TWMS J. Pure Appl. Math., V.13, N.1, 2022, pp.105-118

NUMERICAL SOLUTION OF THE FIRST KIND FREDHOLM INTEGRAL EQUATIONS BY PROJECTION METHODS WITH WAVELETS AS BASIS FUNCTIONS

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Abstract. In this paper, we review new works on approximate methods for solving the first kind Fredholm integral equations. The Galerkin-Bubnov projection method with Legendre wavelets is used for the numerical solution of the first kind Fredholm integral equations. Numerical calculations and the proven theorem show a very strong sensitivity of the solution to the accuracy of calculating double integrals for determining the elements of the matrix and the right-hand side of the system of linear algebraic equations, which are determined by cubature formulas or analytical formulas. Also, in this paper we obtain a priori estimates and convergence of the projection methods with bases in the form of wavelets on half-intervals. The performed comparative analysis shows that the Galerkin method with basis functions in the form of Legendre wavelets is efficient in terms of accuracy and is easy to implement.

Keywords: Fredholm integral equation, Legendre wavelets, Galerkin-Bubnov method, error estimate on half-intervals.

AMS Subject Classification: 45B05, 65R30, 65R32.

1. Introduction

Many inverse problems of science and technology, in particular, inverse problems of mathematical geophysics, can be reduced to solving the first kind Fredholm integral equation. In general, integral equations of this type are ill-posed for the given kernel and right-hand side. These equations may not have a solution, and if a solution exists, then small perturbations of the right-hand side strongly affect the solution.

There are several numerical methods for solving the first kind integral equations. Most of the works are based on projection methods such as the Galerkin-Petrov method, the Bubnov-Galerkin method, the moments method, the collocation method. One of the most attractive developments in recent years is the use of wavelets as basis functions in projection methods. The wavelet technique makes it possible to create very efficient algorithms in comparison with known regularizing algorithms. Various wavelet bases are used in [21, 9, 18, 19, 3, 4, 1, 26]. K. Maleknejad, S. Sohrabi [17] proposed an effective method based on using continuous Legendre wavelet as a basis function for an approximate solution of the first kind Fredholm integral equation. In the Galerkin method, Legendre wavelets are used as basis functions, and a system of linear algebraic equations is obtained to determine the expansion coefficients. This system of linear algebraic equations is solved by the conjugate gradient method. Several specific examples

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Manuscript received July 2021.

of the first kind Fredholm integral equation with known solutions are considered. The results of numerical calculations using Mathematica 5 show very good convergence to the exact solution.

Xufeng Shang, Danfu Han [27] proposed an effective method for solving the first kind Fredholm integral equations, similarly as in [17]. Continuous Legendre multiwavelets are used as a basis in the Galerkin method.

In recent years, a lot of works devoted to the approximate solution of Fredholm integral equations [23, 7, 15] have been published. In [23], two-dimensional Legendre wavelet bases are used to numerically solve two-dimensional Fredholm integral equations. In [7], modern basic numerical methods such as regularization, wavelet analysis, and multilevel iterations methods are described in detail. This paper gives a concise summary of numerical methods for solving the first kind Fredholm integral equations.

In [8], the sought and integrand functions are expanded in a Taylor series, and the first kind Fredholm integral equation is transformed into a system of linear equations for the unknown function and its derivative.

In [18], a numerical method is proposed for solving the first kind Fredholm integral equation based on hybrid block-momentum functions and Legendre polynomials to obtain a high-precision solution. The convergence analysis of the proposed method is given and the convergence rate is determined.

K.Maleknejad, T.Lofti, K.Mahdiani [19] use orthogonal wavelets as a basis in projection methods for the numerical solution. It is shown that the Galerkin wavelet method converges provided the basis has the property of the best approximation.

In [3] wavelets are used as basis functions, and the method of moments is used as a projection method. Special radial functions are considered as the kernel of the integral operator, and the stability of the numerical solution is proved [12].

The properties of Chebyshev wavelets are effectively used to obtain a sparse matrix of a system of linear equations for the expansion coefficients. The given numerical examples show the validity and efficiency of using Chebyshev wavelets for an approximate solution to the first kind Fredholm integral equation [4, 1]. Hybrid block-pulse methods are used to numerically solve nonlinear integro-differential equations. On each half-interval, the sought function is expanded using orthogonal Bernstein polynomials [6]. The concept of ill-posedness of the first kind fuzzy Fredholm integral equations was considered in [11] for the first time. Fuzzy Fredholm integral equations of the first kind are transformed into fuzzy integral equations of the second kind by the Tikhonov regularization method.

