TWMS J. Pure Appl. Math., V.8, N.1, 2017, pp.64-82

CHARACTERIZATIONS OF BI-HYPERIDEALS AND PRIME BI-HYPERIDEALS IN ORDERED KRASNER HYPERRINGS

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Abstract. In this paper, we introduce the concepts of bi-hyperideals and quasi-hyperideals of an ordered Krasner hyperring and present several examples of them. Some properties of bi-hyperideals and quasi-hyperideals in ordered Krasner hyperrings are provided. Moreover, we introduce and analyze the notion of prime bi-hyperideal of an ordered Krasner hyperring. Finally, we discuss some properties of topological bi-hyperideals in ordered Krasner hyperrings.

Keywords: algebraic hyperstructure, ordered Krasner hyperring, quasi-hyperideal, bi-hyperideal, prime bi-hyperideal, regular.

AMS Subject Classification: 16Y99.

1. Introduction and summary

Krasner's hyperring introduced and studied by M. Krasner is a triple $(R, +, \cdot)$ where $(R, +)$ is a canonical hypergroup, $(R, \cdot)$ is a semigroup and the operation $\cdot$ is distributive over the hyperoperation $+$, which means that for all x, y, z of R we have: $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$. We call $(R, +, \cdot)$ a Krasner hyperfield if $(R, +, \cdot)$ is a Krasner hyperring and $(R \setminus \{0\}, \cdot)$ is a group. Some principal notions of hyperring theory can be found in [2, 10, 13, 14, 29, 31]. We invite the readers to [12] to see more about the hyperring theory.

The concept of ideals in ring (semigroup) theory plays the same role as normal subgroups in group theory. Moreover, there exist several kinds of ideals. One of the important kinds of ideals is bi-ideals. Since the concept of ordered Krasner hyperring is a generalizations of the concepts of ring, ordered semigroup and ordered semihypergroup, it is a natural question to ask what does happen for bi-ideals if we consider an ordered Krasner hyperring instead of a ring, ordered semigroup or ordered semihypergroup. The answer to this question is our main motivation to investigate the notion of bi-hyperideals in ordered Krasner hyperrings.

In mathematics, an *ordered semigroup* [4] is a semigroup $(S, \cdot)$ together with a partial order $\leq$ that is compatible with the semigroup operation, meaning that for all $a, b, x \in S$, $a \leq b$ implies that $a \cdot x \leq b \cdot x$ and $x \cdot a \leq x \cdot b$. The concept of a bi-ideal is a very interesting and important thing in semigroups and ordered semigroups. R. A. Good and D. R. Hughes [15] introduced the notion of bi-ideals of a semigroup as early as 1952. Later, bi-ideals of ordered semigroups were studied by many authors, for example, see [20, 21, 30]. We mean by a *bi-ideal* is a subsemigroup A of a semigroup $(S, \cdot)$ such that $ASA \subseteq A$. A subset A of a ring R is called a bi-ideal [26] of R if (1) A is a subring of R and (2) $ARA \subseteq A$. It is easy to see that bi-ideals are a generalization of left (right) ideals. The notion of quasi-ideal was first introduced by O. Steinfeld [28] for rings and semigroups as a generalization of the one-sided ideal. Some principal notions of quasi-ideal theory can be found in [17, 19, 22, 27]. By a *quasi-ideal* of a semigroup

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Manuscript received July 2016.

$(S,\cdot)$ we mean a subsemigroup Q of S satisfying $SQ \cap QS \subseteq Q$.

In [16], D. Heidari and B. Davvaz studied a semihypergroup $(H,\circ)$ besides a binary relation $\leq$, where $\leq$ is a partial order relation such that satisfies the monotone conditin. Indeed, an *ordered semihypergroup* $(H,\circ,\leq)$ is a semihypergroup $(H,\circ)$ together with a partial order $\leq$ that is compatible with the hyperoperation, meaning that for any $x,y,z \in H$,

$$x \leq y \Rightarrow z\circ x \leq z\circ y \text{ and } x\circ z \leq y\circ z.$$

Here, $z\circ x \leq z\circ y$ means for any $a \in z\circ x$ there exists $b \in z\circ y$ such that $a \leq b$. The case $x\circ z \leq y\circ z$ is defined similarly.

J. Chvalina [8] have started the concept of ordered semihypergroups in 1994 as a special class of hypergroups. The concept of ordered semihypergroups is a generalization of the concept of ordered semigroups. The concept of ordering hypergroups extended by many authors, for example, see [8, 9, 16, 18].

In the rest of this section, we provide a brief account of the topic of hyperstructure. Hyperstructure theory was first initiated by F. Marty [24], in 1934 at the 8th Congress of Scandinavian Mathematicians, when he defined the hypergroups and began to investigate their properties with applications to groups, rational fractions and algebraic functions. Hyperstructures have many applications to several sectors of both pure and applied sciences. A comprehensive review of the theory of hyperstructures appears in [6, 7, 11, 12, 29]. Let H be a non-empty set. A mapping $\circ : H\times H \to \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ denotes the family of all non-empty subsets of H, is called a *hyperoperation* on H. The couple $(H,\circ)$ is called a *hyperstructure*. In the above definition, if A and B are two non-empty subsets of H and $x \in H$, then we denote

$$A\circ B = \bigcup_{\substack{a\in A\\ b\in B}} a\circ b,\ A\circ x = A\circ\{x\} \text{ and } x\circ B = \{x\}\circ B.$$

A hyperstructure $(H,\circ)$ is called a *semihypergroup* if for all $x,y,z \in H$, $(x\circ y)\circ z = x\circ(y\circ z)$, which means that

$$\bigcup_{u\in x\circ y} u\circ z = \bigcup_{v\in y\circ z} x\circ v.$$

A non-empty subset K of a semihypergroup $(H,\circ)$ is called a *subsemihypergroup* of H if $K\circ K \subseteq K$. Let $(H,\circ)$ be a semihypergroup. Then, H is called a *hypergroup* if it satisfies the reproduction axiom, for all $x \in H$, $H\circ x = x\circ H = H$. A hypergroup $(H,\circ)$ is called *commutative* if $a\circ b = b\circ a$ for every $a,b \in H$. A non-empty subset K of a hypergroup $(H,\circ)$ is called a *subhypergroup* of H if itself is a hypergroup under hyperoperation $\circ$ restricted to K. It is clear that a subset K of H is a subhypergroup if and only if $K\circ a = a\circ K = K$, under the hyperoperation on H.

2. Review of basic notions

In this section, we give some definitions of the basic notions of Krasner hyperrings, which are necessary for the subsequent section.

Definition 2.1. [25] *A canonical hypergroup is a non-empty set R endowed with an additive hyperoperation $+ : R\times R \to \mathcal{P}^*(R)$, satisfying the following properties:*

1. *For any $x,y,z \in R$, $x+(y+z) = (x+y)+z$;*
2. *For any $x,y \in R$, $x+y = y+x$;*
3. *There exists $0 \in R$ such that $0+x = x$, for any $x \in R$;*
4. *For every $x \in R$, there exists a unique element $x' \in R$ such that $0 \in x+x'$; (we shall write $-x$ for x' and we call it the opposite of x.)*
5. *$z \in x+y$ implies that $y \in -x+z$ and $x \in z-y$, that is $(R,+)$ is reversible.*

The following equalities follow easily from the axioms: $-(-a) = a$, $-(a+b) = -a-b$ and $a + R = R$, for all $a \in R$.

Definition 2.2. [23] *A Krasner hyperring is an algebraic hypersructure* $(R, +, \cdot)$ *which satisfies the following axioms:*

(1) $(R, +)$ *is a canonical hypergroup;*
(2) $(R, \cdot)$ *is a semigroup having zero as a bilaterally absorbing element, i.e.,* $x \cdot 0 = 0 = 0 \cdot x$;
(3) *The multiplication* $\cdot$ *is distributive with respect to the hyperoperation* $+$, *i.e.,* $x \cdot (y+z) = x \cdot y + x \cdot z$ *and* $(x+y) \cdot z = x \cdot z + y \cdot z$ *for all* x, y, z *of* R.

The element 0 is called the *zero element* or simply the *zero* of the Krasner hyperring $(R, +, \cdot)$. It can be easily proved that zero is unique. For $x \in R$, let $-x$ denote the unique inverse of x in $(R, +)$. Then, $-(-x) = x$ and $-(x+y) = -x-y$, where $-A = \{-a \mid a \in A\}$ for all $x, y \in R$. In addition, we have $(x+y) \cdot (z+w) \subseteq x \cdot z + x \cdot w + y \cdot z + y \cdot w$, $(-x) \cdot y = x \cdot (-y) = -(x \cdot y)$, for all $x, y, z, w \in R$. A Krasner hyperring R is called commutative (with unit element) if $(R, \cdot)$ is a commutative semigroup (with unit element). R is called with identity, if there exists an element, say $1 \in R$, such that $1 \cdot x = x = x \cdot 1$, for all $x \in R$. A *Krasner hyperfield* is a Krasner hyperring for which $(R \setminus \{0\}, \cdot)$ is a group.

