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INTEGRAL INEQUALITIES FOR DIFFERENTIABLE RELATIVE HARMONIC PREINVEX FUNCTIONS
(SURVEY)

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Abstract. In this paper, we consider and investigate the relative harmonic preinvex functions, which unifies several new known classes of harmonic preinvex functions. We derive several new integral inequalities such as Hermite-Hadamard, Simpson's, trapezoidal for the relative harmonic preinvex functions. Since the relative harmonic preinvex functions include, convex function, harmonic convex functions, preinvex functions, relative harmonic convex and relative preinvex functions as special cases, results obtained in this paper continue to hold for these problems. Several open problems have been suggested for further research in these areas.

Keywords: harmonic convex functions, preinvex functions, harmonic preinvex functions, h-convex functions, midpoint inequality, trapezoidal inequality, Simpson's inequality, Hermite-Hadamard type inequality.

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1. Introduction

It is well known that inequalities have played a fundamental role in the development of almost all the fields of pure and applied sciences and are continuing to do so. Inequalities present very active and fascinating field of research. In recent years, a wide class of integral inequalities are being derived via different concepts of convexity. These integral inequalities are useful in physics, where upper and lower bounds for natural phenomena described by integrals such as mechanical work (virtual work) are required. Integral inequalities are closely related the convex functions and their variant forms. As a result of interaction between different branches of mathematical and engineering sciences, convex functions have been extended and generalized in several directions from different point of views. The ideas and techniques of convex functions are being used in a variety of diverse areas of sciences and proved to be productive and innovative. These facts have inspired and motivated the researchers to generalize and extend the concept of convexity in various directions. The development of convexity theory can be viewed as the simultaneous pursuit of two different lines of research. It reveals the fundamental facts on the qualitative behavior of the solution to important classes of problems; on the other hand, it also helps us to develop us highly efficient and powerful new numerical techniques to solve complicated and complex problems. In fact, convexity theory provides us a sound basis for computing the approximate and analytical solutions of a large number of seemingly unrelated problems in a general and unified framework. For example, the variational inequalities, which can be regarded as a natural extension of variational principles, are related to the simple fact that the minimum of a differentiable convex function on a convex set in any normed space can be

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characterized by a variational inequality. However, it remarkable and amazing that variational inequalities allow many diversified applications in ever branch of pure and applied sciences. On other hand, a function is a convex function, if and only if, it satisfies the Hermite-Hadamard type inequality. To be more precise, a function f is a convex function on the interval $I=[a,b]$, if and only if, f satisfies the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2}, \qquad x\in[a,b], \tag{1}$$

which is called the Hermite-Hadamard inequality for convex functions, see [13, 11, 25]. For a novel applications of the Hermite-Hadamard inequality (1), see Khattri [15].

Convexity theory is an effective and powerful technique for studying a wide class of problems which arise in various branches of pure and applied sciences. Convex functions have been generalized and extended in several directions using interesting and novel ideas. Several new classes of convex functions and convex sets have been introduced and investigated. Various new inequalities related to these new classes of convex functions have been derived by researchers, see, for example, [4, 6-16, 18-21, 25-29, 31-33, 35-51, 56] and the reference therein. Motivated by the ongoing research in this field, Varosanec [54] introduced a class of convex functions with respect to an arbitrary non-negative function h. This class of convex function is commonly known as h-convex function. She has shown that this class contains several previously known classes of convex functions as special cases.

Hanson [12] introduced and investigated another class of generalized convex functions, which is called invex functions. Ben-Israel and Mond [5] introduced the concepts of invex sets and preinvex functions. They shown that the differentiable preinvex functions are invex functions, but the converse may not true. These preinvex functions are not convex functions, but they enjoy some nice properties, which convex functions have. Noor [27] proved that the minimum of the differentiable preinvex functions on the invex set can be characterized by a class of variational inequalities, which is called the variational-like-inequality or pre-variational inequality. It is known that the invex functions and preinvex functions are equivalent under some suitable conditions, see [35, 36]. For recent applications of the preinvex functions in different branches of pure and applied sciences, see [4, 19, 20, 21, 36, 32, 49, 35, 45]. In 2007, Noor [32] has shown that a function f is a preinvex function, if and only if, it satisfies the inequality of the type,

$$f(\frac{2a+\eta(b,a)}{2}) \leq \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx \leq \frac{f(a)+f(b)}{2}, \quad \forall a,b\in[a,a+\eta(b,a)], \tag{2}$$

which is called the Hermite-Hadamard-Noor type inequality. We remark that if $\eta(b,a)=b-a$, then both inequalities (1) and (2) are the same. Several integral inequalities for various types of preinvex functions have been obtained in recent years, see [47, 48, 49] and the references therein. Noor et. al. [49] introduced a class of preinvex functions with respect to an arbitrary function h. This class of h-preinvex functions is known as relative preinvx functions. They have shown that this class contains several previously known classes of preinvex functions and convex functions as special cases.

It is worth mentioning that the weighted arithmetic mean is used to define the convex set. Related to the arithmetic mean, we have harmonic mean, which has applications in electrical circuit theory and other branches of sciences. It is known that that the total resistance of a set of parallel resisters is obtained by adding up the reciprocal of the individual resistance value and then considering the reciprocal of their total. For example, if r_1 and r_2 are the resistance of two parallel resisters, then the total resistance

$$R = \frac{1}{\frac{1}{r_1} + \frac{1}{r_2}} = \frac{r_1 r_2}{r_1 + r_1},$$

which is half of the harmonic mean. Noor [34] has used harmonic mean to suggest some iterative methods for solving nonlinear equations. Anderson et al. [2] and Iscan [14]. have considered and studied some other properties of the harmonic convex functions. In particular, it has been shown by Isan[14] that a function is a harmonic convex, if and only if, it satisfies the inequality of the type

$$f(\frac{2ab}{a+b}) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}, \quad \forall a, b \in [a, b], \tag{3}$$

which is called the Hermite-Hadamard inequality for harmonic convex function. Noor and Noor [37] have shown that the optimality conditions of the differentiable harmonic convex functions on the harmonic convex set can be expressed by a class of variational inequalities, which is called the harmonic variational inequality. This shows that harmonic convex functions have similar properties, which convex functions have. This allows us to use the analogue results of the convex functions to suggest similar numerical methods for the harmonic convex functions. This is itself an interesting problem. Noor et al. [38] introduced the class of relative harmonic functions with respect to an arbitrary nonnegative function h. This class is more general and contains several known classes of harmonic convex functions as special cases. For the characterization and other aspects of the relative harmonic convex functions, see the references.