In [13], the spectral Legendre wavelet method was proposed for the numerical solution of nonlinear ordinary differential equations on large intervals. The Legendre-Gauss quadrature formula is derived, where the roots of the Legendre polynomial are chosen as the collocation points and the quadrature weights are determined by a formula defined through the derivative of the Legendre polynomial at these points.

In [14], a pseudospectral method is proposed for solving nonlinear singular ordinary Thomas-Fermi differential equations in a semi-infinite interval. The developed pseudospectral method is based on rational Chebyshev functions of the second kind. Application of this method to the Thomas-Fermi equation leads to a nonlinear algebraic system. The authors highlight three main advantages of applying the method to the Thomas-Fermi equation.

In [16], an effective multi-domain pseudospectral method is used for the numerical solution of a nonlinear fractional integro-differential equation. The fractional derivative is considered in the Caputo sense. The original equation is replaced by the singular integro-differential Volterra equation. The transformed problem in subintervals is reduced to systems of algebraic equations.

Further, a pseudospectral method is used based on the shifted points of Legendre-Gauss collocations. In computational experiments, the accuracy of the solution increases either by increasing the number of collocation points within the intervals, either by decreasing the grid step size.

In [20], the first kind classical Volterra integral equations have investigated, where the kernel is discontinuous along continuous curves. The solution of the integral equation is searched in the form of Taylor polynomials and the projection collocation method is used. Several preliminary lemmas and the convergence theorem for the Taylor-collocation method are proved. The examples are solved by using the Taylor-collocation method and the numerical results are presented in table.

In [2], a new computational method is proposed for solving nonlinear Fredholm integral equations of the second kind with weakly singular kernels. The use of a discrete analogue of the Galerkin projection method will lead to the appearance of singular integrals. The solution of the Fredholm integral equation will depend on the accuracy of their calculation. The authors of this paper have overcome this problem by approximating these integrals with non-uniform quadrature Gauss-Legendre formulas.

In [24], a constructive method is proposed for solving the first kind Fredholm integral equation based on the theory of conjugate equations. The solution of the conjugate integral equation with the right-hand side in the form of eigenfunctions of the basic Fredholm integral operator is solved separately. Then the solution of the original integral equation is found as a linear combination of the eigenfunctions of the Fredholm operator. The coefficients of this linear combination are calculated as a definite integral of the product of the right-hand side and the solution of the conjugate equation.

Among the iterative methods for solving Fredholm integral equations, the G.N. Polozhiy's method [21] is effective. This method converges uniformly for any improper values of the parameter λ in an equation of the second kind and converges on average for an equation of the first kind.

From a computational point of view, the constructive method proposed in [21] has great prospects.

In all of the above works, methods with bases in the form of wavelets, hybrid block-impulse functions have been developed, but there is no analysis of the approximation error of the residual, estimates of the error on half-intervals. In the numerical solution of the first kind Fredholm integral equations by projection methods with bases in the form of the wavelet, (1) the function interpolation, (2) requirement of an orthogonal projection is used. It is well known that the piecewise polynomial interpolation process is convergent. Therefore, in this paper, we consider the problem of obtaining a priori estimates and convergence of the projection methods with bases in the form of wavelets on half-intervals. For a comparative analysis, numerical calculations of the solution to the first kind Fredholm integral equation were carried out by the Lavrentyev regularization method, the Polozhiy method, the constructive "tracking" method, and the Bubnov-Galerkin method with Legendre wavelets.

2. Orthonormal system of Legendre wavelets

Consider the first kind Fredholm integral equation

$$K[x,y] \equiv \int_0^1 K(x,y)y(s)ds = f(x), \quad x \in [0,1], \tag{1}$$

where the kernel $K(x,s)$ is a real continuous, square-summable and bounded function in the domain $G=\{0\le x\le 1,\ \ 0\le s\le 1\}$. We apply the Galerkin-Bubnov method with the use of the Legendre wavelets [17] to solve the integral equation (1):

$$\psi_{n,m}(t)=\begin{cases}(2m+1)^{\frac{1}{2}}2^{\frac{k-1}{2}}L_m(2^k t-\hat{n}), & \frac{\hat{n}-1}{2^k}\le t<\frac{\hat{n}+1}{2^k},\\ 0, \ \ for \ \ other \ \ t.\end{cases} \tag{2}$$

where $k=2,3,...;\ \ n=1,2,3,...,2^{k-1};\hat{n}=2n-1;\ \ m=0,1,2,...,M-1.$

The Legendre polynomials of order $l, L_l(t)$, are determined from the following recurrence formula:

$$\begin{gathered}L_0=1,\\ L_1(t)=t,\\ L_{l+1}(t)=\tfrac{2l+1}{l+1}tL_l(t)-\tfrac{l}{l+1}L_{l-1}(t),\ \ l=1,2,3,...\end{gathered} \tag{3}$$

The set $\{L_l(t):l=0,1,2,...\}$ is a complete orthogonal set in the Hilbert space $L_2[-1,1]$. In addition, the Legendre polynomials are bounded, i.e. $|L_l(t)|\le 1,\ \ -1\le t\le 1,\ \ l=0,1,2,\ldots$.