A *subhyperring* of a Krasner hyperring $(R, +, \cdot)$ is a non-empty subset T of R which forms a Krasner hyperring containing 0 under the hyperoperation $+$ and the operation $\cdot$ on R, that is, T is a canonical subhypergroup of $(R, +)$ and $T \cdot T \subseteq T$. Then a non-empty subset T of R is a subhyperring of $(R, +, \cdot)$ if and only if for all $x, y \in T$, $x + y \subseteq T$, $-x \in T$ and $x \cdot y \in T$.

A non-empty subset I of a Krasner hyperring $(R, +, \cdot)$ is called a *left* (resp. *right*) *hyperideal* of R if $(I, +)$ is a canonical subhypergroup of $(R, +)$ and for every $a \in I$ and $r \in R$, $r \cdot a \in I$ (resp. $a \cdot r \in I$). I is called a *hyperideal* if I is both left and right hyperideal. That is, $a + b \subseteq I$ and $-a \in I$, for all $a, b \in I$ and $a \cdot r, r \cdot a \in I$, for all $a \in I$ and $r \in R$.

A *homomorphism* from a Krasner hyperring $(R_1, +_1, \cdot_1)$ into a Krasner hyperring $(R_2, +_2, \cdot_2)$ is a function $\varphi : R_1 \to R_2$ such that we have: (1) $\varphi(a +_1 b) \subseteq \varphi(a) +_2 \varphi(b)$, (2) $\varphi(a \cdot_1 b) = \varphi(a) \cdot_2 \varphi(b)$. Also φ is called a good (strong) homomorphism if in the previous condition (1), the equality is valid. An *isomorphism* from $(R_1, +_1, \cdot_1)$ into $(R_2, +_2, \cdot_2)$ is a bijective good homomophism from $(R_1, +_1, \cdot_1)$ onto $(R_2, +_2, \cdot_2)$.

Definition 2.3. [1] $(R, +, \cdot)$ *is a partially ordered ring if* R *has a partial order* $\leq$ *satisfying the following conditions:*

(1) *For all* $a, b, c \in R$, $a \leq b$ *implies that* $a + c \leq b + c$.
(2) *For all* $a, b, c \in R$, $a \leq b$ *and* $0 \leq c$ *implies that* $a \cdot c \leq b \cdot c$.

Remark 2.1. *It is easy to verify that* R *has a relation* $\leq$ *satisfying the following conditions:*

(1) $a \geq 0$ *and* $-a \geq 0$ *if and only if* $a = 0$,
(2) $a, b \geq 0$ *implies that* $a + b \geq 0$,
(3) $a, b \geq 0$ *implies that* $a \cdot b \geq 0$.

3. On quasi-hyperideals and bi-hyperideals in ordered Krasner hyperrings

Ordered algebraic structures such as ordered groups, ordered semigroups and ordered rings have been widely studied. For more details on ordered algebraic structures we refer to [5]. Ordered polygroups was introduced in a paper of Bakhshi and Borzooei [3]. Ordered hyperstructures are studied by Heidari and Davvaz [16]. In the following, we deal with ordered Krasner hyperrings.

Definition 3.1. *An algebraic hypersructure $(R, +, \cdot, \leq)$ is called an ordered Krasner hyperring if $(R, +, \cdot)$ is a Krasner hyperring with a partial order relation $\leq$ such that for all $a, b, c \in R$:*

(1) *If $a \leq b$, then $a + c \leq b + c$, meaning that for any $x \in a + c$, there exists $y \in b + c$ such that $x \leq y$.*
(2) *If $a \leq b$ and $0 \leq c$, then $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$.*

An element $x \in R$ is called *positive* if $0 \leq x$. The set of all positive elements of R is called the *positive cone* of R and is denoted by $P = R^+$. $x \in R$ is called *negative* if $x \leq 0$. The set of all negative elements of R is called the *negative cone* of R and is denoted by R^-. If P is the positive cone of an ordered Krasner hyperring, then $P \cap (-P) = \{0\}$, $P + P \subseteq P$, and $P \cdot P \subseteq P$. Note that every Krasner hyperring is an ordered Krasner hyperring for the trivial order.

Definition 3.2. *Let $(R, +, \cdot, \leq)$ be an ordered Krasner hyperring. A non-empty subset I of R is called a left (resp. right) hyperideal of R if it satisfies the following conditions:*

(1) *$(I, +)$ is a canonical subhypergroup of $(R, +)$;*
(2) *$R \cdot I \subseteq I$ (resp. $I \cdot R \subseteq I$);*
(3) *If $a \in I$ and $b \in R$ such that $b \leq a$, then $b \in I$.*

By *two-sided hyperideal* or simply *hyperideal*, we mean a non-empty subset I of R which is both a left and a right hyperideal of R.

Definition 3.3. *Let $(R_1, +_1, \cdot_1, \leq_1)$ and $(R_2, +_2, \cdot_2, \leq_2)$ be two ordered Krasner hyperrings and $\varphi : R_1 \to R_2$ be an isotone function, that is, $a, b \in R_1$, $a \leq_1 b$ implies $\varphi(a) \leq_2 \varphi(b)$. Then,*

(1) *φ is said to be an order homomorphism if φ is a homomorphism of Krasner hyperrings $(R_1, +_1, \cdot_1)$ and $(R_2, +_2, \cdot_2)$.*
(2) *φ is said to be an order isomorphism if φ is an isomorphism of Krasner hyperrings and φ^{-1} is isotone.*

Also φ is called a good (strong) order homomorphism if φ is a good (strong) homomorphism of Krasner hyperrings $(R_1, +_1, \cdot_1)$ and $(R_2, +_2, \cdot_2)$.

Example 3.1. *Let $R = \{a, b, c, d\}$ be a set with the hyperaddition $\oplus$ and the multiplication $\odot$ defined as follows:*

$\oplus$	a	b	c	d
a	a	b	c	d
b	b	R	$\{b, c, d\}$	$\{b, c, d\}$
c	c	$\{b, c, d\}$	R	$\{b, c, d\}$
d	d	$\{b, c, d\}$	$\{b, c, d\}$	R

and

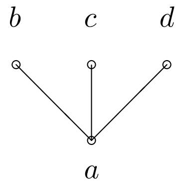
$\odot$	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	d	b
d	a	d	b	c

Then, $(R, \oplus, \odot)$ is a Krasner hyperfield. We have $(R, \oplus, \odot, \leq)$ is an ordered Krasner hyperfield where the order relation $\leq$ is defined by:

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (a, b), (a, c), (a, d)\}.$$

The covering relation and the figure of R are given by:

$$\prec = \{(a, b), (a, c), (a, d)\}.$$



Lemma 3.1. *Let* $(R_1, +_1, \cdot_1, \leq_1)$ *and* $(R_2, +_2, \cdot_2, \leq_2)$ *be two ordered Krasner hyperrings. Then,*

(1) *If* $\varphi : R_1 \to R_2$ *is an order homomorphism, then we have* $\varphi({R_1}^+) \subseteq {R_2}^+$.
(2) *If* $\varphi : R_1 \to R_2$ *is a good (strong) hyperring homomorphism and* $\varphi({R_1}^+) \subseteq {R_2}^+$, *then* φ *is isotone.*

Proof. (1): If $x \in {R_1}^+$, i.e., $0 \leq_1 x$ then $0_{R_2} = \varphi(0_{R_1}) \leq_2 \varphi(x)$. This means that $\varphi(x) \in {R_2}^+$. Hence $\varphi({R_1}^+) \subseteq {R_2}^+$.

(2): Assume that $x \leq_1 y$. Then, $-x+y \subseteq {R_1}^+$ and so $-\varphi(x)+\varphi(y) = \varphi(-x+y) \subseteq \varphi({R_1}^+) \subseteq {R_2}^+$. Hence $\varphi(x) \leq_2 \varphi(y)$. This implies that φ is isotone. □

Definition 3.4. *Let* $(R_1, +_1, \cdot_1, \leq_1)$ *and* $(R_2, +_2, \cdot_2, \leq_2)$ *be two ordered Krasner hyperrings. A function* $\varphi : R_1 \to R_2$ *is said to be exact if* $\varphi({R_1}^+) = {R_2}^+$. *Also* R_1 *is strongly isomorphic to* R_2 *if there is a good (strong) order isomorphism* $\varphi : R_1 \to R_2$. *If* R_1 *is strongly isomorphic to* R_2, *then it is denoted by* $R_1 \cong R_2$.

