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SUPPLY CHAIN INVENTORY MODEL FOR DETERIORATING PRODUCTS WITH MAXIMUM LIFETIME UNDER TRADE-CREDIT FINANCING

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Abstract. In this article, we analyze an economic order quantity (EOQ) model for deteriorating products, in which we assume that the deteriorating items not only deteriorate continuously, but also have their maximum lifetime and we determine the optimal solution of the total annual relevant cost in terms of some lemmas and investigate the obtained properties of the optimal solution, which leads to solution procedures that are simple and easy to understand. We then show that the total annual relevant cost is convex and this result is shown to imply that the optimal replenishment cycle time not only exists, but it also is unique. Finally, numerical examples are given to illustrate the theoretical results of this model. We also present a discussion of the sensitivity analysis of the optimal solution with respect to the key parameters. Furthermore, in the last section on concluding remarks and observations, we have cited many related recent works on the subject-matter of this investigation in order to provide incentive and motivation for making further advances along the lines of the supply chain management and associated inventory problems which we have discussed in this article.

Keywords: deteriorating products, maximum lifetime, trade-credit financing, permissible delay in payments, economic order quantity (EOQ), inventory, supply chain management, mathematical analytic solution procedures.

AMS Subject Classification: 26A06, 26A24, 91B24, 93C15, 26D10, 90B30.

1. Introduction and motivation

In commercial practice, certain types of products either deteriorate, decay or become obsolete in the course of time and are not in a perfect condition to satisfy the demand. The seasonal food, vegetables, fish, fruits, alcohol, gasoline, radio-active chemicals and electronic substances are examples of such products in which sufficient deterioration can take place during the normal storage period of the units because of poor storage and preservation. Apparently, deterioration

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has a crucial impact and to control and maintain the inventory of deteriorating products in inventory control system in order to satisfy the customer's demand or the retailer's order, each of which is important in the business. Ghare and Schrader [14] developed an EOQ model with constant rate of deterioration. Goyal and Giri [16] presented a review of the inventory literature for deteriorating items since the early 1990s. Bakker et al. [2] provided an excellent review of inventory systems with deterioration since 2001. Sarkar et al. [31] explored an inventory model with variable demand and selling price for deteriorating items. In addition, in the real-life situation, we see that if the retailer has an option to pay for products with a delay without interest charges, this type of trade credit is equivalent to offering the retailers short-term interest-free finance in stock. Furthermore, the trade credit should affect the retailer's conduct of ordering significantly. That is, through a real practice regarding the supplier, he/she allows certain fixed period for settling the account and does not charge any interest from the retailer on the amount owed during this period, but the retailer will pay an additional opportunity cost. Obviously, the trade credit is a common commercial strategy in business and the strategic aims are to enhance the elasticity of the capital. An inventory model with permissible delay in payments was first studied by Goyal [17]. Liao [21] developed an inventory control model with non-instantaneous receipt and exponential deteriorating item under two levels of the trade-credit policy. Chen et al. [9] explored an economic production quantity (EPQ) model for deteriorating items with two levels of the trade-credit financing. Bhunia et al. [4] developed a two-warehouse inventory model for single deteriorating items with trade credit in which shortages are allowed and partially backlogged. Studied optimal strategy for deteriorating items with capacity constraints under upstream and downstream trade credits. Some relevant recent articles in trade-credit financing were developed by Taleizadeh (see [36] and [37]), Wu et al. [48]), Mahata and Mahata [26], Mahata et al. [28], Bera et al. [3], Chen et al. [6], Mahata [24], Yang and Chang [50], Singh et al. [34], Giri and Sharma [15], Jaggi et al. [19], and many other authors who care cited in each of these works.

In fact, most of the above-mentioned articles assumed that the retailers were credit-worthy customers who may be qualified to get a permissible delay on the entire purchasing quantity without collateral deposits from the supplier. However, trade-credit risk is a common supply chain problem in the class of credit risks, especially, when the retailer's default or credit risk increases which will impact the supplier, thereby increasing the supplier's another credit risk. Furthermore, the offer of a delay period leads to default risk for the supplier, related to the potential situation when the retailer declares his/her inability to pay off. Based upon the above arguments, this article explores the situation when the credit-risk retailer reduces the financed loan from constant sales and revenue credited gradually and he/she still can utilize the sales revenue to earn interest when he/she pays off all accounts.