In (2) and (3) m,l denote the degree of the Legendre polynomial, and k determines the number of half-intervals on which the wavelets are defined.

If $k=2,\ \ M=2$, then $n=1,2;\ \ \hat{n}=1,3$ and $m=0,1$. For $k=2,\ \ M=2$, we obtain the following Legendre wavelets:

$$\psi_{10}(t)=\begin{cases}\sqrt{2}, \ \ 0\le t<1/2,\\ 0, \ \ 1/2\le t<1.\end{cases} \tag{4}$$

$$\psi_{11}(t)=\begin{cases}\sqrt{6}(4t-1), \ \ 0\le t<1/2,\\ 0, \ \ 1/2\le t<1.\end{cases} \tag{5}$$

$$\psi_{20}(t)=\begin{cases}0, \ \ 0\le t<1/2,\\ \sqrt{2}, \ \ 1/2\le t<1.\end{cases} \tag{6}$$

$$\psi_{21}(t)=\begin{cases}0, \ \ 0\le t<1/2,\\ \sqrt{6}(4t-3), \ \ 1/2\le t<1.\end{cases} \tag{7}$$

If $M=3$, then additionally it is necessary to define

$$\begin{gathered}\psi_{12}(t)=\begin{cases}\sqrt{\frac{5}{2}}(3(4t-1)^2-1), \ \ 0\le t<1/2,\\ 0, \ \ 1/2\le t<1.\end{cases}\\ \psi_{22}(t)=\begin{cases}0, \ \ 0\le t<1/2,\\ \sqrt{\frac{5}{2}}(3(4t-3)^2-1), \ \ 1/2\le t<1.\end{cases}\end{gathered} \tag{8}$$

It is easy to verify [27] that the set $\{\psi_{nm}(t)\}$ forms an orthonormal system in $L_2[0,1]$.

3. The proposed numerical method

The orthonormal system (2) is used for the best approximation of the solution to equation (1) in the Hilbert space, and the problem of its finding formally reduces to solving a system of linear equations. The goal of best approximation is to develop representations of the function $y(x)$ with different levels of resolution.

Let $\{V_i\}$ be a sequence of nested subspaces such that $\{0\}\subset ...\subset V_{-1}\subset V_0\subset V_1\subset L_2(R)$.

The system $\{\psi_{n,m}(t)\}$ introduced in Section 2 forms a Riesz basis for the space V_n [19].

In the general case, any function $f \in L_2(R)$ can be approximated by its orthogonal projection $P_n f$ on the space V_n

$$P_n f(t) = \sum_m (f, \psi_{nm}(t))\psi_{nm}(t), \tag{9}$$

where $P_n f$ is an approximation of f with the resolution 2^{-n}. The best approximation is provided by a sequence of approximations $P_n f$ of increased accuracy up to a given function f [19].

We represent the solution of the integral equation (1) in the form

$$y(s) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{nm}\psi_{nm}(s). \tag{10}$$

The coefficients $\{C_{nm}\}$ are unknown. By substituting (10) into (1), we have

$$\int_0^1 K(x,s) \left(\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{nm}\psi_{nm}(s) \right) ds = f(x). \tag{11}$$

By virtue of linearity of (11), we integrate the function under the sum to obtain

$$\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{nm} \int_0^1 K(x,s)\psi_{nm}(s)ds = f(x). \tag{12}$$

According to the Bubnov-Galerkin method, the coefficients C_{nm} are determined from the requirement that the left-hand side of equation (12) becomes orthogonal to functions ψ_{ij}. Therefore, we multiply (12) scalarly by $\psi_{ij}(x)$, $i = 1, 2, ..., 2^{k-1}$, $j = 0, 1, .., M-1$ to obtain

$$\begin{gathered} \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{nm} \left[\int_0^1 \int_0^1 K(x,s)\psi_{nm}(s)\psi_{ij}(x)dsdx \right] = \int_0^1 f(x)\psi_{ij}(x)dx, \\ i = 1, 2, ..., 2^{k-1}; \ j = 0, 1, ..., M-1. \end{gathered} \tag{13}$$

