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# ON AN OPTIMAL SHAPE PROBLEM FOR THE EIGENFREQUENCY OF THE CLAMPED PLATE

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ABSTRACT. Minimization problem with respect to domain is considered for the eigenvalues of the biharmonic operator describing the tranverse vibrations of the clamped plate. Using the variation formula a necessary optimality condition is derived for the considered domain functional.

Keywords: Shape optimization, clamped plate, convex domain, support function, necessary condition, eigenfrequency.

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## 1. Introduction

In continuum mechanics, plate theories are mathematical descriptions of the mechanics of flat plates that draws on the theory of beams. Plates are defined as plane structural elements with a small thickness compared to the planar dimensions. The typical thickness to width ratio of a plate structure is less than 0.1. A plate theory takes advantage of this disparity in length scale to reduce the full three-dimensional solid mechanics problem to a two-dimensional problem.

Of the numerous plate theories that have been developed since the late 19th century, two are widely accepted and used in engineering. These are the Kirchhoff–Love theory of plates (classical plate theory) and the Mindlin-Reissner theory of plates (first-order shear plate theory). In both these theories the area (domain) of the plate is fixed and the aim is to calculate the deformation and stresses in a plate subjected to loads [11].

But some practical situations put a problem to find and then optimize (minimize or maximize) the eigenfrequency of the plate under across vibrations. The optimization may be done by choosing the physical characteristics of the plate. Some engineering solutions require the optimization of the eigenfrequency by varying the form (area, domain) of the plate. For these problems it is expedient to consider a plate with non-fixed, variable area. Then the eigenfrequency of such plate may be considered as functional depending on the plate area (domain). By this way we arrive to the shape optimization problems. Note that the existence in such problems is investigated by various authors [3-5, 13,15]. As is shown these problem are well-posed if the set of admissible domains satisfy some geometrical restrictions, for example, are open sets, or the functional under minimization depends on lower number of eigenvalues (in the case of plates-eigenfrequences).

Here we consider the eigenfrequency of the clamped plate under transverse vibrations as a functional of the plate domain, calculate its first domain variation, proof an explicit formula for

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the eigenfrequency. Note that the obtained formula shows that boundary values of the function $u_j(x)$ uniquely define the eigenfrequency and may have an important value in many applications. At the end we consider one particular case and find an optimal domain.

## 2. Statement of the problem

It is known that the function $\omega(x_1, x_2, t)$ describing the transverse vibrations of the plate with area $D$ satisfies to the following partial differential equation of the fourth order

$$\omega_{x_1x_1x_1x_1} + 2\omega_{x_1x_1x_2x_2} + \omega_{x_2x_2x_2x_2} + \omega_{tt} = 0, x \in D, \tag{1}$$

where $D$ is a convex bounded domain from Euclidian space $E^n$ [10]. Assuming the process stable the solution (eigenvibration) of this equation may be sought in the form

$$\omega\left(x_1, x_2, t\right) = u\left(x_1, x_2\right)\cos\mu t,$$

where $\lambda$ stands for the eigenfrequency.

Substitution this into the equation (1) gives

$$\Delta^2 u = \lambda u, \tag{2}$$

where $\Delta^2 = \Delta\Delta$, $\Delta$ is the Laplace operator, $\lambda = \mu^2$ .

This equation may be equipped by different boundary conditions for different plates. In the case of clamped plate those conditions indeed are

$$u = 0, \quad \frac{\partial u}{\partial n} = 0\,, \quad x \in S_D, \tag{3}$$

where $S_D$ is a boundary of the domain $D$.

Let

$$K = \left\{D \in E^n : \overline{D} \in K_0,\ S_D \in C^2\right\}\,,$$

where $\overline{D}$ is a closure of $D$, $K_0$ is some subset of convex bounded domains from $E^n$. Note that $K_0$ may be defined by various ways, as well as, by fixing the area, the length of boundary or by the condition type of $K_1 \subset K_0 \subset K_2$, where $K_1$ , $K_2$ are given domains.

The problem is: to find a domain $D \in K$ that is a solution of the problem

$$\lambda_1\left(D\right) \rightarrow \min, \tag{4}$$

under the restriction

$$|S_D| = 2\pi, \tag{5}$$

where $\lambda_1\left(D\right)$ is the first eigenvalue of the problem (2)-(3) in the domain $D$ (indeed the first eigenfrequency of the clamped plate with domain $D$), $|S_D|$ is a length of $S_D$.

