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ON THE INFLUENCE OF INTEGRAL PERTURBATIONS ON THE BOUNDEDNESS OF SOLUTIONS OF A FOURTH-ORDER LINEAR DIFFERENTIAL EQUATION

SAMANDAR ISKANDAROV[1,2] , ELENA KOMARTSOVA[3]

Abstract. Sufficient conditions for the boundedness on the half-axis of all solutions of the fourth-order linear Volterra integro-differential equation are established. Moreover, it is shown that the corresponding linear homogeneous and inhomogeneous differential equations can have unbounded solutions on the half-axis. Illustrative examples are given.

Keywords: integro-differential equation, differential equation, boundedness of solutions, influence of integral perturbations.

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1. Introduction

We assume that all appearing functions and their derivatives are continuous and the relations are true when $t \geq t_0$, $t \geq \tau \geq t_0$; $J = [t_0, \infty)$. We abbreviate IDE - integro-differential equation; DE - differential equation.

Problem 1.1. *To establish sufficient conditions of boundedness on half-interval J of all solutions of the fourth-order IDE:*

$$x^{(4)}(t) + \sum_{k=0}^{3} \left[a_k(t)x^{(k)}(t) + \int_{t_0}^{t} Q_k(t,\tau)x^{(k)}(\tau)d\tau \right] = f(t), \;\; t \geq t_0, \tag{1}$$

in the case where the corresponding linear fourth-order DE

$$L(t,x) \equiv x^{(4)}(t) + a_3(t)x'''(t) + a_2(t)x''(t) + a_1(t)x'(t) + a_0(t)x(t) = 0, \;\; t \geq t_0, \tag{1_0}$$

$$L(t,x) = f(t), \;\; t \geq t_0, \tag{1_1}$$

can have unbounded solutions on J. As far as we know, this problem has not been studied by anyone before. We are talking about solutions $x(t) \in C^4(J,R)$ of the IDE (1) with any initial data $x^{(k)}(t_0)$ $(k = 0,1,2,3)$. Every such solution exists and unique.

[1]Laboratory of Theory of Integro-Differential Equations, Institute of Mathematics of the National Academy of Sciences of Kyrgyzstan, Bishkek, Kyrgyzstan

[2]Kyrgyz-Turkish Manas University, Bishkek, Kyrgyzstan

[3]Kyrgyz-Russian Slavic University, Bishkek, Kyrgyzstan

e-mail: mrmacintosh@list.ru, komartsovnm@mail.ru

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2. Main results

To solve the above problem, we use a non-standard method of reduction to the system [1], method of cutting functions [2, p.41], method of integral inequalities [6] are developed and Lemma 3.3 about integral inequalities of the first kind by $q(t)c \equiv q(t,c)$ [2, p.110 - 111].

Similar to [1] in the IDE (1) we make the following non-standard substitution:

$$x''(t) + \lambda^2 x(t) = W(t)y(t), \tag{2}$$

where $0 \neq \lambda$ – some auxiliary parameter, and $\lambda \in R$; $0 < W(t)$ – some auxiliary weighting function.

Let us denote
$b_3(t) \equiv a_3(t) + 2W'(t)\big(W(t)\big)^{-1}$, $b_2(t) \equiv a_2(t) + a_3(t)W'(t)\big(W(t)\big)^{-1} + W''(t)\big(W(t)\big)^{-1} - \lambda^2$, $b_1(t) \equiv [a_1(t) - \lambda^2 a_3(t)]\big(W(t)\big)^{-1}$, $b_0(t) \equiv [a_0(t) - \lambda^2 a_2(t) + \lambda^4]\big(W(t)\big)^{-1}$, $P_0(t,\tau) \equiv \big(W(t)\big)^{-1}\big[Q_0(t,\tau) - \lambda^2 Q_2(t,\tau)\big]$, $P_1(t,\tau) \equiv \big(W(t)\big)^{-1}\big[Q_1(t,\tau) - \lambda^2 Q_3(t,\tau)\big]$, $P_2(t,\tau) \equiv \big(W(t)\big)^{-1}\big[Q_2(t,\tau)W(\tau) + Q_3(t,\tau)W'(\tau)\big]$, $K(t,\tau) \equiv \big(W(t)\big)^{-1}Q_3(t,\tau)W(\tau)$, $F(t) \equiv \big(W(t)\big)^{-1}f(t)$.

