R\{x_1, x_2\}$.

Definition 1.5. *Let $f_1, f_2, \dots, f_k \in R\{x_1, x_2\}^G$. If the differential algebra generated by these functions is equal to $R\{x_1, x_2\}^G$, then these functions will be called the generator set of $R\{x_1, x_2\}^G$.*

2. Equi-affine invariants of a pair of curves

Let $x_1, x_2, \dots, x_n \in R^n$. We will be denoted the determinant $\begin{vmatrix} x_{11} & \dots & x_{n1} \\ \dots & \dots & \dots \\ x_{1n} & \dots & x_{nn} \end{vmatrix}$ by $[x_1 \dots x_n]$. In here, k. column of this determinant is consist of the components of x_k, which are $x_{k1}, x_{k2}, \dots, x_{kn}$.

Lemma 2.1. *Let $x_0, x_1, \dots, x_n, y_2, \dots, y_n$ be vectors in R^n. Then the following equality holds:*

$$\begin{aligned} &[x_1 x_2 \dots x_n][x_0 y_2 \dots y_n] - [x_0 x_2 \dots x_n][x_1 y_2 \dots y_n] - \dots \\ &\qquad - [x_1 x_2 \dots x_0][x_n y_2 \dots y_n] = 0 \end{aligned} \tag{1}$$

Proof. Page 173 in [8]. □

Definition 2.1. *A curve x in R^n will be called $SAff(n, R)$-regular (briefly regular) if $\left[x' x'' \dots x^{(n)}\right] \neq 0$. Hence for all $t \in I$, $\left[x'(t) x''(t) \dots x^{(n)}(t)\right] \neq 0$.*

Let G be the group $SAff(n, R)$.

Theorem 2.1. *Let x_1 and x_2 be two curve in R^n such that x_1 is regular. Then the generator set of $R\{x_1,x_2\}^G$ is*

$$\left[x_1' x_1'' \dots x_1^{(n)}\right], \left[x_1' \dots x_1^{(i-1)} x_1^{(n+1)} x_1^{(i+1)} \dots x_1^{(n)}\right],$$
$$\left[x_1' \dots x_1^{(i-1)}\ x_2 - x_1\, x_1^{(i+1)} \dots x_1^{(n)}\right]$$

for $i = 1, \dots, n$.

Proof. The equi-affine differential polynomial P is in the form of

$$P\{x_1, x_2\} = P(x_1, x_2, x_1', x_2', \dots, x_1^{(k)}, x_2^{(k)})$$

for some $k \in N$. Since P is $G-$invariant, we get

$$P\{gx_1 + b, gx_2 + b\} = P(gx_1 + b, gx_2 + b, gx_1', gx_2', \dots, gx_1^{(k)}, gx_2^{(k)}) =$$
$$= P(x_1, x_2, x_1', x_2', \dots, x_1^{(k)}, x_2^{(k)})$$

for all $g \in SL(n, R)$ and $b \in R^n$. If we take g as identity element of nxn matrix e, then

$$P(x_1 + b, x_2 + b, x_1', x_2', \dots, x_1^{(k)}, x_2^{(k)})\ = P(x_1, x_2, x_1', x_2', \dots, x_1^{(k)}, x_2^{(k)}).$$

We want to show that the differential polynomial $P(x_1 + b, x_2 + b, x_1', x_2', \dots, x_1^{(k)}, x_2^{(k)})$ is equal to $\varphi(x_2 - x_1, x_1', x_2', \dots, x_1^{(k)}, x_2^{(k)})$ for some differential polynomial φ. Then we get the invariance condition as

$$\varphi(g(x_2 - x_1),\ gx_1', gx_2', \dots, gx_1^{(k)}, gx_2^{(k)}) =\ \varphi((x_2 - x_1),\ x_1', x_2', \dots, x_1^{(k)}, x_2^{(k)}).$$

Let $y_2 = x_2 - x_1$ and $y_1 = x_1'$. So we have from above equality

$$\varphi(y_1,\ y_2, y_1', y_2', \dots, y_1^{(k)}, y_2^{(k)}) =\ \varphi(gy_1,\ gy_2, gy_1', gy_2', \dots, gy_1^{(k)}, gy_2^{(k)}).$$