None of the above-mentioned articles was formulated for deteriorating items with expiration for which deterioration is usually observed. In fact, certain types of products either deteriorate or become obsolete in the course of time because of poor storage or preservation. Then the product will have a maximum lifetime which is time bound. Furthermore, expiration plays a major role as in the case of deteriorating items in management of inventory. It is necessary to consider the inventory problems for deteriorating items with maximum lifetime. Chen and Teng [7] developed the retailer's optimal ordering policy for deteriorating items with maximum lifetime under the supplier's trade-credit financing. Wu et al. [47] obtained the optimal credit period and lot size for deteriorating items with expiration dates under the two-level trade-credit financing. Wang et al. [43] explored the seller's optimal credit period and cycle time in a supply chain for deteriorating items with maximum lifetime. Yadav et al. [49] developed manufacturing inventory model for deteriorating items with maximum lifetime under the two-level trade-credit financing. Sarkar et al. [32] formulated a trade-credit policy with variable deterioration for fixed lifetime products. Mahata and De [25] explored the inventory model for deteriorating items with maximum lifetime and partial trade credit to credit-risk customers. There are several interesting and relevant papers related to inventory models with maximum lifetime such as those by Wu and Chan [46], Mahata and Mahata [27], Chen and Teng [8], Chen et al. [9], Teng et al. [39] and Wu et al. [45].

Owing to the complexity of the calculus, most of the above articles investigate the case when the inventory model for deteriorating products with maximum lifetime, thereby proving that the total annual relevant cost is pseudo-convex in the replenishment cycle time T. In this article, we show that the total cost per unit time is convex, with convexity leading to a simple optimization procedure to improve the observations described by the above-mentioned articles.

By combining the aforementioned arguments, this article explores two important factors in the purchase decision of the inventory model, one of which is that the product not only deteriorates continuously, but also has its maximum lifetime. The other factor is that, in order to reduce the debt risks, the supplier offers another kind of trade credit to settle the credit-risk retailer's accounts. Mathematical theorems have been established to show that the total annual relevant cost is convex and this result provides that the optimal replenishment cycle time T not only exists, but also is unique. Finally, numerical examples are provided to validate the proposed models. Numerical analysis of the parameters involved in the proposed model has also been carried out and the implications are discussed.

We remark in passing that, in the last section (Section 6) on concluding remarks and further observations, we have chosen to include citations of a number of related recent works on the subject-matter of our present investigation with a view to providing incentive and motivation for making further advances along the lines of the supply chain management and the associated inventory problems which we have discussed in this article.

2. Notation and assumptions

o	ordering cost per order in dollars
c	purchasing cost per unit in dollars
p	selling price per unit in dollars, with $p > c$
h	holding cost per unit per year in dollars excluding interest charge
I_e	interest earned per dollar per year
I_p	interest charged per dollar per year
$I(t)$	inventory level at time t
$\theta(t)$	deterioration rate at time t, which is a non-decreasing function in t
m	maximum lifetime or expiration time in years
M	trade-credit period in years by the supplier
D	demand rate in units per year
Q	order quantity in units per replenishment cycle
T	the replenishment cycle time in years
$Z(T)$	the annual total relevant cost
T^*	the optimal replenishment cycle time of $Z(T)$
S	$\dfrac{pM(2+I_eM)}{2c}$

Next, the mathematical model in this article is developed under the following assumptions:

(1) A deteriorating item deteriorates continuously and cannot be sold after its maximum lifetime or expiration rate. Hence, the deterioration rate must be closed to 1 when time is approaching to the expiration date m. Then, it is assumed without loss of generality (WLOG) that the deterioration rate $\theta(t)$ at time t $(0 \le t \le m)$ satisfies the following conditions:

$$0 \le \theta(t) \le 1, \quad \theta'(t) \le 0 \qquad \text{and} \qquad \theta(m) = 1.$$

Now, to make the problem tractable, we assume that the deterioration rate is the same as that in Wang et al. [43] as follows:

$$\theta(t) = \frac{1}{1+m-t} \qquad (0 \le t \le T \le m). \tag{1}$$

We note that it is clear that the replenishment cycle time T is less than or equal to m.