Theorem 3.1. *Let* $(R_1, +_1, \cdot_1, \leq_1)$ *and* $(R_2, +_2, \cdot_2, \leq_2)$ *be two ordered Krasner hyperrings. Then, the following assertions are equivalent:*

(1) $R_1 \cong R_2$.
(2) *There is an exact hyperring isomorphism* $\varphi : R_1 \to R_2$.

Proof. (1) ⇒ (2): Assume that (1) holds. Then, there is a good (strong) order isomorphism $\varphi : R_1 \to R_2$. By (1) of Lemma 3.1, $\varphi({R_1}^+) \subseteq {R_2}^+$. Let $\psi = \varphi^{-1}$. Then, ψ satisfies the condition (1) of Lemma 3.1 and so $\psi({R_2}^+) \subseteq {R_1}^+$. Hence ${R_2}^+ = \varphi(\psi({R_2}^+)) \subseteq \varphi({R_1}^+)$. Thus $\varphi({R_1}^+) = {R_2}^+$.

(2) ⇒ (1): This proof is straightforward. □

Our aim in the following is to introduce and study the concept of a quasi-hyperideal of ordered Krasner hyperrings.

Definition 3.5. *A non-empty subset* Q *of an ordered Krasner hyperring* $(R, +, \cdot, \leq)$ *is called a quasi-hyperideal of* R *if the following conditions hold:*

(1) $(Q, +)$ *is a canonical subhypergroup of* $(R, +)$;
(2) $(Q \cdot R) \cap (R \cdot Q) \subseteq Q$;
(3) *When* $x \in Q$ *and* $y \in R$ *such that* $y \leq x$, *imply that* $y \in Q$.

Example 3.2. *Let $R = \{a, b, c, d, e, f, g, h\}$ be a set with the hyperaddition $\oplus$ and the multiplication $\odot$ defined as follows:*

$\oplus$	a	b	c	d	e	f	g	h
a	a	b	c	d	e	f	g	h
b	b	$\{a,b\}$	d	$\{c,d\}$	f	$\{e,f\}$	h	$\{g,h\}$
c	c	d	$\{a,e\}$	$\{b,f\}$	$\{c,g\}$	$\{d,h\}$	e	f
d	d	$\{c,d\}$	$\{b,f\}$	$\{a,b,e,f\}$	$\{d,h\}$	$\{c,d,g,h\}$	f	$\{e,f\}$
e	e	f	$\{c,g\}$	$\{d,h\}$	$\{a,e\}$	$\{b,f\}$	c	d
f	f	$\{e,f\}$	$\{d,h\}$	$\{c,d,g,h\}$	$\{b,f\}$	$\{a,b,e,f\}$	d	$\{c,d\}$
g	g	h	e	f	c	d	a	b
h	h	$\{g,h\}$	f	$\{e,f\}$	d	$\{c,d\}$	b	$\{a,b\}$

and

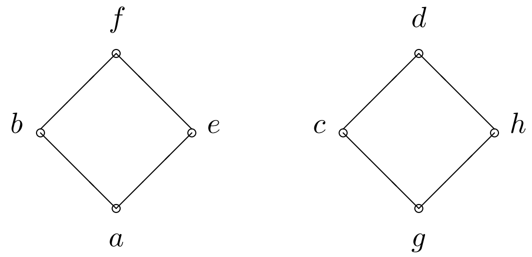
$\odot$	a	b	c	d	e	f	g	h
a	a	a	a	a	a	a	a	a
b	a	b	a	b	a	b	a	b
c	a	a	c	c	e	e	g	g
d	a	b	c	d	e	f	g	h
e	a	a	e	e	e	e	a	a
f	a	b	e	f	e	f	a	b
g	a	a	g	g	a	a	g	g
h	a	b	g	h	a	b	g	h

Then, $(R, \oplus, \odot)$ is a Krasner hyperring. We have $(R, \oplus, \odot, \leq)$ is an ordered Krasner hyperring where the order relation $\leq$ is defined by:

$$\begin{aligned} \leq \ := \{&(a,a),(b,b),(c,c),(d,d),(e,e),(f,f),(g,g),\\ &(h,h),(a,b),(a,e),(a,f),(b,f),(c,d),(e,f),\\ &(g,c),(g,d),(g,h),(h,d)\}. \end{aligned}$$

The covering relation and the figure of R are given by:

$$\prec \ = \{(a,b),(a,e),(b,f),(c,d),(e,f),(g,c),(g,h),(h,d)\}.$$



It is easy to see that $\{a\}$, $\{a,b\}$, $\{a,e\}$, $\{a,g\}$, $\{a,b,e,f\}$, $\{a,b,g,h\}$, $\{a,c,e,g\}$ and $\{a,b,c,d,e,f,g,h\}$ are quasi-hyperideals of R.

Every left, right and two-sided hyperideal of an ordered Krasner hyperring R is a quasi-hyperideal of R. The converse is not true, in general, that is, a quasi-hyperideal may not be a left, right or a two-sided hyperideal of R.

Example 3.3. *Let $R = \{a, b, c, d\}$ be a set with a hyperoperation $\oplus$ and a binary operation $\odot$ as follows:*

$\oplus$	a	b	c	d
a	a	b	c	d
b	b	$\{a, b\}$	d	c
c	c	d	$\{a, c\}$	b
d	d	c	b	$\{a, d\}$

and

$\odot$	a	b	c	d
a	a	a	a	a
b	a	b	b	b
c	a	c	c	c
d	a	d	d	d

Then, $(R, \oplus, \odot)$ is a Krasner hyperring [31]. We have $(R, \oplus, \odot, \leq)$ is an ordered Krasner hyperring where the order relation $\leq$ is defined by:

$$\leq := \{(a, a), (b, b), (c, c), (d, d)\}.$$

Now, it is easy to see that $Q_1 = \{a, b\}$, $Q_2 = \{a, c\}$ and $Q_3 = \{a, d\}$ are quasi-hyperideals of R, but they are not left hyperideals of R.

Proposition 3.1. *Let $(R, +, \cdot, \leq)$ be an ordered Krasner hyperring. Then, the following statements are hold:*

(1) *Every quasi-hyperideal of R is a subhyperring of R.*
(2) *The intersection of quasi-hyperideals set of R is a quasi-hyperideal of R.*
(3) *If I is a left hyperideal and J a right hyperideal of R, then $Q = I \cap J$ is a quasi-hyperideal of R.*
(4) *If Q is a quasi-hyperideal of R and T is a subhyperring of R, then $Q \cap T$ is a quasi-hyperideal of T.*

Proof. (1): Let Q be a quasi-hyperideal of an ordered Krasner hyperring $(R, +, \cdot, \leq)$. Then, $Q \cdot Q \subseteq R \cdot Q$ and $Q \cdot Q \subseteq Q \cdot R$. Hence $Q \cdot Q \subseteq R \cdot Q \cap Q \cdot R \subseteq Q$. Therefore, Q is a subhyperring of R.

(2): Let $\{Q_k : k \in \Lambda\}$ be a family of quasi-hyperideals of R and $Q = \bigcap_{k \in \Lambda} Q_k$. Since $0 \in \bigcap_{k \in \Lambda} Q_k$, it follows that $\bigcap_{k \in \Lambda} Q_k \neq \emptyset$. We show that Q is a quasi-hyperideal of R. Let $x, y \in Q$. Then $x, y \in Q_k$ for every $k \in \Lambda$. By assumption, we have $x + y \subseteq Q_k$ and $-x \in Q_k$ for each $k \in \Lambda$. So we have $x + y \subseteq Q$ and $-x \in Q$. Thus $(Q, +)$ is a canonical subhypergroup of $(R, +)$. Also for all Q_k, $k \in \Lambda$, we have $(R \cdot Q) \cap (Q \cdot R) \subseteq (R \cdot Q_k) \cap (Q_k \cdot R) \subseteq Q_k$. Now, let $x \in Q$ and $y \in R$ such that $y \leq x$. Then for every $k \in \Lambda$, $y \in Q_k$. Hence $y \in Q$. Therefore, $Q = \bigcap_{k \in \Lambda} Q_k$ is a quasi-hyperideal of R.

(3): Let $x, y \in Q = I \cap J$. Then $x, y \in I$ and $x, y \in J$. So we have $x + y \subseteq I \cap J = Q$ and $-x \in I \cap J = Q$. Thus $(Q, +)$ is a canonical subhypergroup of $(R, +)$. Since I is a left hyperideal and J a right hyperideal of R, we have $IJ \subseteq I$ and $IJ \subseteq J$. So $IJ \subseteq I \cap J$. Thus $I \cap J = Q \neq \emptyset$. Also we have

$$(R \cdot Q) \cap (Q \cdot R) = (R \cdot (I \cap J)) \cap ((I \cap J) \cdot R) \subseteq (R \cdot I) \cap (J \cdot R) \subseteq I \cap J = Q.$$

Now, let $x \in Q$ and $y \in R$ such that $y \leq x$. Then, we have $x \in I$ and $x \in J$. So $y \in I$ and $y \in J$. Hence $y \in I \cap J = Q$. Therefore, Q is a quasi-hyperideal of R.