We get that the invariance condition depend only on g. Since $g \in SL(n, R)$ and invariant generator set of $\{y_1, y_2\}$ for the group $SL(n, R)$ in [13] is given by

$$\left[y_1 y_1' \dots y_1^{(n-1)}\right],\ \left[y_1 \dots y_1^{(i-1)}\ y_1^{(n)}\ y_1^{(i+1)} \dots y_1^{(n-1)}\right],$$
$$\left[y_1 \dots y_1^{(i-1)}\ y_2\ y_1^{(i+1)} \dots y_1^{(n-1)}\right]$$

for $i = 0, \dots, n-1$. Since $y_2 = x_2 - x_1$ and $y_1 = x_1'$, we get that the generator set of $R\{x_1, x_2\}^G$ is

$$\left[x_1' x_1'' \dots x_1^{(n)}\right],\ \left[x_1' \dots x_1^{(i-1)}\ x_1^{(n+1)}\ x_1^{(i+1)} \dots x_1^{(n)}\right],$$
$$\left[x_1' \dots x_1^{(i-1)}\ x_2\ - x_1\ x_1^{(i+1)} \dots x_1^{(n)}\right]$$

for $i = 1, \dots, n$. □

Theorem 2.2. *Let $G = SAff(n, R)$ and $\{x_1, x_2\}$, $\{y_1, y_2\}$ be two curve families such that x_1 and y_1 are regular. If for $i = 1, \dots, n$*

$$\left[x_1' x_1'' \dots x_1^{(n)}\right] = \left[y_1' y_1'' \dots y_1^{(n)}\right],$$

$$\left[x_1' \dots x_1^{(i-1)}\ x_1^{(n+1)}\ x_1^{(i+1)} \dots x_1^{(n)}\right] = \left[y_1' \dots y_1^{(i-1)}\ y_1^{(n+1)}\ y_1^{(i+1)} \dots y_1^{(n)}\right], \quad (2)$$
$$\left[x_1' \dots x_1^{(i-1)}\ x_2\ - x_1\ x_1^{(i+1)} \dots x_1^{(n)}\right] = \left[y_1' \dots y_1^{(i-1)}\ y_2\ - y_1\ y_1^{(i+1)} \dots y_1^{(n)}\right],$$

then for some $g \in SL(n, R)$ and $b \in R^n$, $y_1(t) = gx_1(t) + b$, $y_2(t) = gx_2(t) + b$, $\forall t \in I$. So $\{x_1, x_2\} \overset{G}{\approx} \{y_1, y_2\}$.

Proof. Let us take $x_1' = z_1$, $y_1' = w_1$, $x_2 - x_1 = z_2$, $y_2 - y_1 = w_2$. Therefore the preceding equations imply

$$\left[z_1 z_1' \dots z_1^{(n-1)}\right] = \left[w_1 w_1' \dots w_1^{(n-1)}\right],$$

$$\left[z_1 \dots \ z_1^{(i-1)} \ z_1^{(n)} \ z_1^{(i+1)} \dots z_1^{(n-1)}\right] = \left[w_1 \dots w_1^{(i-1)} \ w_1^{(n)} \ w_1^{(i+1)} \dots w_1^{(n-1)}\right],$$

$$\left[z_1 \dots z_1^{(i-1)} \ z_2 \ \ z_1^{(i+1)} \dots z_1^{(n-1)}\right] = \left[w_1 \dots w_1^{(i-1)} \ w_2 \ \ w_1^{(i+1)} \dots w_1^{(n-1)}\right]$$

for $i = 0, \dots, n-1$. If we divide these equations,

$$\frac{\left[z_1 \dots \ z_1^{(i-1)} \ z_1^{(n)} \ z_1^{(i+1)} \dots z_1^{(n-1)}\right]}{\left[z_1 z_1' \dots z_1^{(n-1)}\right]} = \frac{\left[w_1 \dots w_1^{(i-1)} \ w_1^{(n)} \ w_1^{(i+1)} \dots w_1^{(n-1)}\right]}{\left[w_1 w_1' \dots w_1^{(n-1)}\right]},$$

$$\frac{\left[z_1 \dots \ z_1^{(i-1)} \ z_2 \ \ z_1^{(i+1)} \dots z_1^{(n-1)}\right]}{\left[z_1 z_1' \dots z_1^{(n-1)}\right]} = \frac{\left[w_1 \dots w_1^{(i-1)} \ w_2 \ \ w_1^{(i+1)} \dots w_1^{(n-1)}\right]}{\left[w_1 w_1' \dots w_1^{(n-1)}\right]}.$$