(2) Shortages are not permitted.
(3) Replenishment rate is instantaneous and lead time is zero.
(4) Demand rate is known and constant.
(5) Time horizon is infinite.
(6) Trade credit between supplier and retailer is as follows:

(a) At initial time, the retailer takes a loan to pay the procurement cost to the supplier, then the retailer must face interest charged during the time interval $(0, M]$.

(b)If the retailer cannot pay off the balance at time M, then the supplier charges the retailer an interest rate of I_p on unpaid balance and the retailer must utilize the sales revenue to pay off the remaining amount owed to the supplier. Once the retailer pays off all accounts, he/she keeps the profit and sells revenue is utilized to earn interest throughout the replenishment cycle time T.

3. Mathematical formulation of the problem

During the replenishment cycle $[0, T]$, we suppose that the inventory level is depleted by the combination effect of demand and deterioration, so that the inventory level is governed by the following differential equation:

$$\begin{aligned} \frac{dI(t)}{dt} &= -D - \theta(t) I(t) \\ &= -D - \left(\frac{1}{1+m-t}\right) I(t), \qquad (0 \le t \le T), \end{aligned} \tag{2}$$

together with the boundary condition $I(T) = 0$. Solving the differential equation (2), we get

$$I(t) = e^{-\delta(t)} \int_t^T e^{\delta(u),} D du \qquad (0 \le t \le T), \tag{3}$$

where

$$\delta(t) = \int_0^t \theta(u)\,du = \int_0^t \frac{1}{1+m-u}du = \ln\left(\frac{1+m}{1+m-t}\right). \tag{4}$$

Substituting from the equation (4) into the equation (3), we get the inventory level at time t as follows:

$$I(t) = D(1+m-t)\ln\left(\frac{1+m-t}{1+m-T}\right), \qquad (0 \le t \le T). \tag{5}$$

Moreover, according to the assumption (6), the retailer takes a loan to pay the supplier the amount cDT at the time $t=0$, then the retailer must face interest charged, cI_pDMT, during the time interval $(0, M]$. Additionally, the retailer keeps the sales revenue and earns interest on sales revenue during the same time interval, that is,

$$pDM + \frac{pI_eDM^2}{2}.$$

Furthermore, if

$$pDM + \frac{pI_eDM^2}{2} < cDT,$$

that is, $S < T$, the retailer cannot pay off the unpaid balance at time M, so he/she must reduce the financed loan from sales revenue. On the other hand, if

$$pDM + \frac{pI_eDM^2}{2} \ge cDT,$$

that is, $S \ge T$, the retailer can pay off the procurement cost to the supplier at time M, so he/she has amount given by

$$pDM + \frac{pI_eDM^2}{2} - cDT$$

on hand and obtains additional interest earned.

Based on the above arguments, there are various sources of costs for the retailer: ordering cost, procurement cost, the holding cost excluding interest charges, and the opportunity cost. Each source of cost is described in detail below:

- Annual ordering cost $= \frac{o}{T}$.
- Annual procurement cost including the cost of deteriorating products

$$\frac{cQ}{T} = \frac{cD(1+m)}{T}\ln\left(\frac{1+m}{1+m-T}\right). \tag{6}$$

- Annual holding cost excluding interest charge

$$\begin{aligned} &= \frac{h}{T}\int_0^T I(t)\,dt = \frac{hD}{T}\int_0^T (1+m-t)\ln\left(\frac{1+m-t}{1+m-T}\right)dt \\ &= \frac{hD}{T}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right]. \end{aligned} \tag{7}$$

- Annual opportunity cost of the capital. According to the above arguments, it is clear that if $T \leq S$, the retailer has enough money to pay off the procurement amount at time M. Otherwise, the retailer must pay off the remain loan from sales revenue after time M, once the retailer pays off all accounts, he/she can keep the profit and sells revenue to earn interest until the replenishment cycle time T. Actually, there are three cases in terms of annual opportunity cost of the capital.