(4): Let $Q_1 = Q \cap T$. We show that Q_1 is a quasi-hyperideal of T. Clearly Q_1 is a canonical subhypergroup of T. Since $Q_1 \subseteq Q$, it follows that $(Q_1 \cdot T) \cap (T \cdot Q_1) \subseteq (Q \cdot R) \cap (R \cdot Q) \subseteq Q$. Since $Q_1 \subseteq T$ and T is a subhyperring of R, we have $(Q_1 \cdot T) \cap (T \cdot Q_1) \subseteq T \cdot T \subseteq T$. So we have checked that $(Q_1 \cdot T) \cap (T \cdot Q_1) \subseteq Q_1$. If $x \in Q_1$ and $y \in T$ such that $y \leq x$, then since $x \in Q$, it follows that $y \in Q$. Hence $y \in Q_1$. Therefore, Q_1 is a quasi-hyperideal of T. □

In the following, we proceed with the study of bi-hyperideals in ordered Krasner hyperrings and give its characterizations.

Definition 3.6. *Let $(R, +, \cdot, \leq)$ be an ordered Krasner hyperring. A non-empty subset A of R is called a bi-hyperideal of R if the following conditions hold:*

(1) *$(A, +)$ is a canonical subhypergroup of $(R, +)$ and $A \cdot A \subseteq A$;*
(2) *$A \cdot R \cdot A \subseteq A$;*
(3) *When $x \in A$ and $y \in R$ such that $y \leq x$, imply that $y \in A$.*

Example 3.4. *Let $R = \{a, b, c\}$ be a set with the hyperaddition $\oplus$ and the multiplication $\odot$ defined as follows:*

$\oplus$	a	b	c
a	a	b	c
b	b	R	b
c	c	b	$\{a, c\}$

and

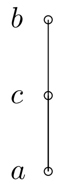
$\odot$	a	b	c
a	a	a	a
b	a	b	c
c	a	c	a

Then, $(R, \oplus, \odot)$ is a Krasner hyperring. We have $(R, \oplus, \odot, \leq)$ is an ordered Krasner hyperring where the order relation $\leq$ is defined by:

$$\leq := \{(a, a), (b, b), (c, c), (a, b), (a, c), (c, b)\}.$$

The covering relation and the figure of R are given by:

$$\prec = \{(a, c), (c, b)\}.$$



It is easy to see that $\{a\}$, $\{a, c\}$ and $\{a, b, c\}$ are bi-hyperideals of R.

Example 3.5. *Let $R = \{a, b, c, d, e, f\}$ be a set with the hyperaddition $\oplus$ and the multiplication $\odot$ defined as follows:*

$\oplus$	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	$\{a,b\}$	d	$\{c,d\}$	f	$\{e,f\}$
c	c	d	$\{a,c,e\}$	$\{b,d,f\}$	c	d
d	d	$\{c,d\}$	$\{b,d,f\}$	R	d	$\{c,d\}$
e	e	f	c	d	$\{a,e\}$	$\{b,f\}$
f	f	$\{e,f\}$	d	$\{c,d\}$	$\{b,f\}$	$\{a,b,e,f\}$

and

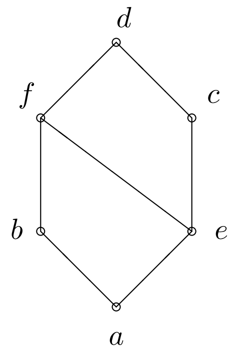
$\odot$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	b	a	b	a	b
c	a	a	c	c	e	e
d	a	b	c	d	e	f
e	a	a	e	e	a	a
f	a	b	e	f	a	b

Then, $(R, \oplus, \odot)$ is a Krasner hyperring. We have $(R, \oplus, \odot, \leq)$ is an ordered Krasner hyperring where the order relation $\leq$ is defined by:

$$\begin{aligned} \leq \ := & \{(a,a),(b,b),(c,c),(d,d),(e,e),(f,f),(a,b), \\ & (a,c),(a,d),(a,e),(a,f),(b,d),(b,f),(c,d), \\ & (e,c),(e,d),(e,f),(f,d)\}. \end{aligned}$$

The covering relation and the figure of R are given by:

$$\prec \ = \{(a,b),(a,e),(b,f),(c,d),(e,c),(e,f),(f,d)\}.$$



It is easy to see that $\{a\}$, $\{a, b\}$, $\{a, e\}$, $\{a, c, e\}$, $\{a, b, e, f\}$ and $\{a, b, c, d, e, f\}$ are bi-hyperideals of R.

The concept of bi-hyperideals of an ordered Krasner hyperring is a generalization of the concept of hyperideals (left hyperideals, right hyperideals) of an ordered Krasner hyperring. Obviously, every left (resp. right) hyperideal of an ordered Krasner hyperring R is a bi-hyperideal of R, but the converse need not be true. Indeed, If A is a left (right) hyperideal of R, then $(A, +)$ is a canonical subhypergroup of $(R, +)$. Since $AA \subseteq RA \subseteq A$, it follows that A is a subhyperring of R.

Example 3.6. *Let $(R, +, \cdot)$ be a Krasner hyperring and $M(R) = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in R \right\}$ be a collection of 2×2 matrices over R. The hyperaddition $\oplus$ and the multiplication $\odot$ are defined on $M(R)$ by:*

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} : x \in a + c, y \in b + d \right\},$$
$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \odot \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} ac & ad \\ 0 & 0 \end{pmatrix},$$

for all $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \in M(R)$. Then, $(M(R), \oplus, \odot)$ is a Krasner hyperring [2]. *Moreover, $(M(R), \oplus, \odot, \preceq)$ is an ordered Krasner hyperring, where $A = (a_{ij}) \preceq B = (b_{ij}) \Leftrightarrow a_{ij} = b_{ij}$ for all $1 \leqslant i, j \leqslant 2$. Here $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in R \right\}$ is a bi-hyperideal of $M(R)$, but it is not a right hyperideal of $M(R)$.*

Lemma 3.2. *Let $(R, +, \cdot, \leq)$ be an ordered Krasner hyperring. Then,*

(1) *If A_k is a bi-hyperideal of R for all $k \in \Lambda$, then $\bigcap\limits_{k\in\Lambda} A_k$ is a bi-hyperideal of R.*
(2) *If $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$ is a ascending chain of bi-hyperideals of R and A is the union of these bi-hyperideals, then A is a bi-hyperideal of R.*

Proof. (1): Let $\{A_k : k \in \Lambda\}$ be a family of bi-hyperideals of R and $A = \bigcap\limits_{k\in\Lambda} A_k$. Since $0 \in \bigcap\limits_{k\in\Lambda} A_k$, it follows that $\bigcap\limits_{k\in\Lambda} A_k \neq \emptyset$. It is easy to check that $(A, +)$ is a canonical subhypergroup of $(R, +)$ and $A \cdot A \subseteq A$. Now, let $x \in A \cdot R \cdot A$. Then $x = a_1 \cdot r \cdot a_2$ for some $a_1, a_2 \in A$ and $r \in R$. Since each A_k is a bi-hyperideal of R, we have $x \in A_k \cdot R \cdot A_k \subseteq A_k$ for all $k \in \Lambda$. Thus $x \in A_k$ for all $k \in \Lambda$. Hence $x \in \bigcap\limits_{k\in\Lambda} A_k = A$. Since x was chosen arbitrarily, we have $A \cdot R \cdot A \subseteq A$. If $x \in A$ and $y \in R$ such that $y \leq x$, then $x \in A_k$ for all $k \in \Lambda$. Since each A_k is a bi-hyperideal of R, we have $y \in A_k$ for all $k \in \Lambda$. Thus $y \in \bigcap\limits_{k\in\Lambda} A_k = A$. Therefore, A is a bi-hyperideal of R.