There is a 4-index matrix A in the square bracket which we expand into a 2-index matrix. The matrices $||C_{ij}||$ and $\|F_{ij}\|$ of the wavelet basis $\psi_{ij}(x)$, $i = 1, 2, ..., 2^{k-1} = N$, $j = 0, 1, .., M-1$ can be represented as vectors $\overrightarrow{\alpha}_l, b_l$, and a vector function $\overrightarrow{\phi}_l(x)$ of length l by using the following ordering of indices:

$$l = (i-1)M + j + 1, \ i = 1, 2, ..., N, \ j = 0, 1, ..., M-1.$$

Then the vector dimension and the number of wavelet basis functions is equal to $L = N \cdot M$. Thus, equation (13) can be written as

$$\sum_{j=1}^{L} \alpha_j \left[\int_0^1 \int_0^1 K(x,s)\phi_j(s)\phi_i(x)dsdx \right] = \int_0^1 f(x)\phi_i(x)dx, \ \ i = 1, 2, ..., L. \tag{14}$$

or

$$A\alpha = b, \tag{15}$$

in a matrix form, where

$$\begin{gathered} a_{ij} = \int_0^1 \int_0^1 K(x,t)\phi_i(x)\phi_j(t)dtdx, \\ b_i = \int_0^1 f(x)\phi_i(x)dx, \\ i = 1, 2, ..., L; \ j = 1, 2, ..., L. \end{gathered} \tag{16}$$

and the desired solution of equation (1) can be written as

$$y(x) \approx \sum_{i=1}^{L} \alpha_i \phi_i(x). \tag{17}$$

4. Convergence study

Assume that the integral operator $K[x, y]$ is compact in Hilbert space V, and rewrite equation (1) in an operator form

$$K[x, y] = f. \tag{18}$$

From (12) we obtain the residual

$$r_n(x) = \sum_{l=1}^{n-1} \sum_{m=0}^{M-1} C_{lm} \int_0^1 K(x, s)\psi_{lm}(s)ds - f(x), \tag{19}$$

where $n = 2^{k-1}, \ldots,$ or it can be written by using (10) and (18) as

$$r_n(x) = K[x, y_n] - f(x). \tag{20}$$

It is necessary to require that the residual $r_n(x)$ tends to zero and the expansion of the function $y_n(x)$ was a good solution to equation (1). We intend to show that a basis of Legendre wavelets has the properties of the best approximation [3], then it follows from Galerkin's methods of solving (1) that $y_n(x) \longrightarrow y(x)$ *as* $n \longrightarrow \infty$.

Let $V = L_2[0, 1]$ and $\langle\cdot\rangle$ denote the scalar product on V. Then by using (19), (20), the Galerkin's method (13) can be written as follows:

$$\langle r_n(x), \psi_{ij}(x)\rangle = 0, \;\; i = 1, 2, .., 2^{k-1}, \;\; j = 0, 1, .., M - 1. \tag{21}$$

From (21) and (20), we have a linear system of the form ((15).

Consider the projection operator (10), which maps V into V_n for each $y \in V$, and define $P_n y$ as a solution to the following problem:

$$\|y - P_n y\| = \max_{g \in V_n} \|y - g\|. \tag{22}$$

The constructed wavelet basis (2) is orthonormal:

$$\langle \psi_{nk}, \psi_{ni}\rangle = \delta_k^i, \;\; k, i = 1, 2, ..., M - 1.$$

Therefore, the projection operator P_n can be written as [15]

$$P_n y(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} \langle \psi_{nm}, y\rangle \psi_{nm}(x). \tag{23}$$

The following lemma is valid for the projection operator P_n [17, 23].

Lemma 4.1. If $y, g \in V$ and P_n is the projection operator defined in the form of (23), then the following relations hold:

$$(P_n y, g) = (y, P_n g), \tag{24}$$

$$P_n^2 y = P_n y, \tag{25}$$

$$(y - P_n y, P_n g) = 0, \tag{26}$$

$$\|y\|^2 = \|P_n\|^2 + \|y - P_n y\|^2, \tag{27}$$

$$\|P_n\| = 1, \tag{28}$$

$$\|y - g\|^2 = \|y - P_n y\|^2 + \|P_n y - g\|^2. \tag{29}$$

Proof. Transform the left-hand side of (24) using formula of the projection operators (23)

$$\begin{aligned}(P_n y, g) &= \int_0^1 \left\{ \sum_{l=1}^{n} \sum_{m=0}^{M-1} \left(\int_0^1 y(t)\cdot\psi_{lm}(t)dt \right) \psi_{lm}(x) \right\} g(x)dx \\ &= \sum_{l=1}^{n} \sum_{m=0}^{M-1} \left(\int_0^1 y(t)\psi_{lm}(t)dt \right) \int_0^1 g(x)\psi_{lm}(x)dx \\ &= \sum_{l=1}^{n} \sum_{m=0}^{M-1} \int_0^1 y(t)\cdot(g,\psi_{lm})\psi_{lm}(t)dt = (y, P_n g).\end{aligned} \tag{30}$$

Thus, (24) is proved.