We would like to emphasize that relative preinvex functions and relative harmonic convex functions are two different extensions and generalizations of the relative convex functions. These classes are quite different in nature and have different applications. It is natural to unify these two classes of relative convex functions. Motivated and inspired by the ongoing research activities in this field, Noor et al. [47, 48] introduced and investigated a new class of convex functions with respective to an arbitrary nonnegative function, which is called relative harmonic preinvex functions. It is shown that this new class unifies several new classes of harmonic preinvex and harmonic convex functions such as Breckner type of s-harmonic preinvex functions, Godunova-Levin type of s-harmonic preinvex functions and harmonic P-preinvex functions.

We remark that the relative harmonic prienvex functions theory is quite broad, we shall content ourselves to give the flavor of the ideas and techniques involved. The techniques used to establish the results are a beautiful blend of ideas of pure and applied mathematical sciences. In this paper, we have presented the results regarding the derivation of integral inequalities such as Hermite-Hadamard, trapezoidal, Simpson's for the relative harmonic preinvex functions. We have included several new results which we and our coworkers have recently obtained such as several Hermite-Hadamard, trapezoidal and Simpson's type inequalities for the relative harmonic preinvex functions. The framework chosen should be seen as a model setting for general results for other classes of convex functions. It is true that each of these areas of applications requires special consideration of peculiarities of the class of convex functions at hand and the inequalities that model it. However, many of the concepts and techniques, we have discussed are fundamental to all of these applications. General and unified framework are important and significant scientific value, both as a means of summarization existing and to provide ideas and tools for explaining relationship and performing investigation. In this paper, we focus basically on presenting the state-art-of generalizing convexity and invexity by means of the harmonic means. We would like to mention that the results obtained and discussed in this paper may motivate and bring a large number of novel, potential applications, extensions, generalizations and interesting topics in these dynamical fields. We have given only the basic and fundamental concepts in this dynamic field.

2. Preliminaries and basic results

In this section, we recall the basic concepts and results in the convex analysis. For more details, see [25, 50] and the references therein. Let K be set in the finite dimensions Euclidean space $\mathbb{R}^n$, whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively.

Definition 2.1. *A set K in $\mathbb{R}^n$ is said to be a convex set, if and only if,*

$$(1-t)u + tv \in K, \quad \forall u, v \in K, \quad t \in [0,1].$$

Definition 2.2. *A function f on the convex set K is said to be a convex function, if and only if,*

$$f((1-t)u + tv) \leq (1-t)f(u) + tf(v), \quad \forall u, v \in K, \quad t \in [0,1].$$

If $t = \frac{1}{2}$, the definition 1.2 reduces to;

$$f(\frac{u+v}{2}) \leq \frac{f(u)+f(v)}{2}, \quad \forall u, v \in K,$$

which is called Jensen convex function.

It is known that a function f is a convex function on the interval $I = [a,b]$, if and only if, it satisfies the Hermite-Hadamard inequality (1).
For the differentiable convex function, we have the following celebrated result.

Theorem 2.1. *Let K be nonempty convex set in $\mathbb{R}^n$. and let f be a differentiable convex function on the set K. Then $u \in K$ is the minimum of f, if and only if $u \in K$ satisfies the inequality*

$$\langle f'(u), v - u \rangle \geq 0, \quad \forall v \in K. \tag{4}$$

The inequality of the type (4) is known as variational inequality, which was introduced and investigated by Stampacchia [53] in 1964. This shows that the variational inequalities are connected with the theory of convex functions. Variational inequalities can be regarded as natural extension of the variational principles, the origin of which can be traced back to Euler, Newton, Bernoulli brothers and Lagrange. Variational inequalities provides us a unified and general framework to study a wide class of unrelated problems, which arise in pure, applied, regional, economics, transportation, structural analysis, game theory and engineering sciences, see [3, 16, 30, 28, 29, 34, 36, 37, 32, 35, 45, 53].

We now recall the concept of relative convex functions with respective to an arbitrary non-negative function h, which is due to [54].

Definition 2.3. [54]. *Let $h : [0,1] \subseteq J \to \mathbb{R}$ be a non-negative function. A function $f : I = [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ is a relative convex function with respect to an arbitrary function h, if*

$$f\big((1-t)x + ty) \leq h(1-t)f(x) + h(t)f(y), \qquad \forall x, y \in I, t \in [0,1].$$

It is clear that, if $h(t) = t$, then the relative convex functions are exactly the convex functions. For different, appropriate and suitable choice of the arbitrary function h, one can obtain several classes of convex functions, which are being investigated by the researchers.

We now consider an other important class of convex functions, which is known as preinvex functions.

Definition 2.4. [12]. *A set $K_\eta \subseteq \mathbb{R}$ is said to be invex set with respect to the bifunction $\eta(\cdot,\cdot)$, if and only if,*

$$x + t\eta(y,x) \in I, \quad \forall x, y \in K_\eta, \quad t \in [0,1].$$

The invex set I is also called η-connected set. Note that, if $\eta(b,a) = b - a$, then invex set becomes the convex set. Clealry, every convex set is an invex set, but the converse is not true.