(2): It is easy to show that $(A, +)$ is a canonical subhypergroup of $(R, +)$ and $A \cdot A \subseteq A$. Now, let $x \in A \cdot R \cdot A$. Then $x \in A_n \cdot R \cdot A_n$ for some bi-hyperideal A_n of R. Hence $x \in A_n \subseteq A$. Thus we have $A \cdot R \cdot A \subseteq A$. If $x \in A$ and $y \in R$ such that $y \leq x$, then $x \in A_n$ for some bi-hyperideal A_n of R. Since each A_n is a bi-hyperideal of R, it follows that $y \in A_n$ for some bi-hyperideal A_n of R. Thus, $y \in A$. Therefore, A is a bi-hyperideal of R. □

Lemma 3.3. *Let $(R, +, \cdot, \leq)$ be an ordered Krasner hyperring. Then,*

(1) *Every quasi-hyperideal Q of a two-sided hyperideal I of R is a bi-hyperideal of R. In particular, every quasi-hyperideal Q of R is a bi-hyperideal of R.*
(2) *If A is a bi-hyperideal of R and T is a subhyperring of R, then $A \cap T$ is a bi-hyperideal of T.*
(3) *If B is a hyperideal of R and Q is a quasi-hyperideal of R, then $B \cap Q$ is a bi-hyperideal and a quasi-hyperideal of B.*

Proof. (1): It is easy to see that $(Q, +)$ is a canonical subhypergroup of $(R, +)$ and $Q \cdot Q \subseteq Q$. Since $Q \subseteq I$, we have

$$Q \cdot R \cdot Q \subseteq Q \cdot R \cdot I \cap I \cdot R \cdot Q \subseteq Q \cdot I \cap I \cdot Q \subseteq Q.$$

Now, let $x \in Q$ and $y \in R$ such that $y \leq x$. Since $Q \subseteq I$, it follows that $x \in I$. By assumption, I is a hyperideal of R. Thus we have $y \in I$. Since Q is a quasi-hyperideal of I, it follows that

$y \in Q$. Hence Q is a bi-hyperideal of R.

(2): Let $A_1 = A \cap T$. We show that A_1 is a bi-hyperideal of T. Clearly, A_1 is a canonical subhypergroup of $(T, +)$ and $A_1 \cdot A_1 \subseteq A_1$. Since $A_1 \subseteq A$, it follows that $A_1 \cdot T \cdot A_1 \subseteq A \cdot R \cdot A \subseteq A$. Since $A_1 \subseteq T$ and T is a subhyperring of R, we have $A_1 \cdot T \cdot A_1 \subseteq T$. So we have checked that $A_1 \cdot T \cdot A_1 \subseteq A_1$. Now, if $x \in A_1$ and $y \in T$ such that $y \leq x$, then since $x \in A$, it follows that $y \in A$. Hence $y \in A_1$. Therefore, A_1 is a bi-hyperideal of T.

(3): This proof is straightforward. □

4. Main results

First, we give certain definitions needed for our purpose. The set of bi-hyperideals of R is totally ordered under inclusion if for all bi-hyperideals A, J either $A \subseteq J$ or $J \subseteq A$. In Example 3.4, the set of bi-hyperideals of R is totally ordered under inclusion, but in Example 3.5, the set of bi-hyperideals of R is not totally ordered under inclusion. In Example 3.2, $\{a\}$, $\{a, b\}$, $\{a, e\}$, $\{a, g\}$, $\{a, b, e, f\}$, $\{a, b, g, h\}$, $\{a, c, e, g\}$ and $\{a, b, c, d, e, f, g, h\}$ are bi-hyperideals of R. So, the set of bi-hyperideals of R is not totally ordered under inclusion. A non-empty subset P of an ordered Krasner hyperring $(R, +, \cdot, \leq)$ is called a *prime hyperideal* of R if the following conditions hold: (1) $A \cdot B \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$ for any two hyperideal A, B of R and (2) If $x \in P$ and $y \leq x$, then $y \in P$ for every $y \in R$. In Example 3.4, $\{a, c\}$ is a prime hyperideal of R. A non-empty subset I of an ordered Krasner hyperring $(R, +, \cdot, \leq)$ is called a *semiprime hyperideal* of R if the following conditions hold: (1) $A \cdot A \subseteq I$ implies that $A \subseteq I$ for any hyperideal A of R and (2) If $x \in I$ and $y \leq x$, then $y \in I$ for every $y \in R$. In Example 3.4, $\{a, c\}$ is a semiprime hyperideal of R, but $\{a\}$ is not a semiprime hyperideal of R. Indeed, $\{a, c\} \odot \{a, c\} = \{a\}$, but $\{a, c\} \nsubseteq \{a\}$.

Definition 4.1. *A bi-hyperideal A of an ordered Krasner hyperring $(R, +, \cdot, \leq)$ is called a prime bi-hyperideal of R if $A_1 \cdot A_2 \subseteq A$ implies either $A_1 \subseteq A$ or $A_2 \subseteq A$ for any bi-hyperideals A_1 and A_2 of R.*

Example 4.1.
(1) *In Example 3.4, $\{a, c\}$ is a prime bi-hyperideal of R, but $\{a\}$ is not a prime bi-hyperideal of R.*
(2) *In Example 3.5, $\{a, b\}$, $\{a, c, e\}$ and $\{a, b, e, f\}$ are prime bi-hyperideals of R. The bi-hyperideal $\{a\}$ is not prime. Indeed, $\{a, b\} \odot \{a, e\} = \{a\}$, but $\{a, b\} \nsubseteq \{a\}$ and $\{a, e\} \nsubseteq \{a\}$. Also, $\{a, e\}$ is not a prime bi-hyperideal of R. Indeed, $\{a, b, e, f\} \odot \{a, c, e\} = \{a, e\}$, but $\{a, b, e, f\} \nsubseteq \{a, e\}$ and $\{a, c, e\} \nsubseteq \{a, e\}$.*
(3) *In Example 3.2, $\{a, b, e, f\}$, $\{a, b, g, h\}$ and $\{a, c, e, g\}$ are prime bi-hyperideals of R, but $\{a\}$, $\{a, b\}$, $\{a, e\}$ and $\{a, g\}$ are not prime bi-hyperideals of R.*

Proposition 4.1. *Let A be a prime bi-hyperideal of an ordered Krasner hyperring R. Then, A is a prime one-sided hyperideal of R.*

Proof. Let A be a prime bi-hyperideal of R. Let I be a right hyperideal of R and J a left hyperideal of R such that $IJ \subseteq A$. Suppose that $I \nsubseteq A$. Assume that $x \in J$ and $s \in I \setminus A$. Then $sIx \subseteq IJ \subseteq A$. Since A is a prime bi-hyperideal of R and $s \notin A$, we have $x \in A$. Hence $J \subseteq A$. So, for any right hyperideal I and left hyperideal J of R, $IJ \subseteq A$ implies $I \subseteq A$ or $J \subseteq A$. Now, we show that A is a one-sided hyperideal of R. Since A is a bi-hyperideal of R, it follows that $(AR)(RA) \subseteq ARA \subseteq A$. Since AR is a right hyperideal and RA is a left hyperideal of R, we have $AR \subseteq A$ or $RA \subseteq A$. Therefore, A is a right hyperideal or a left hyperideal of R. □

Definition 4.2. *A bi-hyperideal A of an ordered Krasner hyperring $(R,+,\cdot,\leq)$ is called a semiprime bi-hyperideal of R if $A_1 \cdot A_1 \subseteq A$ implies $A_1 \subseteq A$, for any bi-hyperideal A_1 of R.*

Notice that every prime bi-hyperideal is a semiprime bi-hyperideal. A semiprime bi-hyperideal is not necessarily prime. In Example 3.2, $\{a\}$ and $\{a,b\}$ are semiprime bi-hyperideals of R, but $\{a\}$ and $\{a,b\}$ are not prime bi-hyperideals of R.

Proposition 4.2. *The intersection of any family of prime bi-hyperideals of an ordered Krasner hyperring $(R,+,\cdot,\leq)$ is a semiprime bi-hyperideal of R.*

Proof. Let $\{A_k : k \in \Lambda\}$ be a family of prime bi-hyperideals of R and $A = \bigcap_{k\in\Lambda} A_k$. By (1) of Lemma 3.2, A is a bi-hyperideal of R. Let B be any bi-hyperideal of R such that $B^2 \subseteq A$. Then $B^2 \subseteq A_k$ for all $k \in \Lambda$. Since each A_k is a prime bi-hyperideal of R, it follows that $B \subseteq A_k$ for all $k \in \Lambda$. Hence $B \subseteq A$. Therefore, A is a semiprime bi-hyperideal of R. □

Proposition 4.3. *Let A be a semiprime bi-hyperideal and B a left (right) hyperideal of an ordered Krasner hyperring $(R,+,\cdot,\leq)$ such that $B^2 \subseteq A$. Then we have $B \subseteq A$.*

Proof. Suppose that $B \nsubseteq A$. Then there exists $a \in B$ such that $a \notin A$. Since B is a left (right) hyperideal of R, we have $aRa \subseteq BRB \subseteq BB \subseteq A$. Since A is a semiprime bi-hyperideal of R, it follows that $a \in A$, that is a contradiction. Hence $B \subseteq A$ and so the proof is completed. □