Since x_1 and y_1 are regular, we can write the above equalities. Take the matrices

$$A_{z_1} = \begin{pmatrix} z_{11}(t) & \dots & z_{11}^{(n-1)}(t) \\ \dots & \dots & \dots \\ z_{1n}(t) & \dots & z_{1n}^{(n-1)}(t) \end{pmatrix} \text{ and } A_{z_1}' = \begin{pmatrix} z_{11}'(t) & \dots & z_{11}^{(n)}(t) \\ \dots & \dots & \dots \\ z_{1n}'(t) & \dots & z_{1n}^{(n)}(t) \end{pmatrix}.$$

Since $\left(A_{w_1} \cdot A_{z_1}^{-1}\right)' = 0$, we get $A_{w_1} = gA_{z_1}$ and $detg \neq 0, g$ is constant. Therefore $w_1(t) = gz_1(t)$, $\forall t \in I$. If we write this equality in first equality in 2.2, we get

$$\left[z_1' z_1'' \dots z_1^{(n)}\right] = \left[w_1' w_1'' \dots w_1^{(n)}\right] = \left[(gz_1)'(gz_1)'' \dots (gz_1)^{(n)}\right] = \left[gz_1' gz_1'' \dots g{z_1}^{(n)}\right] =$$

$$= detg. \left[z_1' z_1^{''} \dots z_1^{(n)}\right]$$

and then $detg = 1$, so g must be element of the group $SL(n, R)$., Let us take the matrix

$$D_{z_2} = \begin{pmatrix} z_{11}(t) & \dots & z_{11}^{(n-2)}(t) & z_{21}(t) \\ \dots & \dots & \dots & \dots \\ z_{1n}(t) & \dots & z_{1n}^{(n-2)}(t) & z_{2n}(t) \end{pmatrix}.$$

Take $A_{z_1}^{-1} \cdot D_{z_2} = H = \|h_{ij}\|$, $i, j = 1, \dots, n$. Let us find the elements of this matrix. We have that $D_{z_2} = A_{z_1} \cdot H$. Then we get the following system of differential equations:

$$z_{11}h_{11} + \dots + z_{11}^{(n-1)}h_{n1} = z_{11},$$

$$z_{12}h_{11} + \dots + z_{12}^{(n-1)}h_{n1} = z_{12},$$

$$\dots$$

$$z_{1n}h_{11} + \dots + z_{1n}^{(n-1)}h_{n1} = z_{1n}.$$

The solution of this equation system in according to Cramer's rule;

$$h_{11}=\frac{\left[z_1 z_1' \dots z_1^{(n-1)}\right]}{\left[z_1 z_1' \dots z_1^{(n-1)}\right]}=1,\quad h_{21}=\frac{\left[z_1 z_1 \dots z_1^{(n-1)}\right]}{\left[z_1 z_1' \dots z_1^{(n-1)}\right]}=0,\ \dots\ ,\ h_{n1}=0.$$

If we go on finding solutions in this way, we obtain;

$$H=\begin{pmatrix} 1 & 0 & \dots & 0 & h_{1n} \\ 0 & 1 & \dots & 0 & h_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & h_{nn} \end{pmatrix}$$

and where the entries of the last column are;

$$h_{1n}=\frac{\left[z_2 z_1' \dots z_1^{(n-1)}\right]}{\left[z_1 z_1' \dots z_1^{(n-1)}\right]},\quad h_{2n}=\frac{\left[z_1 z_2 \dots z_1^{(n-1)}\right]}{\left[z_1 z_1' \dots z_1^{(n-1)}\right]},\ \dots\ ,\ h_{nn}=\frac{\left[z_1 z_1' \dots z_2\right]}{\left[z_1 z_1' \dots z_1^{(n-1)}\right]}.$$

So the matrix H is equal to $A_{z_1}^{-1}\cdot D_{z_2}=A_{w_1}^{-1}\cdot D_{w_2}$. Therefore we have that $D_{z_2}=g^{-1}\cdot D_{w_2}$ and $D_{w_2}=g\cdot D_{z_2}$. This equation implies that

$$w_{21}=g_{11}z_{21}+\dots+g_{1n}z_{2n},$$

$$w_{22}=g_{21}z_{21}+\dots+g_{2n}z_{2n},$$

$$\dots$$

$$w_{2n}=g_{n1}z_{21}+\dots+g_{nn}z_{2n}$$

and we get $w_2=gz_2$, $g\in SL(n,R)$. On the other hand, we know that $w_1=gz_1$, $g\in SL(n,R)$. Since $w_1=y_1'$, $z_1=x_1'$, taking integral, we have that

$$y_1=gx_1+b\ ,\ b\in R^n. \tag{3}$$

And since $w_2=y_2-y_1$, $y_2=x_2-x_1$ and $w_2=gz_2$, we get

$$y_2-y_1=g(x_2-x_1),$$

then

$$y_2=gx_2+b\ ,\ b\in R^n, \tag{4}$$

we get from 3 and 4 for the same $g\in SL(n,R)$ and $b\in R^n$ that $\{x_1,x_2\}\overset{G}{\approx}\{y_1,y_2\}$. □