To prove (25), consider

$$P_n^2 y(x) = \sum_{l=1}^{n} \sum_{m=0}^{M-1} (P_n y(s), \psi_{lm}(s))\cdot\psi_{lm}(x)\ . \tag{31}$$

Transform the scalar product included in (31) using the orthonormality of the functions system $\{\psi_{ij}(x)\}$

$$\begin{aligned}(P_n y(s), \psi_{lm}(s)) &= \int_0^1 P_n y(s)\psi_{lm}(s)ds \\ &= \int_0^1 (\sum_{i=1}^{n} \sum_{j=0}^{M-1} (y,\psi_{ij})\psi_{ij}(s))\psi_{lm}(s)ds \\ &= \sum_{i=1}^{n} \sum_{j=0}^{M-1} (y,\psi_{ij})\cdot\int_0^1 \psi_{ij}(s)\psi_{lm}(s)ds = (y,\psi_{lm}).\end{aligned} \tag{32}$$

Substitute (32) into (31) to obtain

$$P_n^2 y(x) = \sum_{l=1}^{n} \sum_{m=0}^{M-1} (y,\psi_{lm})\psi_{lm} = P_n y(x), \tag{33}$$

which proves (25).

Using (25), we prove the formula (26)

$$(y - P_n y, P_n g) = (y, P_n g) - (P_n y, P_n g) = (y, P_n g) - (y, P_n^2 g) = (y, P_n g) - (y, P_n g) = 0. \tag{34}$$

The formulas (27), (28), and (29) are proved using (26).

The lemma is proved.

From (29) we have

$$\|y - y_n\|^2 = \|y - P_n y\|^2 + \|P_n y - y_n\|^2. \tag{35}$$

Let ε be an arbitrary small positive number, i.e. $\varepsilon > 0$. Then, by the property of best approximation and using the fact that $U_{n=1}^{N} V_n$ is dense in $L_2(R)$, there exists a function $y_n \in V_n$ such that $\|y - y_n\| < \varepsilon$. Then if $m \geq n$, we have

$$\|y - P_m y\| \leq \|y - y_m\| < \varepsilon. \tag{36}$$

Therefore, for any function $y \in L_2(R)$, expanding in the form (23), we have

$P_n y \longrightarrow y$ as $n \longrightarrow \infty$. Now consider the residual $r_l(x)$ on the l -th half-interval

$$r_l(x) = \sum_{m=0}^{M-1} C_{lm} \int_{\frac{l-1}{2^{k-1}}}^{\frac{l}{2^{k-1}}} K(x,s)\psi_{lm}(s)ds - f(x), \ \ x \in \left[\frac{l-1}{2^{k-1}}; \frac{l}{2^{k-1}}\right], \ \ l = 1, 2, .., 2^{k-1}. \tag{37}$$

The kernel and the right-hand side of the integral equation (1) are assumed to be bounded:

$$\begin{gathered} 0 < K_0 \leq |K(x,s)| \leq K_1, \ \ 0 \leq F_0 \leq |f(x)| \leq F_1, \\ for\ \ (x,s) \in G, \ \ l = 1, 2, .., 2^{k-1}; \ \ m = 0, 1, .., M-1. \end{gathered} \tag{38}$$

Lemma 4.2. For $K(x,s)$, $f(x)$ satisfying conditions (38), the following estimate holds:

$$\|r_l(x)\| \leq \left(\frac{2M-1}{2^{2(k-1)}}\right)\frac{K_1F_1}{K_0} + (2M-1)^{\frac{1}{2}}MF_0, for\ \ x \in \left[\frac{l-1}{2^{k-1}}, \frac{l}{2^{k-1}}\right). \tag{39}$$

Proof. Represent the function in the right-hand side of (37) in the form of an expansion by basic functions on the l th half-interval:

$$f(x) \approx \sum_{m=0}^{M-1}(f,\psi_{lm})\psi_{lm}(x). \tag{40}$$

Taking into account (38) and (40), we obtain from (37) that

$$|r_l(x)| \leq \sum_{m=0}^{M-1}\left|C_{lm}\int\limits_{\frac{l-1}{2^{k-1}}}^{\frac{l}{2^{k-1}}} K(x,s)\psi_{lm}(s)ds - \left(\int\limits_{\frac{l-1}{2^{k-1}}}^{\frac{l}{2^{k-1}}} f(s)\psi_{lm}(s)ds\right)\psi_{lm}(x)\right|.$$