Definition 2.5. [55]. *Let K_η be an invex set in $\mathbb{R}^n$. Then a function $f : K_\eta \subseteq \mathbb{R} \to \mathbb{R}$ is said to be preinvex function with respect to the bifunction $\eta(\cdot,\cdot)$, if and only if,*

$$f\big(x + t\eta(y,x)\big) \leq (1-t)f(x) + tf(y), \qquad \forall x, y \in K_\eta, \quad t \in [0,1].$$

Noor[32] has shown that a function f is preinvex function, if and only if, it satisfies the Hermite-Hadamrd-Noor inequality (2).
For differentiable preinvex functions, we have the following result, which is mainly due to Noor [27].

Theorem 2.2. *Let K_η be nonempty inves set in $\mathbb{R}^n$. and let f be a differentiable preinvex function on the set K_η. Then $u \in K_\eta$ is the minimum of f, if and only if $u \in K_\eta$ satisfies the inequality*

$$\langle f'(u), \eta(v,u)\rangle \geq 0, \quad \forall v \in K_\eta. \tag{5}$$

Inequality of type (5) is called the variational-like inequality. For the applications and other properties of the variational-like inequalities, see [27, 31, 36].

We now consider concepts of the harmonic convex set and harmonic convex functions. For more information, see Anderson et al.[2] and Iscan [14].

Definition 2.6. [14]. *A set $K_h \subset \mathbb{R} \setminus \{0\}$ is said to a harmonic convex set, if and only if,*

$$\frac{uv}{v + t(u-v)} \in K_h, \quad \forall u, v \in K_h, \quad t \in [0,1].$$

Definition 2.7. [14]. *A function $f : K_h \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic convex function, if and only if,*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq (1-t)f(x) + tf(y), \qquad \forall x, y \in K_h, \quad t \in [0,1].$$

Noor and Noor [37] have shown that the minimum of a differentiable harmonic convex functions f on the harmonic convex set can be characterized by a class of variational inequalities, which is called harmonic variational inequalities. In this direction, we have the following result.

Theorem 2.3. *Let K_h be nonempty harmonic convex set in $\mathbb{R}^n$. and let f be a differentiable harmonic convex function on the set K_h. Then $u \in K_h$ is a minimum of f, if and only if, $u \in K_h$ satisfies the inequality*

$$\langle f'(u), \frac{uv}{u-v}\rangle \geq 0, \quad \forall v \in K_h. \tag{6}$$

The inequality of the type (6) is called the harmonic variational inequality, see Noor and Noor [34, 37].

Noor et al.[38] and Mihai et al.[23] introduced the relative harmonic convex functions with respect to an arbitrary nonnegative function,

Definition 2.8. *Let $h : [0,1] \subseteq J \to \mathbb{R}$ be a non-negative function. a function on the harmonic convex set K_h is said to relative harmonic convex functions with respect to the nonnegative function h, if and only if,*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq h(1-t)f(x) + h(t)f(y), \qquad \forall x, y \in K_h, \quad t \in [0,1].$$

If $h(t) = t$, then the relative harmonic convex functions become harmonic convex functions. For appropriate choice and suitable choice of the function h, one can obtain several new classes of harmonic convex functions. This shows that the relative harmonic convex functions are quite general and unifying ones.

From the above discussion, it follows that the relative preinvex functions and the relative harmonic convex functions are two different generalization of the relative convex functions. Noor et al . [47, 48] introduced a new class of relative convex functions, which includes all these classes of convex functions as special cases.

Definition 2.9. [48]. *A set $I = [a, a+\eta(b,a)] \subseteq \mathbb{R} \setminus \{0\}$ is said to be a harmonic invex set with respect to the bifunction $\eta(\cdot,\cdot)$, if*

$$\frac{x(x+\eta(y,x))}{x+(1-t)\eta(y,x)} \in I, \quad \forall x, y \in I, t \in [0,1].$$

If $\eta(y,x) = y - x$, then harmonic invex set reduces to harmonic set. Clearly, every harmonic set is invex set but the converse is not true.

We now introduce the concept of the relative harmonic preinvex functions, which are mainly due to Noor et al [47, 48].

Definition 2.10. *Let $h : [0,1] \subseteq J \to \mathbb{R}$ be a non-negative function. A function $f : I = [a, a+\eta(b,a)] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is a relative harmonic preinvex function with respect to an arbitrary nonnegative function h and an arbitrary bifunction $\eta(\cdot,\cdot)$, if*

$$f\left(\frac{x(x+\eta(y,x))}{x+(1-t)\eta(y,x)}\right) \leq h(1-t)f(x) + h(t)f(y), \qquad \forall x, y \in I, t \in [0,1]. \tag{7}$$

Note that for $t = \frac{1}{2}$, we have Jensen type relative harmonic preinvex function.

$$f\left(\frac{2x(x+\eta(y,x))}{2x+\eta(y,x)}\right) \leq h(\frac{1}{2})[f(x)+f(y)], \qquad \forall x, y \in I.$$

We now discuss some special cases of Definition 2.10, which appear to be new ones.

I. If $h(t) = t$ in (7), then Definition 2.10 reduce to the definition of harmonic preinvex functions.

Definition 2.11. [48]. *A function $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic preinvex function with respect to $\eta(\cdot,\cdot)$, if*

$$f\left(\frac{x(x+\eta(y,x))}{x+(1-t)\eta(y,x)}\right) \leq (1-t)f(x) + tf(y), \qquad \forall x, y \in I, t \in [0,1].$$

II. If $h(t) = t^s$ in (7), then Definition 2.10 reduces to the definition of Breckner type of s-harmonic preinvex functions.

Definition 2.12. *A function $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be s-harmonic preinvex function with respect to $\eta(\cdot,\cdot)$ and $s \in (0,1]$, if*

$$f\left(\frac{x(x+\eta(y,x))}{x+(1-t)\eta(y,x)}\right) \leq (1-t)^s f(x) + t^s f(y), \qquad \forall x, y \in I, t \in [0,1].$$

III. If $h(t) = t^{-s}$ in (7), then Definition 2.10 reduces to the definition of Godunova-Levin type of s-harmonic preinvex functions.