Then instead of IDE (1) we have the following system:

$$\begin{cases} x''(t) + \lambda^2 x(t) = W(t)y(t), \\ y''(t) + b_3(t)y'(t) + b_2(t)y(t) + b_1(t)x'(t) + b_0(t)x(t) + \int\limits_{t_0}^{t} \big[P_0(t,\tau)x(\tau) + \\ +P_1(t,\tau)x'(\tau) + P_2(t,\tau)y(\tau) + K(t,\tau)y'(\tau)\big]d\tau = F(t), \quad t \geq t_0. \end{cases} \tag{3}$$

Developing of the non-standard method of reduction to the system on the fourth-order IDE (1) is to obtain the system (3) from the second order DE for $x(t)$ and from the second order IDE for $y(t)$. This reduction is equivalent by virtue of the condition $W(t) > 0$.
Let [2]:

$$K(t,\tau) = \sum_{i=0}^{n} K_i(t,\tau), \tag{K}$$

$$F(t) = \sum_{i=0}^{n} F_i(t), \tag{F}$$

$\psi_i(t)$ $(i = 1..n)$ – some cutting functions,
$R_i(t,\tau) \equiv K_i(t,\tau)\big(\psi_i(t)\psi_i(\tau)\big)^{-1}$, $E_i(t) \equiv F_i(t)\big(\psi_i(t)\big)^{-1}$ ($\psi_i(t)$ cut off, in particular, the growth, non-smoothness, alternation of the terms of $K(t,\tau)$ and $F(t)$),

$$R_i(t,t_0) = A_i(t) + B_i(t) \;\; (i = 1..n), \tag{R}$$

$c_i(t)$ $(i = 1..n)$ – some functions.

For any solution $\big(x(t), y(t)\big)$ of the system (3), its first equation we multiply by $x'(t)$, the second equation – by $y'(t)$ [7, p.194-217], add the resulting relations. Then we integrate between t_0 and t, including by parts, at the same time we introduce the conditions (K),(F),(R) functions $\psi_i(t)$, $R_i(t,\tau)$, $E_i(t)$, $c_i(t)$ $(i = 1..n)$), using lemmas 1.4, 1.5 [3]. Then after some transformations (including, taking account of $b_2(t) \equiv \gamma_0^2 + b_2(t) - \gamma_0^2$, $const\gamma_0 > 0$) we obtain the following identity:

$$\begin{aligned}&\big(x'(t)\big)^2+\lambda^2\big(x(t)\big)^2+\big(y'(t)\big)^2+\gamma_0^2\big(y(t)\big)^2+\sum_{i=1}^{n}\Big\{A_i(t)\big(Y_i(t,t_0)\big)^2-\\&-\int_{t_0}^{t}A_i'(s)\big(Y_i(s,t_0)\big)^2ds+B_i(t)\big(Y_i(t,t_0)\big)^2-2E_i(t)Y_i(t,t_0)+c_i(t)-\\&-\int_{t_0}^{t}\big[B_i'(s)\big(Y_i(s,t_0)\big)^2-2E_i'(s)Y_i(s,t_0)+c_i'(s)\big]ds+\int_{t_0}^{t}R_{i\tau}'(t,\tau)\big(Y_i(t,\tau)\big)^2d\tau-\\&-\int_{t_0}^{t}\int_{t_0}^{s}R_{is\tau}''(s,\tau)\big(Y_i(s,\tau)\big)^2d\tau ds\Big\}\equiv c_*+2\int_{t_0}^{t}W(s)y(s)x'(s)ds+\\&+2\int_{t_0}^{t}y'(s)\Big\{F_0(s)-b_3(s)y'(s)-[b_2(s)-\gamma_0^2]y(s)-b_1(s)x'(s)-b_0(s)x(s)-\\&-\int_{t_0}^{s}\big[P_0(s,\tau)x(\tau)+P_1(s,\tau)x'(\tau)+P_2(s,\tau)y(\tau)+K_0(s,\tau)y'(\tau)\big]d\tau\Big\}ds,\end{aligned}\tag{4}$$

where

$Y_i(t,\tau)\equiv\int_{\tau}^{t}\psi_i(\eta)y'(\eta)d\eta\ \ (i=1..n)$,

$c_*=\big(x'(t_0)\big)^2+\lambda^2\big(x(t_0)\big)^2+\big(y'(t_0)\big)^2+\gamma_0^2\big(y(t_0)\big)^2+\sum_{i=1}^{n}c_i(t_0)$.

Below we give the following lemma, which is applied in proof of our main theorem.