Theorem 2.3. *Let* $G=SAff(n,R)$ *and* $f_0(t)\neq 0,\ f_i(t),\ g_i(t),\ i=1,\dots,n$ *be* C^∞*-functions. Then there exist curves* $x_1,\ x_2$ *which* x_1 *is regular such that*

$$\begin{gathered}\left[x_1' x_1'' \dots x_1^{(n)}\right]=f_0(t),\\ \left[x_1' \dots x_1^{(i-1)}\ x_1^{(n+1)}\ x_1^{(i+1)} \dots x_1^{(n)}\right]=f_i(t)\ \ ,\ i=1,\dots,n\ ,\\ \left[x_1' \dots x_1^{(i-1)}\ x_2\ -x_1\ x_1^{(i+1)} \dots x_1^{(n)}\right]=g_i(t)\ \ ,i=1,\dots,n.\end{gathered} \tag{5}$$

Proof. Let $x_1' = y_1$ and $x_2 - x_1 = y_2$. Then from the equalities 5, we get

$$\left[y_1 y_1' \dots y_1^{(n-1)}\right] = f_0(t),$$

$$\left[y_1 \dots\ y_1^{(i-1)}\ y_1^{(n)}\ y_1^{(i+1)} \dots y_1^{(n-1)}\right] = f_i(t)\ ,\ i = 0, \dots, n-1,$$

$$\left[y_1 \dots y_1^{(i-1)}\ y_2\ \ y_1^{(i+1)} \dots y_1^{(n-1)}\right] = g_i(t)\ , i = 0, \dots, n-1.$$

If we divide these equalities, we have that

$$\frac{\left[y_1 \dots y_1^{(i-1)}\ y_1^{(n)}\ y_1^{(i+1)} \dots y_1^{(n-1)}\right]}{\left[y_1 y_1' \dots y_1^{(n-1)}\right]} = \frac{f_{i+1}(t)}{f_0(t)} = h_{i+1}(t)\ ,\ i = 0, \dots, n-2, \tag{6}$$

$$\frac{\left[y_1 \dots\ y_1^{(i-1)}\ y_2\ \ y_1^{(i+1)} \dots y_1^{(n-1)}\right]}{\left[y_1 y_1' \dots y_1^{(n-1)}\right]} = \frac{g_{i+1}(t)}{f_0(t)} = k_{i+1}(t)\ ,\ i = 0, \dots, n-1, \tag{7}$$

$$\frac{\left[y_1 y_1' \dots y_1^{(n-1)}\right]'}{\left[y_1 y_1' \dots y_1^{(n-1)}\right]} = \frac{f_0(t)'}{f_0(t)} = h_0(t). \tag{8}$$

In the same way with the previous proof, we get the matrix B, taking y instead of x such that $A'_{y_1} = A_{y_1} \cdot B$. Then we have the following system of differential equations from this multiplication;

$$\begin{gathered} y_{11} h_1(t) + y_{11}' h_2(t) + \dots + y_{11}^{(n-1)} h_n(t) - y_{11}^{(n)} = 0, \\ y_{12} h_1(t) + y_{12}' h_2(t) + \dots + y_{12}^{(n-1)} h_n(t) - y_{12}^{(n)} = 0, \\ \dots \\ y_{1n} h_1(t) + y_{1n}' h_2(t) + \dots + y_{1n}^{(n-1)} h_n(t) - y_{1n}^{(n)} = 0. \end{gathered}$$

Let we take $y_{1i} = z$, $i = 1, \dots, n$. So we can write the above system of differential equations as

$$h_1(t) z + h_2(t) z' + \dots + h_n(t) z^{(n-1)} - z^{(n)} = 0.$$

It is known that the theory of differential equations, there exist one solution of this differential equation. Let $(w_1, w_2, \dots, w_n)$ be the solution. Put $y_1(t) = (w_1, w_2, \dots, w_n)$. Then the curve $y_1(t)$ provide the equalities 6 and 8.