Definition 2.13. *A function $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be Godunova-Levin type of s-harmonic preinvex function, where $s \in (0,1)$ with respect to $\eta(\cdot,\cdot)$, if*

$$f\left(\frac{x(x+\eta(y,x))}{x+(1-t)\eta(y,x)}\right) \leq \frac{1}{(1-t)^s} f(x) + \frac{1}{t^s} f(y), \qquad \forall x, y \in I, t \in (0,1).$$

It is obvious that for $s = 0$, s-harmonic Godunova-Levin preinvex functions reduces to harmonic P-preinvex functions. If $s = 1$, s-harmonic Godunova-Levin preinvex functions reduces to harmonic Godunova-Levin preinvex functions.

V. If $h(t) = 1$ in (7), then Definition 2.10 reduces to the definition of harmonic P-preinvex functions.

Definition 2.14. *A function $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic P-preinvex function with respect to $\eta(\cdot,\cdot)$, if*

$$f\left(\frac{x(x+\eta(y,x))}{x+(1-t)\eta(y,x)}\right) \leq f(x) + f(y), \qquad \forall x, y \in I, t \in [0,1].$$

Definition 2.15. [50]. *Two functions f, g are said to be similarly ordered (f is g-monotone), if and only if,*

$$\langle f(x) - f(y), g(x) - g(y) \rangle \geq 0, \qquad \forall x, y \in \mathbb{R}^n.$$

Noor et al [47, 48] have proved that the pproduct of two relative harmonic preinvex functions is again a relative preinvex function.

Lemma 2.1. *Let f and g be two similarly ordered relative harmonic preinvex functions. If $h(t) + h(1-t) \leq 1$, then the product fg is again a relative harmonic preinvex function.*

We need the following assumption about the bifunction $\eta(.,.)$, which is due to Mohan and Neogy[24]. This assumption have been used to prove the existence of a solution of variational-like inequalities. We use to prove the left hand of the Hermire-Hadamard type inequalities;

Condition *C*: Let $I \subset \mathbb{R}$ be an invex set with respect to bifunction $\eta(\cdot,\cdot) : I \times I \to \mathbb{R}$. For any $x, y \in I$ and any $t \in [0,1]$, we have

$$\begin{aligned} \eta(y, y + t\eta(x,y)) &= -t\eta(x,y) \\ \eta(x, y + t\eta(x,y)) &= (1-t)\eta(x,y). \end{aligned}$$

Note that for every $x, y \in I$, $t_1, t_2 \in [0,1]$ and from condition C, we have

$$\eta(y + t_2\eta(x,y), y + t_1\eta(x,y)) = (t_2 - t_1)\eta(x,y).$$

It is remarked that this condition is automatically satisfied for the convex functions.

We now derive Hermite-Hadamard inequalities for relative harmonic preinvex functions.

Theorem 2.4. *Let $f : I = [a, a+\eta(b,a)] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be relative harmonic preinvex function, where $a, a+\eta(b,a) \in I$ with $a < a+\eta(b,a)$. If $f \in L[a, a+\eta(b,a)]$ and condition C holds, then*

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) \leq \frac{a(a+\eta(b,a))}{\eta(b,a)} \int_a^{a+\eta(b,a)} \frac{f(x)}{x^2} \mathrm{d}x \leq [f(a) + f(b)] \int_0^1 h(t)\mathrm{d}t. \quad (8)$$

Proof. Let f be relative harmonic preinvex function with $t = \frac{1}{2}$ in the inequality (7). Then

$$f\left(\frac{2x(x+\eta(y,x)}{2x+\eta(y,x)}\right) \leq h(\frac{1}{2})[f(x) + f(y)], \qquad \forall x, y \in I, t \in [0,1].$$

Let

$$x = \frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)},$$

$$y = \frac{a(a+\eta(b,a))}{a+t\eta(b,a)}.$$

Then, using condition C, we have

$$\frac{1}{h(\frac{1}{2})} f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) \leq \left[f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right) + f\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right)\right].$$

Integrating both sides of the above inequality with respect to t over $[0,1]$, we have

$$\begin{aligned}\frac{1}{2h(\frac{1}{2})} f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) &\leq \frac{a(a+\eta(b,a))}{\eta(b,a)} \int_a^{a+\eta(b,a)} \frac{f(x)}{x^2}\mathrm{d}x \\ &= \int_0^1 f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\mathrm{d}t \\ &\leq \int_0^1 \left[h(1-t)f(a) + h(t)f(a+\eta(b,a))\right]\mathrm{d}t \\ &= [f(a)+f(a+\eta(b,a))]\int_0^1 h(t)\mathrm{d}t \leq [f(a)+f(b)]\int_0^1 h(t)\mathrm{d}t,\end{aligned}$$

which is the required result. □

Now we discuss some special cases of Theorem 2.4, which appear to be new ones.

I. If $h(t) = t$, then Theorem 2.4 reduces to the following result.

Corollary 2.1. [48]. *Let $f : I = [a, a+\eta(b,a)] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be harmonic preinvex function. If $f \in L[a, a+\eta(b,a)]$, then*

$$f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) \leq \frac{a(a+\eta(b,a))}{\eta(b,a)} \int_a^{a+\eta(b,a)} \frac{f(x)}{x^2}\mathrm{d}x \leq \frac{f(a)+f(b)}{2}.$$

II. If $h(t) = t^s$, then Theorem 2.4 reduces to the following result.

Corollary 2.2. *Let $f : I = [a, a+\eta(b,a)] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be s-harmonic preinvex function. If $f \in L[a, a+\eta(b,a)]$, then*

$$2^{s-1} f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) \leq \frac{a(a+\eta(b,a))}{\eta(b,a)} \int_a^{a+\eta(b,a)} \frac{f(x)}{x^2}\mathrm{d}x \leq \frac{f(a)+f(b)}{s+1}.$$

III. If $h(t) = t^{-s}$, then Theorem 2.4 reduces to the following result.

Corollary 2.3. *Let $f : I = [a, a+\eta(b,a)] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be s-harmonic Godunova-Levin preinvex function. If $f \in L[a, a+\eta(b,a)]$, then*

$$\frac{1}{2^{s+1}} f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) \leq \frac{a(a+\eta(b,a))}{\eta(b,a)} \int_a^{a+\eta(b,a)} \frac{f(x)}{x^2}\mathrm{d}x \leq \frac{f(a)+f(b)}{1-s}.$$

IV. If $h(t) = 1$, then Theorem 2.4 reduces to the following result.

Corollary 2.4. *Let $f : I = [a, a+\eta(b,a)] \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be harmonic P-preinvex function. If $f \in L[a, a+\eta(b,a)]$, then*

$$\frac{1}{2} f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) \leq \frac{a(a+\eta(b,a))}{\eta(b,a)} \int_a^{a+\eta(b,a)} \frac{f(x)}{x^2}\mathrm{d}x \leq f(a)+f(b).$$

3. Main results

We need the following result, which plays an important role in the derivation of the main results.