## 3. Main results

Thus we arrive to the shape optimization problem (2)-(5). As one can see the argument of the functional under minimization (4) is a domain. This is the principle difference of the shape optimization problems from the traditional ones. This fact generates serious difficulties in solution of such problems. The first one is the lack of existence of the optimal solutions (optimal shapes). But as above noted in some cases the solution exists due some geometrical restrictions on the domains, or the minimizing functional. In the case of shape optimization for the eigenvalues of some operators (in our case for the operator $\Delta^2 = \Delta\Delta$) the cost functional must depend on lower number of the eigenvalues. Another difficulty is related to the mathematically constructive description of the domain variation. Introduced by Cea J. and developed by Sokoowski J. and Zolesio J.-P. vector filed method allowed one to solve a wide class of functions.

This technique is based mainly on the calculation of the gradient or variations of the considered functionals with respect to domain. But to do this first the mathematical definition of the domain variation must be introduced. The idea of the vector field method is that the domain is varied in a preselected direction. In this point this method has certain disadvantages [3]. For example, it is problematic to establish a connection between the vector field and the set of admissible domains; to check: does satisfy the obtained after each iteration domain to the imposed conditions during the numerical solution of the problem; sometimes the obtained set of the "boundary" points does not form a boundary of any domain in general.

To overcome these and other difficulties in [14] a new definition of the domain variation is introduced using one-to-one mapping between bounded convex sets and continuous positive homogeneous convex functions. Thus the following statements are true. For any convex-bounded domain its support function is continuous convex and positive homogeneous. Also it is known that for each continuous convex positive homogeneous function there exists a convex bounded set, such that this function is a support function for this set. The set coincides with the sub-differential of this function at the origin [6]. This single-valued correspondence between domains and convex and positive homogeneous functions allows us to express the variation of the domain by the variation of the corresponding support function.

To do this it is shown that the set of such domains with boundaries from $C^2$ (we denote this set by $M$) forms a structure of linear space and one can even define a scalar product and norm in it. The obtained space we denote by $ML_2$. The variation of the domain then may be defined in this space and as is shown naturally replaced by the variation of the corresponding support function. Finally a shape derivative formula is derived for the integral cost functional in the class of bounded convex domains.

This approach allows one to avoid some of above noted disadvantages of the existing methods. For instance applying this method in the process of numerical simulation after each iteration we get not only a set of boundary points, but also the values of the support function. The domain then is reconstructed as a sub-differential of its support function in the point 0.

Note that this approach has been extended for more large classes of metrics and functions by other authors. For instance in [1] the shape derivative formula for the integral cost functional with respect to a class of admissible convex domains given in [12] is extended to the case of $W^{1,1}_{loc}$ functions. In [2] the obtained results are implemented in Brunn-Minkowski theory.

Let us give the definition of the variation of the functional in the space $ML_2$ referring to [14]. As is denoted $M = \{D \in R^n : \ S_D \in C^2\}$.

The functional $\lambda(D)$ is called differentiable in the Gateaux sense on $M$ in the direction $K_0$ if for any $D \in M$ there exists the limit

$$\delta\lambda\left(D_0, D\right) = \lim_{\varepsilon \to +0} \frac{\lambda\left(\left(1-\varepsilon\right) D_0 + \varepsilon D\right) - \lambda\left(D_0\right)}{\varepsilon}. \tag{6}$$

Here we give the theorems which will be used in further considerations.
In [13] the following theorem is proved for the first variation of the functional $\lambda_j\left(D\right)$

**Theorem 3.1.** *Eigenfrequency $\lambda_j\left(D\right)$ of the clamped plate under transverse vibrations is differentiable with respect to $D$ in the Gateaux sense on $M$ and the following formula is valid for the its first variation*

$$\delta\lambda_j(D) = -\max_{u_j} \int\limits_{S_D} \left(\Delta u_j\right)^2 \left[P_D(n(x)) - P_{D_0}(n(x))\right] ds$$

*where* $u_j(x)$ *is an eigenfunction (indeed an eigenvibration of the clamped plate) corresponding to the eigenvalue (eigenfrequency)* $\lambda_j$ *of the problem (2)-(3) in the domain* $D$, $P_D(x) = \max_{l\in D}(x,l)$ , $x \in R^n$ *is a support function of the domain* $D$, *max is taken over all eigenfunctions* $u_j$ *in the case of multiplicity of* $\lambda_j$.