Lemma 2.1. [6]. Let for non-negative functions $u(t)$, $\alpha(t)$, $\beta(t)$, $\gamma(t,\tau)$ and $const=c\geq 0$ satisfy the integral inequality

$u(t)\leq c+\int_{t_0}^{t}\big[\alpha(s)\big(u(s)\big)^{\frac{1}{2}}+\beta(s)u(s)+\int_{t_0}^{s}\gamma(s,\tau)\big(u(\tau)u(s)\big)^{\frac{1}{2}}d\tau\big]ds.$

Then

$u(t)\leq\Big\{\sqrt{c}+\int_{t_0}^{t}\alpha(s)exp\big(-\int_{t_0}^{s}V(\eta)d\eta\big)ds\Big\}^2exp\big(\int_{t_0}^{t}V(s)ds\big),$

where

$V(t)\equiv\beta(t)+\int_{t_0}^{t}\gamma(t,\tau)d\tau.$

Theorem 2.1. *Let 1)* $\lambda>0$, $W(t)>0$, $\gamma_0>0$*; the presentation* (K), (F), (R) *are satisfied; 2)* $A_i(t)\geq 0$, $B_i(t)\geq 0$, $B_i'(t)\leq 0$, $R_{i\tau}'(t,\tau)\geq 0$, *there are functions* $A_i^*(t)\geq 0$, $c_i(t)$, $R_i^*(t)\geq 0$ *such that* $A_i'(t)\leq A_i^*(t)A_i(t)$, $\big(E_i^{(k)}(t)\big)^2\leq B_i^{(k)}(t)c_i^{(k)}(t)$, $R_{it\tau}''(t,\tau)\leq R_i^*(t)R_{i\tau}'(t,\tau)$ $(i=1..n;\ k=0,1)$. *Then for any solution* $(x(t),y(t))$ *of the system (3) the following energy estimate is valid:*

$$\begin{aligned}U(t)\equiv\big(x'(t)\big)^2+\lambda^2\big(x(t)\big)^2+\big(y'(t)\big)^2+\gamma_0^2\big(y(t)\big)^2+\sum_{i=1}^{n}\Big[A_i(t)\big(Y_i(t,t_0)\big)^2+\\+\int_{t_0}^{t}R_{i\tau}'(t,\tau)\big(Y_i(t,\tau)\big)^2d\tau\Big]\leq M(t,c_*),\end{aligned}\tag{5}$$

where

$$M(t,c_*)\equiv\Big\{\sqrt{c_*}+\int\limits_{t_0}^{t}|F_0(s)|exp\big(-\frac{1}{2}\int\limits_{t_0}^{s}V(\eta)d\eta\big)ds\Big\}^2 exp\big(\int\limits_{t_0}^{t}V(s)ds\big),$$

$$V(t)\equiv\sum_{i=1}^{n}\big[A_i^*(t)+R_i^*(t)\big]+2\Big\{\gamma_0^{-1}W(t)+|b_3(t)|+\gamma_0^{-1}|b_2(t)-\gamma_0^2|+|b_1(t)|+$$

$$+\lambda^{-1}|b_0(t)|+\int\limits_{t_0}^{t}\big[\lambda^{-1}|P_0(t,\tau)|+|P_1(t,\tau)|+\gamma_0^{-1}|P_2(t,\tau)|+|K_0(t,\tau)|\big]d\tau\Big\}.$$

Let, besides, 3) $A_j(t)>0$, $\psi_j(t)>0$,$\psi_j'(t)\ge 0$ $(1\le j\le n)$, $q_j(t,c_*)\ge 0$, $q_j'(t,c_*)\ge 0$, $q_j'(t,c_*)\big(\psi_j(t)\big)^{-1}\in L^1(J,R_+)$, *where* $q_j(t,c_*)\equiv\big(A_j(t)\big)^{-\frac{1}{2}}\big(M(t,c_*)\big)^{\frac{1}{2}}$. *Then* $y(t)=O(1)$.