Lemma 3.1. *Let $f : I = [a, a+\eta(b,a)] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, a+\eta(b,a)]$ and $\lambda \in [0,1]$, then*

$$\begin{aligned}
&\Upsilon_f(a, a+\eta(b,a);\lambda)\\
=\ &\frac{a(a+\eta(b,a))\eta(b,a)}{2}\Bigg[\int_0^{\frac{1}{2}} \frac{2t-\lambda}{\big(a+(1-t)\eta(b,a)\big)^2} f'\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\mathrm{d}t\\
&+\int_{\frac{1}{2}}^{1} \frac{2t-2+\lambda}{(a+(1-t)\eta(b,a))^2} f'\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\mathrm{d}t\Bigg],
\end{aligned}$$

where

$$\begin{aligned}
&\Upsilon_f(a, a+\eta(b,a);\lambda)\\
=\ &(1-\lambda)f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)+\lambda\left(\frac{f(a)+f(a+\eta(b,a))}{2}\right)\\
&-\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{a+\eta(b,a)} \frac{f(x)}{x^2}\mathrm{d}x.
\end{aligned}$$

Proof. Let

$$\begin{aligned}
I\ =\ &\frac{a(a+\eta(b,a))\eta(b,a)}{2}\int_0^{\frac{1}{2}} \frac{2t-\lambda}{\big(a+(1-t)\eta(b,a)\big)^2} f'\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\mathrm{d}t\\
&+\int_{\frac{1}{2}}^{1} \frac{2t-2+\lambda}{(a+(1-t)\eta(b,a))^2} f'\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\mathrm{d}t\Bigg]\\
=\ &\frac{a(a+\eta(b,a))\eta(b,a)}{2}\int_0^{\frac{1}{2}} \frac{2t-\lambda}{\big(a+(1-t)\eta(b,a)\big)^2} f'\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\mathrm{d}t\\
&+\frac{a(a+\eta(b,a))\eta(b,a)}{2}\int_{\frac{1}{2}}^{1} \frac{2t-2+\lambda}{\big(a+(1-t)\eta(b,a)\big)^2} f'\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\mathrm{d}t\\
=\ &I_1+I_2.
\end{aligned}$$

Now

$$\begin{aligned}
I_1 &= \frac{a(a+\eta(b,a))\eta(b,a)}{2}\int_0^{\frac{1}{2}} \frac{2t-\lambda}{\big(a+(1-t)\eta(b,a)\big)^2} f'\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\mathrm{d}t\\
&= \frac{1}{2}\left|(2t-\lambda)f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\right|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\mathrm{d}t\\
&= \frac{1-\lambda}{2}f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)+\frac{\lambda}{2}f(a) - \int_0^{\frac{1}{2}} f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\mathrm{d}t.
\end{aligned}$$

Similarly, one can show that

$$\begin{aligned}
I_2 &= \frac{a(a+\eta(b,a))\eta(b,a)}{2}\int_{\frac{1}{2}}^{1} \frac{2t-2+\lambda}{(a+(1-t)\eta(b,a))^2} f'\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\mathrm{d}t\\
&= \frac{\lambda}{2}f(a+\eta(b,a))+\frac{1-\lambda}{2}f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) - \int_{\frac{1}{2}}^{1} f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\mathrm{d}t.
\end{aligned}$$

Thus

$$\begin{aligned} I_1 + I_2 &= (1-\lambda) f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) + \lambda\left(\frac{f(a)+f(a+\eta(b,a))}{2}\right) \\ &\quad - \frac{a(a+\eta(b,a))}{\eta(b,a)} \int_a^{a+\eta(b,a)} \frac{f(x)}{x^2} \mathrm{d}x, \end{aligned}$$

which is the required result. □

Theorem 3.1. *Let $f : I = [a, a+\eta(b,a)] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, a+\eta(b,a)]$ and $|f'|^q$ is relative harmonic preinvex function on I for $q \geq 1$ and $\lambda \in [0,1]$, then*

$$\begin{aligned} \left|\Upsilon_f(a, a+\eta(b,a); \lambda)\right| &\leq \frac{a(a+\eta(b,a))\eta(b,a)}{2}\Big[(\kappa_1(a,b;\lambda))^{1-\frac{1}{q}}[\kappa_2(a,b;\lambda,h)|f'(a)|^q \\ &\quad + \kappa_3(a,b;\lambda,h)|f'(b)|^q]^{\frac{1}{q}} + (\kappa_4(a,b;\lambda))^{1-\frac{1}{q}}[\kappa_5(a,b;\lambda,h)|f'(a)|^q \\ &\quad + \kappa_6(a,b;\lambda,h)|f'(b)|^q]^{\frac{1}{q}}\Big], \end{aligned}$$

where one can evaluate these integrals using any mathematical software (i.e, maple).