Definition 4.3. *If $(R,+,\cdot,\leq)$ is an ordered Krasner hyperring and $A \subseteq R$, then $(A]$ is the subset of R defined as follows:*

$$(A] = \{t \in R : t \leq a, \text{ for some } a \in A\}.$$

Note that the condition (3) in Definition 3.2 is equivalent to $A = (A]$. If A and B are non-empty subsets of R, then we have

1. *$A \subseteq (A]$. Hence, $R = (R]$.*
2. *$(A] \cdot (B] \subseteq (A \cdot B]$,*
3. *$((A] \cdot (B]] = (A \cdot B]$,*
4. *$((A]] = (A]$,*
5. *$A \subseteq B$ implies $(A] \subseteq (B]$.*
6. *If A and B are left (respectively, right, two-sided) hyperideals of R, then $(AB]$ is left (respectively, right, two-sided) hyperideal of R.*

Definition 4.4. *Let $(R,+,\cdot,\leq)$ be an ordered Krasner hyperring. An element $a \in R$ is said to be regular if there exists an element $x \in R$ such that $a \leq (a\cdot x)\cdot a$. An ordered Krasner hyperring R is called regular if all elements of R are regular.*

Equivalent definitions*:*

1. *$a \in (aRa]$, $\forall a \in R$.*
2. *$A \subseteq (ARA]$, $\forall A \subseteq R$.*

Example 4.2. *In Example 3.2, $(R,\oplus,\odot,\leq)$ is a regular ordered Krasner hyperring.*

Definition 4.5. *An ordered Krasner hyperring $(R,+,\cdot,\leq)$ is called left (resp. right) regular if for every $a \in R$ there exists an element $x \in R$ such that $a \leq x \cdot a^2$ (resp. $a \leq a^2 \cdot x$). An ordered Krasner hyperring R is called left (resp. right) regular if all elements of R are left (resp. right) regular.*

Equivalent definitions*:*

(1) $a \in (Ra^2]$, *(resp.* $a \in (a^2R]$*)* $\forall a \in R$.
(2) $A \subseteq (RA^2]$, *(resp.* $A \subseteq (A^2R]$*)* $\forall A \subseteq R$.

Example 4.3. (1) *In Example 3.2,* $(R, \oplus, \odot, \leq)$ *is a left regular ordered Krasner hyperring.*
(2) *In Example 3.4 and Example 3.5,* $(R, \oplus, \odot, \leq)$ *is not a left regular ordered Krasner hyperring.*

Definition 4.6. *An ordered Krasner hyperring is called completely regular if it is regular, left regular and right regular.*

Example 4.4. *In Example 3.2,* $(R, \oplus, \odot, \leq)$ *is a completely regular ordered Krasner hyperring, but in Example 3.4 and Example 3.5,* $(R, \oplus, \odot, \leq)$ *is not a completely regular ordered Krasner hyperring.*

Theorem 4.1. *Let* $(R, +, \cdot, \leq)$ *be an ordered Krasner hyperring. Then* R *is regular if and only if every bi-hyperideal* A *of* R *is a semiprime bi-hyperideal.*

Proof. Assume that R is a regular ordered Krasner hyperring. Let A be a bi-hyperideal of R. Let $x \in R$ be such that $xRx \subseteq A$. Since R is regular, there exists $y \in R$ such that $x \leq xyx$. Thus we have $x \leq xyx \in xRx \subseteq A$. Hence $x \in (A] = A$. Therefore, A is a semiprime bi-hyperideal of R. We remark that since A is a bi-hyperideal of R, we have $(xRx] \subseteq A$ if and only if $xRx \subseteq A$.

Conversely, suppose that every bi-hyperideal of R is a semiprime bi-hyperideal. Let $a \in R$. It is easy to check that aRa is a bi-hyperideal of R. By assumption, aRa is a semiprime bi-hyperideal of R. Since $aRa \subseteq (aRa]$, it follows that $a \in (aRa]$. Thus there exists $x \in R$ such that $a \leq axa$. Hence R is a regular ordered Krasner hyperring. □

Theorem 4.2. *Let* $(R, +, \cdot, \leq)$ *be an ordered Krasner hyperring. Then* R *is a regular ordered Krasner hyperring if and only if* $(A \cdot B] = (A \cap B]$ *for every right hyperideal* A *and left hyperideal* B *of* R.

Proof. It is straightforward. □

In the following, some properties and relationships between bi-hyperideals and quasi-hyperideals are investigated.

Theorem 4.3. *Let* $(R, +, \cdot, \leq)$ *be an ordered Krasner hyperring. Then the following conditions are equivalent:*

(1) R *is regular.*
(2) $A = A \cdot R \cdot A$ *for every bi-hyperideal* A *of* R.
(3) $Q = Q \cdot R \cdot Q$ *for every quasi-hyperideal* Q *of* R.

Proof. (1) ⇒ (2): Assume that (1) holds. Let A be any bi-hyperideal of R and a any element of A. Then there exists $x \in R$ such that $a \leq (a \cdot x) \cdot a$. It is easy to see that $(a \cdot x) \cdot a \in A \cdot R \cdot A$. Hence, $A \subseteq A \cdot R \cdot A$. Since A is a bi-hyperideal of R, it follows that $A \cdot R \cdot A \subseteq A$. Therefore, we have $A = A \cdot R \cdot A$.

(2) ⇒ (3): Evidently, every quasi-hyperideal of R is a bi-hyperideal of R. Then by the assumption, we have $Q = Q \cdot R \cdot Q$ for every quasi-hyperideal Q of R.

(3) ⇒ (1): Assume that (3) holds. Let I and J be any right hyperideal and any left hyperideal of R, respectively. Then we have $(I \cap J) \cdot R \cap R \cdot (I \cap J) \subseteq I \cdot R \cap R \cdot J \subseteq I \cap J$ and so it is easy to see that $I \cap J$ is a quasi-hyperideal of R. By the assumption and Theorem 4.2, we have $I \cap J = (I \cap J) \cdot R \cdot (I \cap J) \subseteq I \cdot R \cdot J \subseteq I \cdot J \subseteq I \cap J$. Hence, $I \cdot J = I \cap J$ and so R is regular. □

Corollary 4.1. *Let $(R, +, \cdot, \leq)$ be an ordered Krasner hyperring. Then, the following statements are true:*

(1) *If $Q \cap A = Q \cdot A \cdot Q$ for every quasi-hyperideal Q and every hyperideal A of R, then R is a regular ordered Krasner hyperring.*
(2) *If $B \cap A = B \cdot A \cdot B$ for every bi-hyperideal B and every hyperideal A of R, then R is a regular ordered Krasner hyperring.*
(3) *If the set of all quasi-hyperideals of R is regular, then R is a regular ordered Krasner hyperring.*

Proof. (1): Let Q be any quasi-hyperideal of R. Since R itself is a hyperideal of R, it follows that

$$Q = Q \cap R = Q \cdot R \cdot Q.$$

Then, it follows from Theorem 4.3 that R is regular.

(2): It is obvious.

(3): By assumption, for every quasi-hyperideal Q of R, there exists a quasi-hyperideal B of R such that $Q = Q \cdot B \cdot Q \subseteq Q \cdot R \cdot Q \subseteq (Q \cdot R) \cap (R \cdot Q) \subseteq Q$. Hence for every quasi-hyperideal Q of R, we have $Q = Q \cdot R \cdot Q$. Then, it follows from Theorem 4.3 that R is regular. □

The following proposition shows that the notions of quasi-hyperideal and bi-hyperideal in a regular ordered Krasner hyperring coincide.

Proposition 4.4. *Let $(R, +, \cdot, \leq)$ be a regular ordered Krasner hyperring. Then, the following statements are hold:*

(1) *Every bi-hyperideal A of R is a quasi-hyperideal.*
(2) *For every bi-hyperideal A of a two-sided hyperideal I of R, A is a quasi-hyperideal of R.*

Proof. (1): Let A be a bi-hyperideal of R. It is easy to see that $R \cdot A$ is a left hyperideal and $A \cdot R$ is a right hyperideal of R. By Theorem 4.2, we have $(A \cdot R) \cap (R \cdot A) = A \cdot R \cdot R \cdot A \subseteq A \cdot R \cdot A \subseteq A$. Hence A is a quasi-hyperideal of R.

(2): First, we show that I is a regular subhyperring of R. Let $a \in I$. Then, there exists $x \in R$ such that $a \leq a \cdot x \cdot a \leq a \cdot x \cdot (a \cdot x \cdot a) = a \cdot (x \cdot a \cdot x) \cdot a$. Since $x \cdot a \cdot x \in I$, it follows that a is a regular element of I. Hence I is a regular subhyperring of R. By (1), the bi-hyperideal A of I is a quasi-hyperideal of I. By (1) of Lemma 3.3, A is a bi-hyperideal of R. So by (1), A is a quasi-hyperideal of R. □

A subset A of an ordered Krasner hyperring $(R, +, \cdot, \leq)$ is called idempotent if $A = (A^2]$. In Example 3.2, $\{a, b\}$ is a idempotent subset of R.