Proof. First of all, note that the conditions $\big(E_i^{(k)}(t)\big)^2\le B_i^{(k)}(t)c_i^{(k)}(t)$ $(i=1..n;\ k=0,1)$ are guaranteed the fulfillment of relationships

$$(-1)^k\big[B_i^{(k)}(t)\big(Y_i(t,t_0)\big)^2-2E_i^{(k)}Y_i(t,t_0)+c_i^{(k)}(t)\big]\ge 0\quad (k=0,1;\ i=1..n).\tag{6}$$

Under the conditions 1, 2 also $U(t)\ge 0$, taking into account (6) and inequalities:

$$|x'(t)|\le\big(U(t)\big)^{\frac{1}{2}},\ |x(t)|\le\lambda^{-1}\big(U(t)\big)^{\frac{1}{2}},\ |y'(t)|\le\big(U(t)\big)^{\frac{1}{2}},\ |y(t)|\le\gamma_0^{-1}\big(U(t)\big)^{\frac{1}{2}},$$
$$\sum_{i=1}^{n}A_i(t)\big(Y_i(t,t_0)\big)^2\le U(t),\ \sum_{i=1}^{n}\int\limits_{t_0}^{t}R_{i\tau}'(t,\tau)\big(Y_i(t,\tau)\big)^2d\tau\le U(t),\tag{7}$$

from identity(4) we obtain the following integral inequality:

$$U(t)\le c_*+2\int\limits_{t_0}^{t}\Big\{|F_0(s)|\big(U(s)\big)^{\frac{1}{2}}+\big[\gamma_0^{-1}W(s)+\frac{1}{2}\sum_{i=1}^{n}\big(A_i^*(s)+R_i^*(s)\big)+$$
$$+|b_3(s)|+\gamma_0^{-1}|b_2(s)-\gamma_0^2|+|b_1(s)|+\lambda^{-1}|b_0(s)|\big]U(s)+\tag{8}$$
$$+\big(U(s)\big)^{\frac{1}{2}}\int\limits_{t_0}^{s}\big[\lambda^{-1}|P_0(s,\tau)|+|P_1(s,\tau)|+\gamma_0^{-1}|P_2(s,\tau)|+|K_0(s,\tau)|\big]\big(U(\tau)\big)^{\frac{1}{2}}d\tau\Big\}ds.$$

Applying to the integral inequality (8) the above lemma, we have

$$U(t)\le M(t,c_*).\tag{9}$$

From (9) by virtue of (7) we have the following estimate:

$$\sum_{i=1}^{n}A_i(t)\big(Y_i(t,t_0)\big)^2\le M(t,c_*).$$

Hence for $1\le j\le n$, taking into account the notation $Y_i(t,t_0)$, we get

$$A_j(t)\Big(\int\limits_{t_0}^{t}\psi_j(\eta)y'(\eta)d\eta\Big)^2\le M(t,c_*),$$

from which we obtain the following integral inequality of the first kind:

$$\Big|\int\limits_{t_0}^{t}\psi_j(\eta)y'(\eta)d\eta\Big|\le\big(A_j(t)\big)^{-\frac{1}{2}}\big(M(t,c_*\big)^{\frac{1}{2}}\equiv q_j(t,c_*).\tag{10}$$

To integral inequality (10) we apply Lemma 3.3 [2, p.110-111], which gives an estimate

$$|y(t)| \le |y(t_0)| + \big(\psi_j(t_0)\big)^{-1} q_j(t_0, c_*) + \int\limits_{t_0}^{t} q_j'(s, c_*)\big(\psi_j(s)\big)^{-1} ds. \quad (11)$$

Based on the last of conditions 3) from (11) follows

$$|y(t)| \le |y(t_0)| + \big(\psi_j(t_0)\big)^{-1} q_j(t_0, c_*) + \int\limits_{t_0}^{\infty} q_j'(s, c_*)\big(\psi_j(s)\big)^{-1} ds < \infty,$$

i.e. $y(t) = O(1)$. The theorem is proved.

From this theorem by $B_i(t) \equiv 0$, $A_i(t) \equiv R_i(t, t_0)$, $F_i(t) \equiv E_i(t) \equiv c_i(t) \equiv 0$, $F_0(t) \equiv 0$ follows
Corollary 2.1. If 1) $\lambda > 0$, $W(t) > 0$, $\gamma_0 > 0$; presentation (K) is true; $R_i(t, t_0) \ge 0$, $R'_{i\tau}(t, \tau) \ge 0$, there are functions $A_i^*(t) \ge 0$, $R_i^*(t) \ge 0$, such that $R'_{it}(t, t_0) \le A_i^*(t) R_i(t, t_0)$, $R''_{it\tau}(t, \tau) \le R_i^*(t) R'_{i\tau}(t, \tau)$ $(i = 1..n)$, then for any solution $\big(x(t), y(t)\big)$of system (3) with $F(t) \equiv 0$ the following energy estimate is valid:

$$U(t) \le M_1(t, c_*), \quad (12)$$

where

$$M_1(t, c_*) \equiv c_* exp\bigg(\int\limits_{t_0}^{t} V(s) ds\bigg).$$

If, in addition, 2) $R_j(t, t_0) > 0$, $\psi_j(t) > 0$, $\psi_j'(t) \ge 0$ $(1 \le j \le n)$, $q_{j1}(t, c_*) \ge 0$, $q'_{j1}(t, c_*) \ge 0$, $q'_{j1}(t, c_*)\big(\psi_j(t)\big)^{-1} \in L^1(J, R_+)$,
where $q_{j1}(t, c_*) \equiv \big(R_j(t, t_0)\big)^{-\frac{1}{2}}\big(M_1(t, c_*)\big)^{\frac{1}{2}}$,
then $y(t) = O(1)$.

Theorem 2.2. *If all the conditions of Theorem 2.1 are satisfied and* $W(t) \in L^1(J, R_+ \backslash \{0\})$,*then any solution of the fourth-order IDE (1)* $x(t) = O(1)$, *i.e. it is bounded to a half-interval* J.

The validity of this theorem is obtained from the following integral Cauchy representation [4, p.393-394]:

$$x(t) = x(t_0)\cos\lambda(t - t_0) + \frac{1}{\lambda}x'(t_0)\sin\lambda(t - t_0) + \frac{1}{\lambda}\int\limits_{t_0}^{t} [\sin\lambda(t - s)] W(s) y(s) ds,$$

from replacement (2), i.e. from first equation of system (7) for any initial values $x(t_0), x'(t_0)$.

We give the simplest illustrative examples.

Example 2.1. For the fourth-order linear homogeneous IDE

$$x^{(4)}(t) + \frac{1}{6}(t+1)^{-1}x'''(t) - \frac{1}{2}(t+1)^{-2}x''(t) + (t+1)^{-3}x'(t) - (t+1)^{-4}x(t) +$$
$$+\int\limits_{t_0}^{t} \bigg\{ \big[|\cos(t\tau)| + Q_3(t,\tau)\big]x(\tau) + \bigg[\frac{1}{t - \tau + 2} + Q_3(t,\tau)\bigg]x'(\tau) + \big[1 + Q_3(t,\tau)\big]x''(\tau) + \quad (13)$$
$$+ Q_3(t,\tau)x'''(\tau)\bigg\} d\tau = 0, \quad t \ge 0,$$

where

$$Q_3(t,\tau) \equiv \frac{e^{-t+\tau}}{t - \tau + 1} + e^{-t+\tau}\Big[\exp\bigg(\frac{\sin t}{t+4}\bigg) + \tau\Big]^{\frac{1}{2}} \exp(t^5 + \tau^5 + e^t + e^\tau + \exp e^t + \exp e^\tau),$$

all conditions of the Theorem 2.2 are fulfilled when $\lambda = 1$, $W(t) \equiv e^{-t}$, $\gamma_0 = 1$, here $t_0 = 0$, $b_3(t) \equiv \frac{1}{6}(t+1)^{-1} - 2$, $b_2(t) \equiv -\frac{1}{2}(t+1)^{-2} - \frac{1}{6}(t+1)^{-1}$, $b_1(t) \equiv \left[(t+1)^{-3} - \frac{1}{6}(t+1)^{-1}\right]e^t$, $b_0(t) \equiv \left[1 + \frac{1}{2}(t+1)^{-2} - (t+1)^{-4}\right]e^t$, $P_0(t,\tau) \equiv e^t\left[|\cos(t\tau)| - 1\right]$, $P_1(t,\tau) \equiv \frac{e^t}{t-\tau+2}$, $P_2(t,\tau) \equiv e^{t-\tau}$, $K(t,\tau) \equiv \frac{1}{t-\tau+1} + \left[exp\left(\frac{sint}{t+4}\right) + \tau\right]^{\frac{1}{2}} \exp(t^5 + \tau^5 + e^t + e^\tau + \exp e^t + \exp e^\tau)$, $K_0(t,\tau) \equiv \frac{1}{t-\tau+1}$, $n = 1$,$\psi_1(t) \equiv \exp(t^5 + e^t + \exp e^t)$, $R_1(t,\tau) \equiv \left[exp\left(\frac{\sin t}{t+4}\right) + \tau\right]^{\frac{1}{2}}$, $A_1^*(t) \equiv R_1^*(t) \equiv \frac{t+5}{(t+4)^2}$. So, all solutions of this IDE (13) are bounded on the half-axis R_+. However, the corresponding DE for the IDE (13): $L(t,x) \equiv x^{(4)}(t) + \frac{1}{6}(t+1)^{-1}x'''(t) - \frac{1}{2}(t+1)^{-2}x''(t) + (t+1)^{-3}x'(t) - (t+1)^{-4}x(t) = 0$, $t \geq 0$, has unbounded solutions on the R_+, which follows from the general solution of this DE: $x(t) = (t+1)c_1 + (t+1)^2c_2 + (t+1)^3c_3 + (t+1)^{-\frac{1}{6}}c_4$ (c_1, c_2, c_3, c_4 – arbitrary constants).