$$\kappa_1(a,b;\lambda) = \int_0^{\frac{1}{2}} \frac{\left|2t-\lambda\right|}{(a+(1-t)\eta(b,a))^2} \mathrm{d}t, \tag{9}$$

$$\kappa_2(a,b;\lambda,h) = \int_0^{\frac{1}{2}} \frac{h(1-t)\left|2t-\lambda\right|}{(a+(1-t)\eta(b,a))^2} \mathrm{d}t, \tag{10}$$

$$\kappa_3(a,b;\lambda,h) = \int_0^{\frac{1}{2}} \frac{h(t)\left|2t-\lambda\right|}{(a+(1-t)\eta(b,a))^2} \mathrm{d}t, \tag{11}$$

$$\kappa_4(a,b;\lambda) = \int_{\frac{1}{2}}^{1} \frac{\left|2t-2+\lambda\right|}{(a+(1-t)\eta(b,a))^2} \mathrm{d}t, \tag{12}$$

$$\kappa_5(a,b;\lambda,h) = \int_{\frac{1}{2}}^{1} \frac{h(1-t)\left|2t-2+\lambda\right|}{(a+(1-t)\eta(b,a))^2} \mathrm{d}t, \tag{13}$$

$$\kappa_6(a,b;\lambda,h) = \int_{\frac{1}{2}}^{1} \frac{h(t)\left|2t-2+\lambda\right|}{(a+(1-t)\eta(b,a))^2} \mathrm{d}t. \tag{14}$$

Proof. Using Lemma 3.1 and the power mean inequality, we have

$$
\begin{aligned}
&\Big|\Upsilon_f(a,a+\eta(b,a);\lambda)\Big|\\
\leq\ & \frac{a(a+\eta(b,a))\eta(b,a)}{2}\Bigg[\int_0^{\frac{1}{2}}\frac{|2t-\lambda|}{\big(a+(1-t)\eta(b,a)\big)^2}\Bigg|f'\Bigg(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\Bigg)\Bigg|\mathrm{d}t\\
&+\int_{\frac{1}{2}}^1\frac{|2t-2+\lambda|}{\big(a+(1-t)\eta(b,a)\big)^2}\Bigg|f'\Bigg(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\Bigg)\Bigg|\mathrm{d}t\Bigg]\\
\leq\ & \frac{a(a+\eta(b,a))\eta(b,a)}{2}\Bigg[\Bigg(\int_0^{\frac{1}{2}}\frac{|2t-\lambda|}{\big(a+(1-t)\eta(b,a)\big)^2}\mathrm{d}t\Bigg)^{1-\frac{1}{q}}\\
&\Bigg(\int_0^{\frac{1}{2}}\frac{|2t-\lambda|}{\big(a+(1-t)\eta(b,a)\big)^2}\Bigg|f'\Bigg(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\Bigg)\Bigg|^q\mathrm{d}t\Bigg)^{\frac{1}{q}}\\
&+\Bigg(\int_{\frac{1}{2}}^1\frac{|2t-2+\lambda|}{\big(a+(1-t)\eta(b,a)\big)^2}\mathrm{d}t\Bigg)^{1-\frac{1}{q}}\\
&\Bigg(\int_{\frac{1}{2}}^1\frac{|2t-2+\lambda|}{\big(a+(1-t)\eta(b,a)\big)^2}\Bigg|f'\Bigg(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\Bigg)\Bigg|\mathrm{d}t\Bigg)^{\frac{1}{q}}\Bigg]\\
\leq\ & \frac{a(a+\eta(b,a))\eta(b,a)}{2}\Bigg[\Bigg(\int_0^{\frac{1}{2}}\frac{|2t-\lambda|}{\big(a+(1-t)\eta(b,a)\big)^2}\mathrm{d}t\Bigg)^{1-\frac{1}{q}}\\
&\Bigg(\int_0^{\frac{1}{2}}\frac{|2t-\lambda|}{\big(a+(1-t)\eta(b,a)\big)^2}\big[h(1-t)|f'(a)|^q+h(t)|f'(b)|^q\big]\mathrm{d}t\Bigg)^{\frac{1}{q}}\\
&+\Bigg(\int_{\frac{1}{2}}^1\frac{|2t-2+\lambda|}{\big(a+(1-t)\eta(b,a)\big)^2}\mathrm{d}t\Bigg)^{1-\frac{1}{q}}\\
&\Bigg(\int_{\frac{1}{2}}^1\frac{|2t-2+\lambda|}{\big(a+(1-t)\eta(b,a)\big)^2}\big[h(1-t)|f'(a)|^q+h(t)|f'(b)|^q\big]\mathrm{d}t\Bigg)^{\frac{1}{q}}\Bigg]\\
\leq\ & \frac{a(a+\eta(b,a))\eta(b,a)}{2}\Big[(\kappa_1(a,b;\lambda))^{1-\frac{1}{q}}[\kappa_2(a,b;\lambda,h)|f'(a)|^q+\kappa_3(a,b;\lambda,h)|f'(b)|^q]^{\frac{1}{q}}\\
&+(\kappa_4(a,b;\lambda))^{1-\frac{1}{q}}[\kappa_5(a,b;\lambda,h)|f'(a)|^q+\kappa_6(a,b;\lambda,h)|f'(b)|^q]^{\frac{1}{q}}\Big],
\end{aligned}
$$

which is the required result. □

If $q=1$, then, Theorem 3.1 reduces to the following result, which appears to be a new one.

Corollary 3.1. *Let $f : I = [a, a+\eta(b,a)] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, a+\eta(b,a)]$ and $|f'|$ is relative harmonic preinvex function on I and $\lambda \in [0,1]$, then*

$$
\begin{aligned}
\Big|\Upsilon_f(a,a+\eta(b,a);\lambda)\Big| \leq\ & \frac{a(a+\eta(b,a))\eta(b,a)}{2}\big[[\kappa_2(a,b;\lambda,h)+\kappa_3(a,b;\lambda,h)]|f'(a)|\\
&+[\kappa_5(a,b;\lambda,h)+\kappa_6(a,b;\lambda,h)]|f'(b)|\big],
\end{aligned}
$$

where $\kappa_2(a,b;\lambda,h), \kappa_3(a,b;\lambda,h), \kappa_5(a,b;\lambda,h), \kappa_6(a,b;\lambda,h)$ are given by (10), (11), (13) *and*(14) *respectively.*