Estimate $K(x,s)$ in the first term from above, and f and $\psi_{lm}(x)$ in the second term from below to obtain

$$|r_l(x)| \leq \sum_{m=0}^{M-1}\left|C_{lm}K_1 - F_0 2^{\frac{k-1}{2}}\right|(2M-1)^{1/2}2^{\frac{k-1}{2}}\frac{1}{2^{k-1}}.$$

Thus, we obtain the following inequality:

$$|r_l(x)| \leq \left(\frac{2M-1}{2^{k-1}}\right)^{1/2}\sum_{m=0}^{M-1}\left|C_{lm}K_1 - 2^{\frac{k-1}{2}}F_0\right|. \tag{41}$$

Then it follows that

$$|r_l(x)| \leq \left(\frac{2M-1}{2^{k-1}}\right)^{1/2} M(\|C\|K_1 + 2^{\frac{k-1}{2}}F_0), \tag{42}$$

where C is the solution to the system of linear equations (13), i.e.

$$AC = F,$$

where C is a matrix with elements C_{ij}, and F is a matrix with elements (f,ψ_{ij}), where $i = 1,2,,2^{k-1}$; $j = 0,1,,M-1$.

Let us estimate the norm of the four-index matrix A. According to the definition of the matrix norm and the form of the matrix, we have from (13):

$$\|A\| = \max_{1\leq l\leq 2^{k-1},\ 0\leq m\leq M-1}\left(\sum_{i=1}^{2^{k-1}}\sum_{j=0}^{M-1}\int\limits_0^1\int\limits_0^1 K(x,s)\psi_{ij}(s)\psi_{lm}(s)dsdx\right).$$

Further, by summing up under the integral sign, we have

$$\|A\| = \max_{1\leq l\leq 2^{k-1},\ 0\leq m\leq M-1}\left(\int\limits_0^1\psi_{lm}(x)dx\int\limits_0^1\left(\sum_{i=1}^{2^{k-1}}\sum_{j=0}^{M-1}\psi_{ij}(s)\right)K(x,s)ds\right). \tag{43}$$

We obtain an upper bound for the norm of the matrix A

$$\|A\| = K_1\max_{1\leq l\leq 2^{k-1},\ 0\leq m\leq M-1}\left\{\int\limits_0^1\psi_{lm}(x)dx\int\limits_0^1\left[\sum_{i=1}^{2^{k-1}}\sum_{j=0}^{M-1}(2j+1)^{\frac{1}{2}}2^{\frac{k-1}{2}}L_j(2^ks-\hat{n})\right]ds\right\} \leq$$

$$\leq 2^{\frac{k-1}{2}}K_1(2M-1)^{\frac{1}{2}}M2^{k-1}\max_{1\leq l\leq 2^{k-1},\ 0\leq m\leq M-1}\left|\int\limits_0^1\psi_{lm}(x)dx\right| \leq$$

$$\leq 2^{\frac{3(k-1)}{2}}K_1(2M-1)^{\frac{1}{2}}M(2M-1)^{\frac{1}{2}}2^{\frac{k-1}{2}} = 2^{2(k-1)}K_1(2M-1)M.$$

Thus

$$\|A\| \le 2^{2(k-1)} K_1 (2M-1) M. \tag{44}$$

Let us estimate the norm of matrix (43) from below:

$$\|A\| \ge K_0 2^{\frac{k-1}{2}} M 2^{k-1} \max_{1\le l\le 2^{k-1},\ 0\le m\le M-1} \left| \int_0^1 \psi_{lm}(x)dx \right| \ge$$
$$\ge 2^{\frac{3(k-1)}{2}} K_0 M 2^{\frac{k-1}{2}} = 2^{2(k-1)} K_0 M.$$

Taking into account inequality (44), we have from this relation that

$$2^{2(k-1)} K_0 M \le \|A\| \le 2^{2(k-1)} K_1 (2M-1) M. \tag{45}$$

The matrix A is symmetric therefore [5]

$$\|A^{-1}\| \le \frac{1}{\|A\|}. \tag{46}$$

From (45), (46) we have

$$\|A^{-1}\| \le \frac{1}{2^{2(k-1)} K_0 M}. \tag{47}$$

We have the following estimate for the solution of the system of linear algebraic equations (13):

$$\|C\| \le \|A^{-1}\| \|F\| \le \|A^{-1}\| \|f\| \|\psi\| = \frac{F_1 (2M-1)^{1/2} 2^{\frac{k-1}{2}}}{2^{2(k-1)} K_0 M}.$$

Thus,

$$\|C\| \le \frac{(2M-1)^{1/2} F_1}{2^{\frac{3(k-1)}{2}} K_0 M}. \tag{48}$$

Substitute (48) into (42) to obtain

$$\|r_l(x)\| \le \left(\frac{2M-1}{2^{k-1}} \right)^{1/2} M \left(\frac{(2M-1)^{1/2} F_1 K_1}{2^{3/2(k-1)} K_0 M} + 2^{\frac{k-1}{2}} F_0 \right).$$

This gives estimate (39).