(i) $T \leq M < S$

In this case, the retailer can pay off the procurement cost at time M, so the annual interest payable is given by

$$\frac{I_p cDTM}{T} = I_p cDM.$$

On the other hand, the retailer sells this items and continues to accumulate sales revenue and earns the interest with rate I_e from $t = 0$ to $t = T$. So, the interest earned is given by

$$\frac{pI_e DT^2}{2}.$$

Additionally, the interest earned starts from the time $t = T$ to $t = M$ is given by

$$I_e\left(pDT + \frac{pI_e DT^2}{2}\right)(M - T).$$

Furthermore, the annual interest earned is given by

$$\frac{1}{T}\left[\frac{pI_e DT^2}{2} + I_e\left(pDT + \frac{pI_e DT^2}{2}\right)(M - T)\right]$$
$$= pI_e D\left[\frac{T}{2} + \left(1 + \frac{I_e T}{2}\right)(M - T)\right]$$

Consequently, the annual opportunity cost of capital is given by

$$I_p cDM - pI_e D\left[\frac{T}{2} + \left(1 + \frac{I_e T}{2}\right)(M - T)\right]. \tag{8}$$

The graphical representation of the inventory level is shown in Figure 1 below.

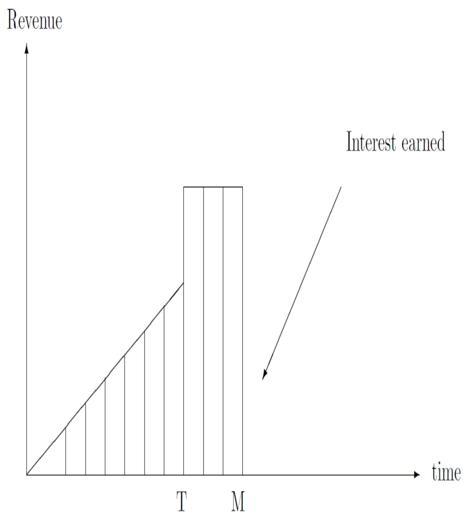

Figure 1. Annual opportunity cost of capital excluding $I_p cDM$ when $T \leq M < S$.

(ii) $M < T \leq S$

In this case, the retailer can pay off the procurement cost at time M, so the annual interest payable is $I_p cDM$. Conversely, the retailer uses the sales revenue to earn interest as well from $t = 0$ to $t = M$. So, the interest earned is given by

$$\frac{pI_e DM^2}{2}.$$

As mentioned before, due to the retailer pays off the procurement amount at time M, the retailer can use the remain revenue and the sales revenue to earn interest from $t = M$ to $t = T$ is given by

$$I_e\left(pDM + \frac{pI_e DM^2}{2} - cDT\right)(T - M) + \frac{pI_e D}{2}(T - M)^2.$$

Furthermore, the annual interest earned is given by

$$\frac{pI_e DM^2}{2T} + \frac{I_e}{T}\left(pDM + \frac{pI_e DM^2}{2} - cDT\right)(T - M) + \frac{pI_e D}{2T}(T - M)^2.$$

Consequently, the annual opportunity cost of capital is given by

$$I_p cDM - \frac{pI_e DM^2}{2T} - \frac{I_e}{T}\left(pDM + \frac{pI_e DM^2}{2} - cDT\right)(T - M) - \frac{pI_e D}{2T}(T - M)^2. \tag{9}$$

The graphical representation of the inventory level is shown in Figure 2 below.

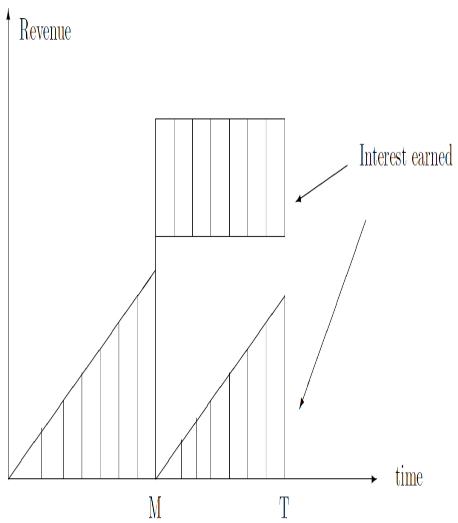

Figure 2. Annual opportunity cost of capital.