Theorem 3.2. *Let $f : I = [a, a+\eta(b,a)] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, a+\eta(b,a)]$ and $|f'|^q$ is relative harmonic preinvex function on I for $p, q > 1$, $\frac{1}{p}+\frac{1}{q}=1$ and $\lambda \in [0,1]$, then*

$$\begin{aligned}
&\Big|\Upsilon_f(a, a+\eta(b,a);\lambda)\Big| \\
\leq\ & \frac{a(a+\eta(b,a))\eta(b,a)}{2}\Bigg[(\kappa_7(a,b;p,\lambda))^{\frac{1}{p}}\Bigg(\frac{|f'(a)|^q + \big|f'\big(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\big)\big|^q}{2}\int_0^1 h(t)\mathrm{d}t\Bigg)^{\frac{1}{q}} \\
&+(\kappa_8(a,b;p,\lambda))^{\frac{1}{p}}\Bigg(\frac{\big|f'\big(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\big)\big|^q + |f'(b)|^q}{2}\int_0^1 h(t)\mathrm{d}t\Bigg)^{\frac{1}{q}}\Bigg], \qquad (15)
\end{aligned}$$

where

$$\kappa_7(a,b;p,\lambda) = \int_0^{\frac{1}{2}} \frac{|2t-\lambda|^p}{(a+(1-t)\eta(b,a))^2}\mathrm{d}t, \qquad (16)$$

$$\kappa_8(a,b;p,\lambda) = \int_{\frac{1}{2}}^1 \frac{|2t-2+\lambda|^p}{(a+(1-t)\eta(b,a))^2}\mathrm{d}t. \qquad (17)$$

Proof. Using Lemma 3.1 and the Holder's integral inequality, we have

$$\begin{aligned}
&\Big|\Upsilon_f(a, a+\eta(b,a);\lambda)\Big| \\
\leq\ & \frac{a(a+\eta(b,a))\eta(b,a)}{2}\Bigg[\int_0^{\frac{1}{2}} \frac{\big|2t-\lambda\big|}{(a+(1-t)\eta(b,a))^2}\Bigg|f'\Bigg(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\Bigg)\Bigg|\mathrm{d}t \\
&+\int_{\frac{1}{2}}^1 \frac{\big|2t-2+\lambda\big|}{(a+(1-t)\eta(b,a))^2}\Bigg|f'\Bigg(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\Bigg)\Bigg|\mathrm{d}t\Bigg] \\
\leq\ & \frac{a(a+\eta(b,a))\eta(b,a)}{2}\Bigg[\Bigg(\int_0^{\frac{1}{2}} \frac{\big|2t-\lambda\big|^p}{(a+(1-t)\eta(b,a))^2}\mathrm{d}t\Bigg)^{\frac{1}{p}}\Bigg(\Bigg|f'\Bigg(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\Bigg)\Bigg|^q\mathrm{d}t\Bigg)^{\frac{1}{q}} \\
&+\Bigg(\int_{\frac{1}{2}}^1 \frac{\big|2t-2+\lambda\big|^p}{(a+(1-t)\eta(b,a))^2}\mathrm{d}t\Bigg)^{\frac{1}{p}}\Bigg(\Bigg|f'\Bigg(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\Bigg)\Bigg|^q\mathrm{d}t\Bigg)^{\frac{1}{q}}\Bigg] \\
=\ & \frac{a(a+\eta(b,a))\eta(b,a)}{2}\Bigg[\Bigg(\int_0^{\frac{1}{2}} \frac{\big|2t-\lambda\big|^p}{(a+(1-t)\eta(b,a))^2}\mathrm{d}t\Bigg)^{\frac{1}{p}} \\
&\Bigg(\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}} \frac{|f'(x)|^q}{x^2}\mathrm{d}x\Bigg)^{\frac{1}{q}} \\
&+\Bigg(\int_{\frac{1}{2}}^1 \frac{\big|2t-2+\lambda\big|^p}{(a+(1-t)\eta(b,a))^2}\mathrm{d}t\Bigg)^{\frac{1}{p}}\Bigg(\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_{\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}}^{a+\eta(b,a)} \frac{|f'(x)|^q}{x^2}\mathrm{d}t\Bigg)^{\frac{1}{q}}\Bigg]. \qquad (18)
\end{aligned}$$

Using the relative harmonic preinvexity of $|f'|^q$, we obtain the following inequalities from inequality (8)

$$\frac{2a(a+\eta(b,a))}{\eta(b,a)}\int_a^{\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}} \frac{|f'(x)|^q}{x^2}\mathrm{d}x \leq \Bigg[|f'(a)|^q + \Bigg|f'\Bigg(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\Bigg)\Bigg|^q\Bigg]\int_0^1 h(t)\mathrm{d}t, \qquad (19)$$

and

$$\frac{2a(a+\eta(b,a))}{\eta(b,a)}\int_{\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}}^{a+\eta(b,a)} \frac{|f'(x)|^q}{x^2}\mathrm{d}x \leq \Bigg[\Bigg|f'\Bigg(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\Bigg)\Bigg|^q + |f'(b)|^q\Bigg]\int_0^1 h(t)\mathrm{d}t. \qquad (20)$$