According to the Bubnov-Galerkin method, the coefficients C_{lm} are determined from the system of linear algebraic equations (13) with a symmetric matrix A. The matrix and right-hand side of this system are calculated inaccurately, i.e. integrals are calculated with an error. Then, instead of the system (15) presented for the solution, in fact, the following system is solved:

$$A_1 \alpha^* = b_1, \ \ A_1 = A + \delta A, \ \ b_1 = b + \delta b. \tag{49}$$

Let the estimates $\|\delta A\|$ and $\|\delta b\|$ be known. Let us estimate the error of the solution to the system (15).

Let us denote solutions (15) and (41) by α and α^*, and the difference $\alpha^* - \alpha$ denote by $\delta\alpha$. Substituting the expressions $A_1,\ b_1$, and α^* into (49) to have

$$(A + \delta a)(\alpha + \delta\alpha) = b + \delta b.$$

Subtracting (15) from this equality, we obtain

$$A(\delta a) + \delta A \alpha + \delta A(\delta\alpha) = \delta b,$$

whence we get

$$A(\delta a) = \delta b - \delta A \alpha - \delta A(\delta\alpha),$$

and

$$\delta\alpha = A^{-1}(\delta b - \delta A\alpha - \delta A(\delta\alpha)). \tag{50}$$

To estimate the error of an approximate solution, the following theorem is valid.

Theorem 4.1. Let the matrix A have an inverse matrix, and let the condition

$$\|A^{-1}\|\|\delta A\| < 1, \tag{51}$$

hold. Then the matrix $A_1 = A + \delta A$ has an inverse matrix, and the following error estimate is valid

$$\|\delta\alpha\| \leq \frac{\|A\|^{-1}(\|\delta b\| + \|\delta A\|\|\alpha\|)}{1 - \|A^{-1}\|\|\delta A\|}. \tag{52}$$

The proof will use the following lemma [25].

Lemma 4.3. Let B be a square matrix satisfying the condition $\|B\| < 1$ and E be the identity matrix. Then there exists a matrix $(E+B)^{-1}$, and

$$\|(E+B)^{-1}\| \leq \frac{1}{1-\|B\|}. \tag{53}$$

Proof. For any vector $x \in V$ we have

$$\|(E+B)x\| = \|x + Bx\| \geq \|x\| - \|Bx\| \geq \|x\| - \|B\|\|x\| = (1-\|B\|)\|x\| = \delta\|x\|,$$

where $\delta = 1 - \|B\| > 0$.

In the inequality

$$\|(E+B)x\| \geq \delta\|x\| \tag{54}$$

denote $(E+B)x = y$, $x = (E+C)^{-1}y$ and rewrite (54) as

$$\|y\| \geq \delta\|(E+B)^{-1}y\|.$$

From this we get

$$\|(E+B)^{-1}y\| \leq \frac{1}{\delta}\|y\| = \frac{1}{1-\|B\|}\|y\|.$$

The lemma is proved.

Proof of Theorem 4.1. Let us prove that there is an inverse matrix $(A+\delta A)^{-1}$.

$$A_1 = A + \delta A = A(E + A^{-1}\delta A) = A(E+B),$$

where $B = A^{-1}\delta A$. By condition (52), we have

$$\|B\| = \|A^{-1}\delta A\| \leq \|A^{-1}\|\|\delta A\| < 1,$$

therefore, according to Lemma 4.3, there exists $(E+B)^{-1}$, and hence A_1^{-1} also exists.

We now prove inequality (52). From relation (50) we obtain the inequality

$$\|\delta\alpha\| \leq \|A^{-1}\|\|\delta b\| + \|A^{-1}\|\|\delta A\|\|\alpha\| + \|A^{-1}\|\|\delta A\|\|\delta\alpha\|.$$

By condition (51) of the theorem, we have $\|A^{-1}\|\|\delta A\| < 1$. Collecting the terms containing $\|\delta\alpha\|$ in the left-hand side, we obtain the estimate

$$\|\delta\alpha\| \leq \frac{\|A^{-1}\|(\|\delta b\| + \|\delta A\|\|\alpha\|)}{1 - \|A^{-1}\|\|\delta A\|}.$$

The theorem is proved.