(iii) $M < S \leq T$

In this case, the retailer cannot pay off the procurement amount at time M, so the retailer must pay off the remain loan from sales revenue at the time t given by

$$M + \frac{cDT - pDM - \dfrac{pI_e DM^2}{2}}{pD}.$$

Hence the annual interest payable is given by

$$U = I_p cDM + \frac{I_p}{2pcDM}\left[cDT - pDM - \frac{pI_e DM^2}{2}\right]^2.$$

Additionally, the retailer uses the sales revenue to earn interest from $t = 0$ to $t = M$ is given by

$$\frac{pI_e DM^2}{2}.$$

In addition, the interest earned starts from the time t given by

$$t = M + \frac{cDT - pDM - \frac{pI_e DM^2}{2}}{pD}$$

to $t = T$ is as follows:

$$\frac{pI_e D}{2}\left(T - M - \frac{cDT - pDM - \frac{pI_e DM^2}{2}}{pD}\right)^2.$$

Moreover, the annual interest earned is given by

$$\frac{pI_e DM^2}{2T} + \frac{pI_e D}{2T}\left[T - M - \frac{cDT - pDM - \frac{pI_e DM^2}{2}}{pD}\right]^2$$
$$= \frac{pI_e DM^2}{2T} + \frac{pI_e D}{2T}\left[\left(1 - \frac{c}{p}\right)T + \frac{I_e M^2}{2}\right]^2.$$

Consequently, the annual opportunity cost of capital is given by

$$I_p cDM + \frac{I_p}{2pDT}\left(cDT - pDM - \frac{pI_e DM^2}{2}\right)^2 - \frac{pI_e DM^2}{2T} - \frac{pI_e D}{2T}\left[\left(1 - \frac{c}{p}\right)T + \frac{I_e M^2}{2}\right]^2. \quad (10)$$

The graphical representation of the inventory level is shown in Figure 3 below.

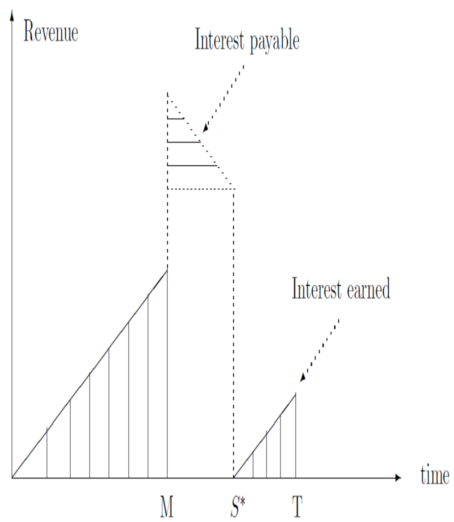

Figure 3. Annual opportunity cost of capital excluding U when $M < S \leq T$.

where

$$s^* + M = \frac{cT - pM - \dfrac{pI_eDM^2}{2}}{p}.$$

Based upon the above arguments, the annual total relevant cost can be expressed as follows:

$$Z(T) = \begin{cases} Z_1(T), & \text{if } 0 < T \leq M < S, \qquad (12a) \\ Z_2(T), & \text{if } M < T \\ leqS(12\text{b}), & \\ Z_3(T), & \text{if } S < T, \qquad (12\text{c}) \end{cases}$$

where

$$Z_1(T) = \frac{1}{T}\left\{cD(1+m)\ln\left(\frac{1+m}{1+m-T}\right) + o + hD\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right] + I_pcDMT - pI_eD\left[\frac{T^2}{2} + \left(T + \frac{I_eT^2}{2}\right)(M-T)\right]\right\}, \qquad (13)$$