A combination of (18)-(20) gives the required inequality (15). $\square$

Theorem 3.3. *Let $f : I = [a, a + \eta(b, a)] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, a + \eta(b, a)]$ and $|f'|^q$ is relative harmonic preinvex function on I for $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\lambda \in [0, 1]$, then*

$$\begin{aligned}
&\Big|\Upsilon_f(a, a + \eta(b, a); \lambda)\Big| \\
\leq\ & \frac{a(a + \eta(b, a))\eta(b, a)}{2} \times \left(\frac{\lambda^{p+1} + (1 - \lambda)^{p+1}}{2(p + 1)}\right)^{\frac{1}{p}} \big[\big(\kappa_9(a, b; q, h)|f'(a)|^q \\
+\ & \kappa_{10}(a, b; q, h)|f'(b)|^q\big)^{\frac{1}{q}} + \big(\kappa_{11}(a, b; q, h)|f'(a)|^q + \kappa_{12}(a, b; q, h)|f'(b)|^q\big)^{\frac{1}{q}}\big],
\end{aligned}$$

where

$$\kappa_9(a, b; q, h) = \int_0^{\frac{1}{2}} \frac{h(1 - t)}{(a + (1 - t)\eta(b, a))^{2q}}\mathrm{d}t, \tag{21}$$

$$\kappa_{10}(a, b; q, h) = \int_0^{\frac{1}{2}} \frac{h(t)}{(a + (1 - t)\eta(b, a))^{2q}}\mathrm{d}t, \tag{22}$$

$$\kappa_{11}(a, b; q, h) = \int_{\frac{1}{2}}^1 \frac{h(1 - t)}{(a + (1 - t)\eta(b, a))^{2q}}\mathrm{d}t, \tag{23}$$

$$\kappa_{12}(a, b; q, h) = \int_{\frac{1}{2}}^1 \frac{h(t)}{(a + (1 - t)\eta(b, a))^{2q}}\mathrm{d}t. \tag{24}$$

Proof. Using Lemma 3.1 and the Holder's integral inequality, we have

$$\begin{aligned}
&\Big|\Upsilon_f(a, a + \eta(b, a); \lambda)\Big| \\
\leq\ & \frac{a(a + \eta(b, a))\eta(b, a)}{2}\left[\int_0^{\frac{1}{2}} |2t - \lambda|\left|\frac{f'\big(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\big)}{[a + (1 - t)\eta(b, a)]^2}\right|\mathrm{d}t\right. \\
&\left.+\int_{\frac{1}{2}}^1 |2t - 2 + \lambda|\left|\frac{f'\big(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\big)}{[a + (1 - t)\eta(b, a)]^2}\right|\mathrm{d}t\right] \\
\leq\ & \frac{a(a + \eta(b, a))\eta(b, a)}{2}\left[\left(\int_0^{\frac{1}{2}} |2t - \lambda|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\left|\frac{f'\big(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\big)}{[a + (1 - t)\eta(b, a)]^2}\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1 |2t - 2 + \lambda|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\left|\frac{f'\big(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\big)}{[a + (1 - t)\eta(b, a)]^2}\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right] \\
=\ & \frac{a(a + \eta(b, a))\eta(b, a)}{2}\left[\left(\int_0^{\frac{1}{2}} |2t - \lambda|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\frac{\big|f'\big(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\big)\big|^q}{[a + (1 - t)\eta(b, a)]^{2q}}\mathrm{d}t\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1 |2t - 2 + \lambda|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\frac{\big|f'\big(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\big)\big|^q}{[a + (1 - t)\eta(b, a)]^{2q}}\mathrm{d}t\right)^{\frac{1}{q}}\right]
\end{aligned}$$

$$\leq \frac{a(a+\eta(b,a))\eta(b,a)}{2}\Bigg[\Bigg(\int_0^{\frac{1}{2}}\big|2t-\lambda\big|^p\mathrm{d}t\Bigg)^{\frac{1}{p}}\Bigg(\frac{h(1-t)|f'(a)|^q+h(t)|f'(b)|^q}{[a+(1-t)\eta(b,a)]^{2q}}\Bigg)^{\frac{1}{q}}$$

$$+\Bigg(\int_{\frac{1}{2}}^1\big|2t-2+\lambda\big|^p\mathrm{d}t\Bigg)^{\frac{1}{p}}\Bigg(\frac{h(1-t)|f'(a)|^q+h(t)|f'(b)|^q}{[a+(1-t)\eta(b,a)]^{2q}}\Bigg)^{\frac{1}{q}}\Bigg]$$

$$= \frac{a(a+\eta(b,a))\eta(b,a)}{2}\times\Bigg(\frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{2(p+1)}\Bigg)^{\frac{1}{p}}\Big[\big(\kappa_9(a,b;q,h)|f'(a)|^q$$

$$+\kappa_{10}(a,b;q,h)|f'(b)|^q\big)^{\frac{1}{q}}+\big(\kappa_{11}(a,b;q,h)|f'(a)|^q+\kappa_{12}(a,b;q,h)|f'(b)|^q\big)^{\frac{1}{q}}\Big],$$

which is the required result. □

Remark 3.1.

(1) For $h(t) = t$, $h(t) = t^s$, $h(t) = t^{-s}$ and $h(t) = 1$, the class of relative harmonic preinvex functions reduces to the class of harmonic preinvex functions, s-harmonic convex functions, s-harmonic Godunova-Levin functions and harmonic P-preinvex functions respectively. This shows that the class of relative harmonic preinvex functions is quite general and unifying one. Consequently, results obtained in this paper continue to hold for all these new classes of harmonic preinvex functions.

(2) With the suitable and appropriate choice value of λ, one can obtain integral inequalities for midpoint, trapezoidal, Simpson's rule and three point trapezoidal rule, respectively. We leave this to the interested readers.

Conclusion and future research

In this paper, we have presented the state-of-the art in convexity theory and several aspects of relative harmonic preinvex functions. These new concepts are very recent ones and offer great opportunities for future research. It is expected that the interplay among all these classes of convex functions will certainly lead to some innovative, interesting and significant results. In this paper, our main aim have been to describe the basic ideas and techniques, which have used to derive the integral inequalities, the foundation, we have laid is quite broad and flexible. Quantum calculus is branch of mathematics and has many applications in physics. Noor et al. [40, 41] has established some quantum estimates for the convex functions. These quantum Hermite-Hadamard inequalities and their variant forms are useful for quantum mechanics where upper and lower bounds for natural phenomena described by integrals are frequently required. In spite of their importance, little research has been carried out in this direction. To the best of knowledge, no quantum estimates have been derived for harmonic convex functions. This field s new one and efforts are needed to develop a sound basis for applications. The study of these aspects of the integral inequalities is fruitful and growing field of intellectual endeavor.

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