Using this theorem the following important result is obtained in [9] for the eigenfrequency of the clamped plate.

**Theorem 3.2.** *For the eigenfrequency of the clamped plate in the domain* $D$ *the formula*

$$\lambda_j = \frac{1}{4}\max_{u_j}\int_{S_D}(u_j(x))^2 P_D(n(x))ds. \tag{7}$$

*is valid.*

Now we prove the following

**Theorem 3.3.** *Let* $D^* \in K$ *gives minimum to the functional* $\lambda_j(D)$. *Then for any* $D \in K$ *is valid the inequality*

$$\int_{S_D}(\Delta u_j(x))^2\left[P_D(n(x)) - P_{D^*}(n(x))\right]ds \le 0, \tag{8}$$

*where* $u_j^*(x)$ *is an eigenfunction corresponding to the eigenvalue* $\lambda_j^*$ *of the problem* (1)-(2) *in the domain* $D^*$, $P_D(x) = \max_{l\in D}(x,l)$ , $x \in R^n$ *is a support function of the domain* $D$, *max is taken over all eigenfunctions* $u_j^*$ *in the case of multiplicity of* $\lambda_j$.

*Proof.* Let us take arbitrary $D \in K$, $\varepsilon \in (0,1)$ and denote $D_\varepsilon = (1-\varepsilon)D^* + \varepsilon D$. Since $K$ is convex we can write $D_\varepsilon \in K$. Then according to the formula (8) for the eigenfrequency $\lambda_j(D)$ one can have

$$\lambda_j(D_\varepsilon) - \lambda_j(D^*) = -\max_{u_j}\textstyle\int_{S_{D_\varepsilon}}\left|\Delta u_j^*(x)\right|^2 \times$$

$$\times\left[P_{D_\varepsilon}(n(x)) - P_{D^*}(n(x))\right]ds + o(\varepsilon).$$

Considering

$$P_{D_\varepsilon} - P_{D^*} = (1-\varepsilon)P_{D^*} + \varepsilon P_D - P_{D^*} = \varepsilon(P_D - P_{D^*}),$$

from the last we obtain

$$\lambda_j(D_\varepsilon) - \lambda_j(D^*) = -\varepsilon\max_{u_j}\textstyle\int_{S_{D_\varepsilon}}\left|\Delta u_j^*(x)\right|^2 \times$$

$$\times\left[P_D(n(x)) - P_{D^*}(n(x))\right]ds + o(\varepsilon).$$

Since according to the statement of the theorem $D^*$ gives minimum to the functional $\lambda_j(D)$ the following inequality is true

$$\max_{u_j}\int_{S_{D_\varepsilon}}\left|\Delta u_j^*(x)\right|^2\left[P_D(n(x)) - P_{D^*}(n(x))\right]ds + \frac{o(\varepsilon)}{\varepsilon} \le 0. \tag{9}$$

From this by $\varepsilon \to +0$ we come to the statement of the theorem. □

For our case the condition (8) takes the form

$$\int_{S_{D^*}}\left|\Delta u_1^*(x)\right|^2 P_D(n(x))\,ds \le \int_{S_{D^*}}\left|\Delta u_1^*(x)\right|^2 P_{D^*}(n(x))\,ds, \tag{10}$$

Let us show that unit ball $B$ satisfies this condition. It is known that on the surface $S_B$ of $B$ is true

$$|\Delta u_1^*(x)| = const, \quad x \in S_B.$$

Considering this in (5) we get

$$\int_{S_B} P_D(n(x))\,ds \le \int_{S_B} P_B(n(x))\,ds.$$

In [6] is proved that

$$\int_{S_B} P_D(n(x))\,ds = \int_{S_D} dx = |S_D|\,. \tag{11}$$

From the other hand [6]

$$\int_{S_B} P_B(n(x))\,ds = 2\pi. \tag{12}$$

Since $|S_D| = 2\pi$ in the considered problem, from (11) and (12) we obtain that unit ball satisfies to the condition (10).

**Note.** As is shown in [7] in some cases $\lambda_j(D)$ is a quasi-convex functional with respect to $D$. In such cases (8) is also a sufficient condition.

## 4. Conclusion

In the paper a shape optimization problem is considered for the eigenfrequency of the clamped plate under across vibrations. The first eigenfrequency is considered as a domain functional. A necessary condition for the optimal domain is derived. A particular case is analyzed and optimal shape is found.

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