Take the matrix

$$A_2 = \begin{pmatrix} y_{11} & \dots & y_{11}^{(n-2)} & y_{21} \\ y_{12} & \dots & y_{12}^{(n-2)} & y_{22} \\ \dots & \dots & \dots & \dots \\ y_{1n} & \dots & y_{1n}^{(n-2)} & y_{2n} \end{pmatrix}$$

and let $A_{y_1}^{-1} \cdot A_2 = H$. Then the matrix H satisfies the equality $A_2 = A_{y_1} \cdot H$. Hence

$$\begin{gathered} y_{21} = y_{11} k_1(t) + y_{11}' k_2(t) + \dots + y_{11}^{(n-1)} k_n(t), \\ y_{22} = y_{12} k_1(t) + y_{12}' k_2(t) + \dots + y_{12}^{(n-1)} k_n(t), \\ \dots \\ y_{2n} = y_{1n} k_1(t) + y_{1n}' k_2(t) + \dots + y_{1n}^{(n-1)} k_n(t). \end{gathered}$$

So we have the curve y_2. Take $\left[y_1 y_1' \cdots y_1^{(n-1)}\right] = \varphi(t)$. By $detA_{y_1} \neq 0$, we get $\varphi(t) \neq 0$ for all $t \in I$. Let $d'(t) = \frac{\varphi'(t)}{\varphi(t)} = \frac{f_0(t)'}{f_0(t)} = h_0(t)$. Taking integral, we have that $d(t) = \int\limits_0^t h_0(t)\,dt + c$, $c \in R^n$. So $ln\varphi(t) = \int\limits_0^t h_0(t)\,dt + c$. Therefore

$$\varphi(t) = e^{\int\limits_0^t h_0(t)dt + c} = e^c . e^{\int\limits_0^t h_0(t)dt} = \lambda_1 . e^{\int\limits_0^t h_0(t)dt}, \; e^c = \lambda_1 \neq 0.$$

In the same way, there exists $\lambda_2 \neq 0$ such that $f_0(t) = \lambda_2 .\, e^{\int_0^t h_0(t)dt}$. Let us take $\lambda = \frac{\lambda_2}{\lambda_1} \neq 0$. So $f_0(t) = \lambda.\varphi(t)$. Let $g \in GL(n, R)$ and $detg = \lambda$. Therefore

$$\left[(gy_1)(gy_1)' \cdots (gy_1)^{(n-1)}\right] = \left[gy_1 gy_1' \cdots gy_1{}^{(n-1)}\right] = detg. \left[y_1 y_1' \cdots y_1^{(n-1)}\right] = f_0(t).$$

If we take $gy_1 = z_1$ and $gy_2 = z_2$, these curves satisfy required conditions. Because

$$f_0(t) = \left[x_1' x_1'' \cdots x_1^{(n)}\right] = \lambda. \left[y_1 y_1' \cdots y_1^{(n-1)}\right] = \left[z_1 z_1' \cdots z_1^{(n-1)}\right] \neq 0.$$

Then the curve $z_1(t)$ is regular. On the other hand for $i = 0, \ldots, n-1$

$$\frac{\left[z_1 \ldots\; z_1^{(i-1)}\; z_1^{(n)}\; z_1^{(i+1)} \ldots z_1^{(n-1)}\right]}{\left[z_1 z_1' \ldots z_1^{(n-1)}\right]} = h_{i+1}(t)$$

and

$$\frac{\left[z_1 \ldots\; z_1^{(i-1)}\; z_2\; z_1^{(i+1)} \ldots z_1^{(n-1)}\right]}{\left[z_1 z_1' \ldots z_1^{(n-1)}\right]} = k_{i+1}(t).$$

So we have the following equalities:

$$\left[z_1 z_1' \ldots z_1^{(n-1)}\right] = f_0(t) \neq 0,$$

$$\left[z_1 \ldots\; z_1^{(i-1)}\; z_1^{(n)}\; z_1^{(i+1)} \ldots z_1^{(n-1)}\right] = f_i(t)\;,\; i = 0, \ldots, n-1,$$

$$\left[z_1 \ldots z_1^{(i-1)}\; z_2\; z_1^{(i+1)} \ldots z_1^{(n-1)}\right] = g_i(t)\;, i = 0, \ldots, n-1.$$

Hence the curves z_1 and z_2 satisfy the above equalities. So there exist the curves z_1 and z_2 such that the above equalities are hold. Since $x_1' = y_1$ and $x_2 - x_1 = y_2$, we get

$$x_1(t) = \int\limits_0^t y_1(t)\,dt + b,$$

$$x_2(t) = y_2(t) + x_1(t) = y_2(t) + \int\limits_0^t y_1(t)\,dt + b.$$

Then the curves $x_1(t)$ and $x_2(t)$ satisfy the hypotheses of the theorem. So the proof is completed. □

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