Example 2.2. The fourth-order linear nonhomogeneous equation (1), where functions $a_k(t)$ $(k = 0,1,2,3)$, $Q_j(t,\tau)$ $(j = 0,1,2)$ same as in IDE (13),

$$Q_3(t,\tau) \equiv \frac{e^{-t+\tau}}{t-\tau+1} + e^{-t+\tau}\left\{\left[\exp\left(\frac{\sin t}{t+4}\right) + \tau\right]^{\frac{1}{2}} + \frac{1}{t-\tau+5}\right\}\exp(t^5 + \tau^5 +$$

$$+e^t + e^\tau + \exp e^t + \exp e^\tau),\quad f(t) \equiv -\frac{e^{-t}}{t+10} - \frac{\exp(t^5 + e^t + \exp e^t)}{t+6},\ t \geq 0,$$

is satisfied all conditions of Theorem 2.2 for the same $\lambda = 1$, $W(t) \equiv e^{-t}$, $\gamma_0 = 1$, $b_k(t)$ $(k = 0,1,2,3)$, $P_j(t,\tau)$ $(j = 0,1,2)$, $K_0(t,\tau), n = 1$, $\psi_1(t) \equiv \exp(t^5 + e^t + \exp e^t)$, as in example 1, here $R_1(t,\tau) \equiv \left[\exp\left(\frac{\sin t}{t+4}\right) + \tau\right]^{\frac{1}{2}} + \frac{1}{t-\tau+5}$, $A_1(t) \equiv \left[\exp\left(\frac{\sin t}{t+4}\right) + \tau\right]^{\frac{1}{2}}$, $B_1(t) = \frac{1}{t+5}$, $F_0(t) \equiv -\frac{1}{t+10}$, $E_1(t) \equiv -\frac{1}{t+6}$, $c_1(t) \equiv \frac{1}{t+5}$. Consequently, all solutions to such an IDE are bounded on the half-axis. At the same time, the corresponding linear nonhomogeneous DE:

$$L(t,x) \equiv -\frac{e^{-t}}{t+10} - \frac{exp(t^5 + e^t + expe^t)}{t+6},\ t \geq 0 \qquad (1_1^*)$$

has unbounded solutions. This follows from the structure of the general solution of the DE (1_1^*): $x(t) = x_0(t) + x_r(t)$, where $x_0(t)$ – general solution of homogeneous DE for IDE (13), $x_r(t)$ – particular solution of DE (1_1^*), which will also be nonbounded on the half axis R_*, that is obtained from the Cauchy formula for solving the Cauchy problem for (1_1^*) with any $x_k(t_0)$ $(k = 0,1,2,3)$.

3. Conclusion

As examples 1 and 2 are showed, we managed to find a class of IDE of the form (1) for which the above problem is solvable. Note that cutting functions $\psi_i(t)$ $(i = 1..n)$ play the main role in solving the problem posed above. They help to investigate the asymptotic properties of solutions of new classes of Volterra type integro-differential equations on semiaxes.

Solvability connection of integral inequality of the first kind [2, p.110-111] and solvability of integral equation of the first kind [5] can be investigated.

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Samandar Iskandarov was born in Leilek region of Kyrgyzstan in 1951. He graduated from the Kyrgyz State University in 1974. He received his Ph.D. degree at the Institute of Mathematics and Mechanics of Azerbaijan Academy of Sciences in 1989 and Doctorate of Sciences degree at the Institute of Mathematics of NAS of Kyrgyz Republic in 2003. He is a professor of mathematics (2015) and at the present time he is a head of Laboratory of Theory of Integro-Differential Equations.

Elena Komartsova was born in 1979, graduated from Kyrgyz National University named after J. Balasagyn in 2002. Presently, she works as a senior lecturer at the Kyrgyz-Russian Slavic University.