$$Z_2(T) = \frac{1}{T}\left\{cD(1+m)\ln\left(\frac{1+m}{1+m-T}\right) + o + hD\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right] + I_pcDMT - \left[\frac{pI_eDM^2}{2} + I_e\left(pDM + \frac{pI_eDM^2}{2} - cDT\right) \cdot (T-M) + \frac{pI_eD}{2}(T-M)^2\right]\right\} \qquad (14)$$

and

$$Z_3(T) = \frac{1}{T}\left\{cD(1+m)\ln\left(\frac{1+m}{1+m-T}\right) + o + hD\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right] + I_pcDMT + \frac{I_p}{2pD}\left(cDT - pDM - \frac{pI_eDM^2}{2}\right)^2 - \frac{pI_eDM^2}{2} - \frac{pI_eD}{2}\left[\left(1 - \frac{c}{p}\right)T + \frac{I_eM^2}{2}\right]^2\right\} \qquad (15)$$

As a matter of fact, we have

$$Z_1(M) = Z_2(M) \qquad \text{and} \qquad Z_2(S) = Z_3(S),$$

so $Z(T)$ is continuous and well-defined.

4. Theoretical results and optimal solutions

The problem is to determine the replenishment cycle time T^* to minimize the annual total relevant cost. Furthermore, taking the first-order and the second-order derivatives of $Z_1(T)$, $Z_2(T)$ and $Z_3(T)$ with respect to T, we obtain

$$Z_1'(T) = \frac{1}{T^2}\Bigg\{\left(cD(1+m) + \frac{Dh(1+m)^2}{2}\right)\cdot\left[\frac{T}{1+m-T} - \ln\left(\frac{1+m}{1+m-T}\right)\right] + \frac{hDT^2}{4} - o - \frac{pI_eD}{2}\Big[I_eM - 2I_eT - 1\Big]T^2\Bigg\}, \tag{16}$$

$$Z_1''(T) = \frac{1}{T^3}\Bigg\{\left(cD(1+m) + \frac{Dh(1+m)^2}{2}\right)\cdot\left[\frac{T^2}{(1+m-T)^2} - \frac{2T}{1+m-T} + 2\ln\left(1+\frac{T}{1+m-T}\right)\right] + 2o + pI_e^2DT^3\Bigg\}, \tag{17}$$

$$Z_2'(T) = \frac{1}{T^2}\Bigg\{\left(cD(1+m) + \frac{Dh(1+m)^2}{2}\right)\cdot\left[\frac{T}{1+m-T} - \ln\left(\frac{1+m}{1+m-T}\right)\right] + \frac{hDT^2}{4} - o - I_e\left(-cDT^2 + \frac{pD}{2}T^2 + \frac{pI_eDM^3}{2}\right)\Bigg\}, \tag{18}$$

$$Z_2''(T) = \frac{1}{T^3}\Bigg\{\left(cD(1+m) + \frac{Dh(1+m)^2}{2}\right)\cdot\left[\frac{T^2}{(1+m-T)^2} - \frac{2T}{1+m-T} + 2\ln\left(1+\frac{T}{1+m-T}\right)\right] + 2o + pI_e^2DM^3\Bigg\}, \tag{19}$$

$$Z_3'(T) = \frac{1}{T^2}\Bigg\{\left(cD(1+m) + \frac{Dh(1+m)^2}{2}\right)\cdot\left[\frac{T}{1+m-T} - \ln\left(\frac{1+m}{1+m-T}\right)\right] + \frac{hDT^2}{4} - o - \frac{pDI_e^2M^4(I_e-I_p)}{8} - \frac{pDM^2(I_p-I_e)}{2} - \frac{pI_eI_pDM^3}{2} - \frac{D}{2}\left(\frac{c^2}{p}(I_e-I_p) - (2c-p)I_eT^2\right)\Bigg\} \tag{20}$$

and

$$Z_3''(T) = \frac{1}{T^3}\Bigg\{\left(cD(1+m) + \frac{Dh(1+m)^2}{2}\right)\cdot\left[\frac{T^2}{(1+m-T)^2} - \frac{2T}{1+m-T} + 2\ln\left(1+\frac{T}{1+m-T}\right)\right] + 2o + \frac{pI_e^2DM^4}{4}(I_p-I_e) + pDM^2(I_p-I_e) + pI_eI_pDM^3\Bigg\}. \tag{21}$$