5. Numerical analysis

To illustrate the possibilities of the proposed numerical solution to the first kind Fredholm integral equation, consider the following example:

$$\int_0^1 \left(xs + \frac{x+s}{3} + \frac{1}{5} \right) y(s)ds = 3x + \frac{19}{15}, \;\; 0 \le x \le 1.$$

with a symmetric kernel and exact solution

$$y(x) = 1 + 6x^2.$$

In calculations, we assumed $k = 2$ and $M = 2$. In this case, the number of basis functions and the dimension of the unknowns vector α will be $L = 4$. Cubature formulas were used to calculate the double integrals

$$\int_0^1 \int_0^1 K(x,s)\phi_j(s)\phi_i(x)dsdx,$$

where $h_1 = \frac{1}{N_1}$, $h_2 = \frac{1}{N_2}$ the cubature formulas of the rectangles were used. The number of integration nodes was chosen by the Runge rule, and the most optimal number was turned out to be $n = 51$. The results of numerical calculations at 11 nodes are shown in the following table.

Table 1.

i	x_i	$\widetilde{y}(x_i)$	$y_E(x_i)$	$\lvert\widetilde{y}(x_i) - y_E(x_i)\rvert$
1	0	0.98730	1	0.012700
2	0.1	1.40862	1.06	0.348622
3	0.2	1.82994	1.24	0.589944
4	0.3	2.25127	1.54	0.711266
5	0.4	2.67259	1.96	0.712588
6	0.5	2.50174	2.5	0.001739
7	0.6	3.27233	3.16	0.112327
8	0.7	4.04291	3.94000	0.102915
9	0.8	4.8135	4.84000	0.026496
10	0.9	5.58409	5.86000	0.275908
11	1	6.35468	7	0.645320

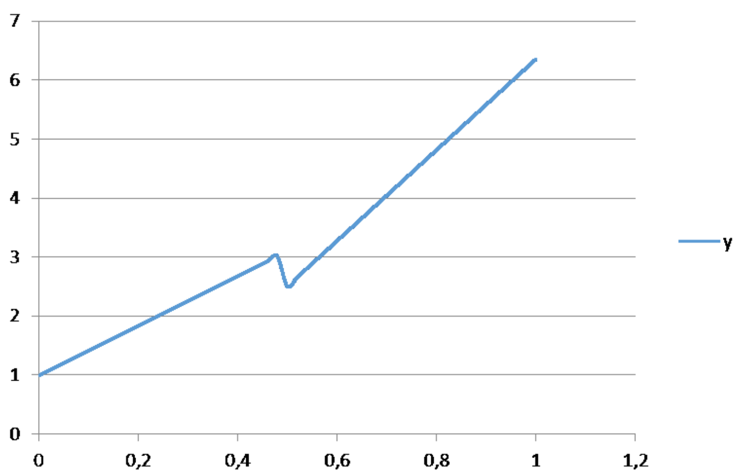

Figure 1. Approximate solution

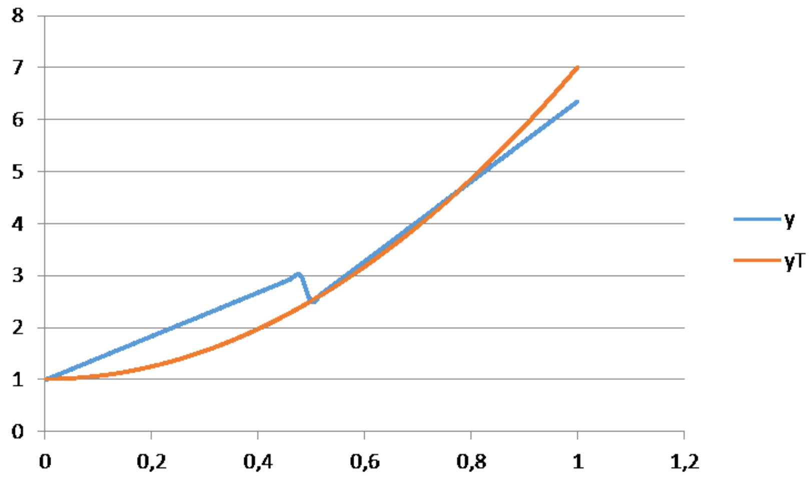

Figure 2. The approximate and exact solutions

This example was numerically solved by many methods, including the Tikhonov regularization method [25, 22], the Polozhiy method [21], and a constructive method with an integral ”tracking” operator with Poisson kernel proposed in [10].

6. Conclusion

The use of wavelets for solving the first kind Fredholm integral equations by the Galerkin method has shown enough efficiency. In addition, numerical calculations show the use of Legendre wavelets as basis functions has a positive effect for the numerical or analytical calculation of integrals in a computational scheme.

7. Acknowledgment

This research is funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP AP08856012), ”Development of a module of a geoinformation system based on the methods of intelligent detection of anomalies for in-depth exploration of deposits”.

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