Theorem 4.4. *Let $(R, +, \cdot, \leq)$ be an ordered Krasner hyperring. Then the following are equivalent:*

(1) *R is regular.*
(2) *Every hyperideal of R is idempotent.*
(3) *Every hyperideal of R is semiprime.*

Proof. (1) ⇒ (2): Assume that (1) holds. Let A be any hyperideal of R. Since $A^2 = A \cdot A \subseteq A \cdot R \subseteq A$, implies $A^2 \subseteq A$. Now, let $a \in A$. Since R is regular, so there exists an element $x \in R$ such that $a \leq (a \cdot x) \cdot a$. Since $a \in A$, it follows that $a \cdot x \in A$, for all $x \in R$. It is easy to see that $(a \cdot x) \cdot a \in A \cdot A = A^2$. Hence $A \subseteq A^2$. Thus we have $A = A^2$. Therefore, Every hyperideal of R is idempotent.

(2) ⇒ (3): Let A and J be any hyperideals of R such that $A^2 \subseteq J$. Since $A = A \cdot A \subseteq J$, it

follows that $A \subseteq J$.

(3) ⇒ (1): Assume that (3) holds. Let A and J be any right hyperideal and any left hyperideal of R, respectively. It is obvious that $A \cdot J \subseteq A \cap J$. Also $A \cap J \subseteq A$ and $A \cap J \subseteq J$. Thus we have $(A \cap J)^2 \subseteq A \cdot J$. By the assumption and $(A \cap J)^2 \subseteq A \cdot J$, we have $A \cap J \subseteq A \cdot J$. Hence $A \cap J = A \cdot J$. By Theorem 4.2, R is regular. □

Definition 4.7. *A bi-hyperideal A of an ordered Krasner hyperring $(R, +, \cdot, \leq)$ is called an irreducible bi-hyperideal if for any bi-hyperideals I and J of R, $I \cap J = A$ implies that either $I = A$ or $J = A$. The bi-hyperideal A is strongly irreducible if for bi-hyperideals I and J of R, $I \cap J \subseteq A$ implies that either $I \subseteq A$ or $J \subseteq A$.*

Proposition 4.5. *Let $(R, +, \cdot, \leq)$ be an ordered Krasner hyperring. Then, the following conditions are equivalent:*

(1) *Every bi-hyperideal of R is strongly irreducible.*
(2) *Every bi-hyperideal of R is irreducible.*
(3) *The bi-hyperideals of R form a chain under inclusion.*

Proof. (1) ⇒ (2): Assume that (1) holds. Let I and J be any two bi-hyperideals of R. Let A be a bi-hyperideal of R such that $I \cap J = A$. Then we have $A \subseteq I$ and $A \subseteq J$. Since A is strongly irreducible, it follows that $I \subseteq A$ or $J \subseteq A$. So we have $I = A$ or $J = A$. Hence A is an irreducible bi-hyperideal of R.

(2) ⇒ (3): Let I and J be any two bi-hyperideals of R. By Lemma 3.2, $I \cap J$ is a bi-hyperideal of R. Since $I \cap J = I \cap J$, by assumption we have $I = I \cap J$ or $J = I \cap J$. This implies that $I \subseteq J$ or $J \subseteq I$. Therefore, the set of bi-hyperideals of R form a chain under inclusion.

(3) ⇒ (1): Assume that (3) holds. Let A be a bi-hyperideal of R. Let I and J be any two bi-hyperideals of R such that $I \cap J \subseteq A$. By assumption, we have $I \subseteq J$ or $J \subseteq I$. Thus either $I \cap J = I$ or $I \cap J = J$. This implies that $I \subseteq A$ or $J \subseteq A$. Hence A is a strongly irreducible bi-hyperideal of R. □

Definition 4.8. *A bi-hyperideal A of an ordered Krasner hyperring $(R, +, \cdot, \leq)$ is called a strongly prime bi-hyperideal of R if $A_1 \cdot A_2 \cap A_2 \cdot A_1 \subseteq A$ implies either $A_1 \subseteq A$ or $A_2 \subseteq A$ for any bi-hyperideals A_1 and A_2 of R.*

Proposition 4.6. *Every strongly irreducible, semiprime bi-hyperideal of an ordered Krasner hyperring $(R, +, \cdot, \leq)$ is a strongly prime bi-hyperideal of R.*

Proof. Let A be a strongly irreducible, semiprime bi-hyperideal of an ordered Krasner hyperring R. Let A_1 and A_2 be any two bi-hyperideals of R such that $A_1 \cdot A_2 \cap A_2 \cdot A_1 \subseteq A$. Since $(A_1 \cap A_2)^2 \subseteq A_1 \cdot A_2$ and $(A_1 \cap A_2)^2 \subseteq A_2 \cdot A_1$, we have $(A_1 \cap A_2)^2 \subseteq A_1 \cdot A_2 \cap A_2 \cdot A_1 \subseteq A$. Since A is a semiprime bi-hyperideal, it follows that $A_1 \cap A_2 \subseteq A$. Since A is a strongly irreducible bi-hyperideal of R, so either $A_1 \subseteq A$ or $A_2 \subseteq A$. Therefore, A is a strongly prime bi-hyperideal of R. □

Theorem 4.5. *Let A be a bi-hyperideal of an ordered Krasner hyperring $(R, +, \cdot, \leq)$ and $a \in R$ such that $a \notin A$. Then, there exists a strongly irreducible bi-hyperideal I of R such that $A \subseteq I$ and $a \notin I$.*

Proof. Let $\mathcal{C} = \{I : I \text{ is a bi-hyperideal of } R,\ A \subseteq I,\ a \notin I\}$. Since $A \in \mathcal{C}$, it follows that $\mathcal{C} \neq \emptyset$. Also $\mathcal{C}$ is a partially ordered set under the usual inclusion. Let $\{I_k : k \in \Lambda\}$ be a chain in $\mathcal{C}$. Consider $B = \bigcup_{k \in \Lambda} I_k$. We show that B is a bi-hyperideal of R and $A \subseteq B$. If $x, y \in B$, then

$x \in I_i$ and $y \in I_j$ for some $i, j \in \Lambda$. Since $\{I_k : k \in \Lambda\}$ be a totally ordered set, it follows that $I_i \subseteq I_j$ or $I_j \subseteq I_i$. If, say, $I_i \subseteq I_j$, then both x, y are inside I_j, so we have $x + y \subseteq I_j \subseteq B$. This implies that $x + y \subseteq B$. If, say, $I_j \subseteq I_i$, then both x, y are inside I_i, so we have $x + y \subseteq I_i \subseteq B$. This implies that $x + y \subseteq B$. If $x \in B$, then $x \in I_i$ for some $i \in \Lambda$. Thus we have $-x \in I_i \subseteq B$. This implies that $-x \in B$. Now, let $x, y \in B$, then $x \in I_i$ and $y \in I_j$ for some $i, j \in \Lambda$. Since $I_i \subseteq I_j$ or $I_j \subseteq I_i$, it follows that $x, y \in I_i$ or $x, y \in I_j$. Thus $x \cdot y \in I_i \subseteq B$ or $x \cdot y \in I_j \subseteq B$. This implies that $B \cdot B \subseteq B$. Therefore, $(B, +)$ is a canonical subhypergroup of $(R, +)$ and $B \cdot B \subseteq B$. Let $x, y \in B$ and $z \in R$. Then, $x \in I_r$ and $y \in I_s$ for some $r, s \in \Lambda$. Without any loss of generality we assume that $x, y \in I_s$. Hence $x \cdot z \cdot y \in I_s \subseteq B$. This implies that $B \cdot R \cdot B \subseteq B$. If $x \in B$ and $y \in R$ such that $y \leq x$, then $x \in I_k$ for some $k \in \Lambda$. So we have $y \in I_k \subseteq B$. This implies that $y \in B$. Therefore, $B = \bigcup_{k \in \Lambda} I_k$ is a bi-hyperideal of R. Since each $I_k \in \mathcal{C}$ contains A and $a \notin I_k$, we have $A \subseteq \bigcup_{k \in \Lambda} I_k = B$ and $a \notin B$. Hence $B \in \mathcal{C}$ is an upper bound for chain $\{I_k : k \in \Lambda\}$. By Zorn's lemma, $\mathcal{C}$ has a maximal element, say N. We show that N is a strongly irreducible bi-hyperideal of R. Let A_1 and A_2 be two bi-hyperideal of R such that $A_1 \nsubseteq N$ and $A_2 \nsubseteq N$. By the maximality of N, we have $a \in A_1$ and $a \in A_2$. Thus $a \in A_1 \cap A_2$, implies $A_1 \cap A_2 \nsubseteq N$. This shows that N is strongly irreducible. Hence the proof is completed. □