We also note that

$$\left(cD\left(1+m\right)+\frac{Dh\left(1+m\right)^2}{2}\right)\cdot\left[\frac{T^2}{\left(1+m-T\right)^2}-\frac{2T}{1+m-T}+2\ln\left(1+\frac{T}{1+m-T}\right)\right]$$
$$>\left(cD\left(1+m\right)+\frac{Dh(1+m)^2}{2}\right)\cdot\left[\frac{T^2}{\left(1+m-T\right)^2}-\frac{2T}{1+m-T}+2\left(\frac{T}{1+m-T}-\frac{T^2}{2\left(1+m-T\right)^2}\right)\right]$$
$$=0. \tag{22}$$

Furthermore, it is clear from the equation (22) that $Z_1\left(T\right)$, $Z_2\left(T\right)$ and $Z_3\left(T\right)$ are strictly convex functions in T. Moreover, the equations (16), (18) and (20) yield

$$Z_1'\left(M\right)=\left(\frac{1}{M^2}\right)f_1\left(M\right), \tag{23}$$

$$Z_2'\left(M\right)=\left(\frac{1}{M^2}\right)f_2\left(M\right) \tag{24}$$

and

$$Z_2'\left(S\right)=Z_3'\left(S\right)=\left(\frac{1}{S^2}\right)f\left(S\right), \tag{25}$$

where

$$f_1\left(M\right)=\left(cD\left(1+m\right)+\frac{Dh\left(1+m\right)^2}{2}\right)\cdot\left[\frac{M}{1+m-M}-\ln\left(\frac{1+m}{1+m-M}\right)\right]$$
$$+\frac{hDM^2}{4}-o+\frac{pI_e^2DM^3}{2}+\frac{pI_eDM^2}{2}, \tag{26}$$

$$f_2\left(M\right)=\left(cD\left(1+m\right)+\frac{Dh\left(1+m\right)^2}{2}\right)\cdot\left[\frac{M}{1+m-M}-\ln\left(\frac{1+m}{1+m-M}\right)\right]$$
$$+\frac{hDM^2}{4}-o+cDI_eM^2-\frac{pI_eD}{2}M^2-\frac{pI_e^2DM^3}{2} \tag{27}$$

and

$$f\left(S\right)=\left(cD\left(1+m\right)+\frac{Dh\left(1+m\right)^2}{2}\right)\cdot\left[\frac{S}{1+m-S}-\ln\left(\frac{1+m}{1+m-S}\right)\right]$$
$$+\frac{hDS^2}{4}-o+\frac{pDI_e^2M^4\left(I_e-I_p\right)}{8}-\frac{pDM^2\left(I_p-I_e\right)}{2}-\frac{pI_eI_pDM^3}{2}$$
$$-\frac{D}{2}\left(\frac{c^2}{p}\left(I_e-I_p\right)-\left(2c-p\right)I_e\right)S^2. \tag{28}$$

We thus find that

$$f_1\left(M\right)>f_2\left(M\right) \qquad \text{and} \qquad f_2\left(M\right)<f\left(S\right),$$

which means that

$$Z_1'\left(M\right)>Z_2'\left(M\right) \qquad \text{and} \qquad Z_2'\left(M\right)<Z_2'\left(S\right)=Z_3'\left(S\right).$$

Based upon the above arguments, we have the following results in which, for convenience, we let T_i^* denote the optimal solution of $Z_i(T)$ on $T > 0$ for $i = 1, 2, 3$, respectively.