Corollary 4.2. *A bi-hyperideal A of an ordered Krasner hyperring $(R, +, \cdot, \leq)$ is the intersection of all strongly irreducible bi-hyperideals of R containing A.*

Proof. Let $\mathcal{I} = \{I_k : k \in \Lambda\}$ be the set of all strongly irreducible bi-hyperideals of R containing A. Clearly, $A \subseteq \bigcap_{k \in \Lambda} I_k$. Suppose $B = \bigcap_{k \in \Lambda} I_k$ and $A \in \mathcal{I}$. Let $0 \neq a \in R$ such that $a \notin A$. By Theorem 4.5, there exists a strongly irreducible bi-hyperideal J of R such that $A \subseteq J$ and $a \notin J$. Hence $J \in \mathcal{I}$ and so $a \notin B$. Thus we have $B = \bigcap_{k \in \Lambda} I_k \subseteq A$. Hence $A = \bigcap_{k \in \Lambda} I_k$ and so the proof is completed. □

Let $\mathcal{P}_R$ denote the set of strongly irreducible proper bi-hyperideals of an ordered Krasner hyperring $(R, +, \cdot, \leq)$. For a bi-hyperideal A of R, define the set $\mathcal{E}_A = \{J \in \mathcal{P}_R : A \nsubseteq J\}$ and $\xi(\mathcal{P}_R) = \{\mathcal{E}_A : A \text{ is a bi-hyperideal of } R\}$.

Theorem 4.6. *Let $(R, +, \cdot, \leq)$ be an ordered Krasner hyperring. If the set of bi-hyperideals of R form a chain under inclusion, then the set $\xi(\mathcal{P}_R)$ forms a topology on the set $\mathcal{P}_R$.*

Proof. Since $\{0\}$ is a bi-hyperideal of R and $\mathcal{E}_{\{0\}} = \{J \in \mathcal{P}_R : \{0\} \nsubseteq J\} = \emptyset$, we have $\emptyset \in \xi(\mathcal{P}_R)$. Thus $\mathcal{E}_{\{0\}}$ is an empty subset of $\xi(\mathcal{P}_R)$. Since R is a bi-hyperideal of itself and strongly irreducible bi-hyperideals are proper, we have $\mathcal{E}_R = \{J \in \mathcal{P}_R : R \nsubseteq J\} = \mathcal{P}_R$. So we obtain $\mathcal{P}_R \in \xi(\mathcal{P}_R)$. Hence the first axiom for the topology is hold. Now, let $\mathcal{E}_{A_1}, \mathcal{E}_{A_2} \in \xi(\mathcal{P}_R)$. We shall show that $\mathcal{E}_{A_1} \cap \mathcal{E}_{A_2} = \mathcal{E}_{A_1 \cap A_2}$. If $J \in \mathcal{E}_{A_1} \cap \mathcal{E}_{A_2}$, then $J \in \mathcal{P}_R$ and $A_1 \nsubseteq J$, $A_2 \nsubseteq J$. Suppose $A_1 \cap A_2 \subseteq J$. Since J is a strongly irreducible bi-hyperideal of R, we have $A_1 \subseteq J$ or $A_2 \subseteq J$, a contradiction. Hence $A_1 \cap A_2 \nsubseteq J$, which implies that $J \in \mathcal{E}_{A_1 \cap A_2}$. Thus $\mathcal{E}_{A_1} \cap \mathcal{E}_{A_2} \subseteq \mathcal{E}_{A_1 \cap A_2}$. Now, if $J \in \mathcal{E}_{A_1 \cap A_2}$, then $J \in \mathcal{P}_R$ and $A_1 \cap A_2 \nsubseteq J$. Thus $A_1 \nsubseteq J$ and $A_2 \nsubseteq J$. Hence $J \in \mathcal{E}_{A_1}$ and $J \in \mathcal{E}_{A_2}$, which implies that $J \in \mathcal{E}_{A_1} \cap \mathcal{E}_{A_2}$. Therefore, we have $\mathcal{E}_{A_1 \cap A_2} \subseteq \mathcal{E}_{A_1} \cap \mathcal{E}_{A_2}$. Now, consider an arbitrary family $\{A_\alpha : \alpha \in \Lambda\}$ of bi-hyperideals of R. Let $\{\mathcal{E}_{A_\alpha} : \alpha \in \Lambda\} \subseteq \xi(\mathcal{P}_R)$. Then, we have $\bigcup_{\alpha \in \Lambda} \mathcal{E}_{A_\alpha} = \bigcup_{\alpha \in \Lambda} \{J \in \mathcal{P}_R : A_\alpha \nsubseteq J\} = \{J \in \mathcal{P}_R : A_\alpha \nsubseteq J \text{ for some } \alpha \in \Lambda\} = \{J \in \mathcal{P}_R : \langle \bigcup A_\alpha \rangle \nsubseteq J\} = \mathcal{E}_{\bigcup A_\alpha}$, where $\langle \bigcup A_\alpha \rangle$ is a bi-hyperideal of R generated by $\bigcup A_\alpha$ and by the proof of Theorem 4.5, $\bigcup A_\alpha$ is a bi-hyperideal of R. Thus we have $\bigcup_{\alpha \in \Lambda} \mathcal{E}_{A_\alpha} \in \xi(\mathcal{P}_R)$. Hence $\xi(\mathcal{P}_R)$ forms a topology on the set $\mathcal{P}_R$ and so the proof is completed. □

Definition 4.9. *Let $(R, +, \cdot, \leq)$ be an ordered Krasner hyperring. A non-zero bi-hyperideal A of R is called a minimal bi-hyperideal of R if there is no non-zero bi-hyperideal B of R such that $B \subset A$. A bi-hyperideal A of R is said to be maximal if for any proper bi-hyperideal B of R, $A \subseteq B$ implies that $A = B$.*

Proposition 4.7. *Let A be a proper bi-hyperideal of an ordered Krasner hyperring $(R, +, \cdot, \leq)$. Then, A is contained in a maximal bi-hyperideal of R.*

Proof. Let $\mathcal{C} = \{J : J$ is a proper bi-hyperideal of R and $A \subseteq J\}$. Since $A \in \mathcal{C}$, it follows that $\mathcal{C} \neq \emptyset$. Also $\mathcal{C}$ is an ordered set by inclusion. Let $\{J_k : k \in \Lambda\}$ be a totally ordered subset in $\mathcal{C}$. Consider $B = \bigcup\limits_{k\in\Lambda} J_k$. It is easy to see that B is in $\mathcal{C}$. Hence by Zorn's lemma, $\mathcal{C}$ has a maximal element, say M. Now, let K be a proper bi-hyperideal of R containing M. Then, K contains A and so it belongs to $\mathcal{C}$. Since M is maximal in $\mathcal{C}$, it follows that $K = M$. Therefore, M is a maximal bi-hyperideal of R. □

Theorem 4.7. *Let $(R, +, \cdot, \leq)$ ba an ordered Krasner hyperring. If the set of bi-hyperideals of R is totally ordered under inclusion, then any maximal bi-hyperideal of R is strongly irreducible.*

Proof. Let A be a maximal bi-hyperideal of R. Let I and J be bi-hyperideals of R such that $I \cap J \subseteq A$ and $I \nsubseteq A$. Consider $B = I \cup A$. We show that B is a bi-hyperideal of R. Let $b_1, b_2 \in B$. By the assumption, we have $A \subseteq I$. So both b_1, b_2 are inside I. Thus we have $b_1 + b_2 \subseteq I \subseteq B$, $-b_1 \in I \subseteq B$ and $b_1 \cdot b_2 \in I \subseteq B$. This implies that $b_1 + b_2 \subseteq B$, $-b_1 \in B$ and $b_1 \cdot b_2 \in B$. Therefore, $(B, +)$ is a canonical subhypergroup of $(R, +)$ and $B \cdot B \subseteq B$. Now, let $b_1, b_2 \in B$ and $r \in R$. Then $b_1, b_2 \in I$. Hence $b_1 \cdot r \cdot b_2 \in I \subseteq B$. This implies that $B \cdot R \cdot B \subseteq B$. If $x \in B$ and $y \in R$ such that $y \leq x$, then $x \in I$. So we have $y \in I \subseteq B$. This implies that $y \in B$. Therefore, $B = I \cup A$ is a bi-hyperideal of R such that $A \subset I \cup A$, so $I \cup A = R$. Thus we have $J = J \cap R = J \cap (I \cup A) = (J \cap I) \cup (J \cap A) \subseteq A$. Hence A is a strongly irreducible bi-hyperideal of R and so the proof is completed. □

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