Lemma 4.1. *Each of the following assertions holds true*:

(1) $f_1(M) > 0$ *if and only if* $T_1^* < M$, *so that* $Z_1(T)$ *is decreasing on* $(0, T_1^*]$ *and increasing on* $[T_1^*, M)$.
(2) $f_1(M) \leq 0$ *if and only if* $T_1^* \geq M$, *so that* $Z_1(T)$ *is decreasing on* $(0, M)$.
(3) $f_2(M) > 0$ *if and only if* $T_2^* < M$, *so that* $Z_2(T)$ *is increasing on* $[M, S]$.
(4) $f_2(M) \leq 0$ *if and only if* $T_2^* \geq M$.
(5) $f(S) > 0$ *if and only if* $T_2^* < S$.
(6) $f(S) \leq 0$ if and only if $T_2^* \geq S$, so that $Z_2(T)$ is decreasing on $[M, S]$.
(7) $f(S) > 0$ *if and only if* $T_3^* < S$, *so that* $Z_3(T)$ *is increasing on* $[S, \infty)$.
(8) $f(S) \leq 0$ *if and only if* $T_3^* \geq S$, *so that* $Z_3(T)$ *is decreasing on* $[S, T_3^*]$ *and increasing on* $[T_3^*, \infty)$.

The above Lemma reveals that the meanings of $f_1(M), f_2(M)$ and $f(S)$ can be used to infer functional behaviors of all corresponding $Z_j(T)$ $(j = 1, 2, 3)$ on the intervals $[0, M], [M, S]$ and $[S, \infty)$, respectively, in order to locate optimal solutions for $Z(T)$. Additionally, since $Z_1(T)$, $Z_2(T)$ and $Z_3(T)$ are convex on $T > 0$ if T_i^* exists, we have

$$Z_i'(T) \begin{cases} < 0; & \text{if } 0 < T < T_i^*, & (29a) \\ = 0; & \text{if } T = T_i^*, & (29\text{b}) \\ > 0; & \text{if } T > T_i^*, & (29\text{c}). \end{cases}$$

So, $Z_i'(T)$ is decreasing on $(0, T_i^*]$ and increasing on $[T_i^*, \infty)$ for all $i = 1, 2, 3$ and the *Intermediate Value Theorem* (see Thomas et al. [40]) can be used to locate T_i^* if T_i^* exists for $i = 1, 2, 3$. Finally, in conjunction with the above Lemma, we have the following results.

Theorem 4.1. *Each of the following assertions holds true*:

(1) *If* $f_1(M) > 0$, $f_2(M) > 0$ *and* $f(S) > 0$, *then* $T^* = T_1^*$.
(2) *If* $f_1(M) \leq 0$, $f_2(M) \leq 0$ *and* $f(S) > 0$, *then* $T^* = T_2^*$.
(3) *If* $f_1(M) \leq 0$, $f_2(M) \leq 0$ *and* $f(S) \leq 0$, *then* $T^* = T_3^*$.
(4) *If* $f_1(M) > 0$, $f_2(M) \leq 0$ *and* $f(S) > 0$, *then* T^* *is* T_1^* *or* T_2^* *associated with the least cost.*
(5) *If* $f_1(M) > 0$, $f_2(M) \leq 0$ *and* $f(S) \leq 0$, *then* T^* *is* T_1^* *or* T_3^* *associated with the least cost.*

Proof. We consider each of the following cases.

(1) If $f_1(M) > 0$, $f_2(M) > 0$ and $f(S) > 0$, with the above Lemma $(1, 3, 5, 7)$,

$$Z_1(M) = Z_2(M) \qquad \text{and} \qquad Z_2(S) = Z_3(S),$$

and the equations $(12)(a, b, c)$ and $(29)(a, b, c)$, we conclude that $T^* = T_1^*$.

(2) If $f_1(M) \leq 0, f_2(M) \leq 0$ and $f(S) > 0$, together with the above Lemma $(2, 4, 5, 7)$,

$$Z_1(M) = Z_2(M) \qquad \text{and} \qquad Z_2(S) = Z_3(S),$$

and the equations $(12)(a, b, c)$ and $(29)(a, b, c)$, we conclude that $T^* = T_2^*$.

(3) If $f_1(M) \leq 0, f_2(M) \leq 0$ and $f(S) \leq 0$, together with the above Lemma $(2, 4, 6, 8)$,

$$Z_1(M) = Z_2(M) \qquad \text{and} \qquad Z_2(S) = Z_3(S),$$

and the equations $(12)(a, b, c)$ and $(29)(a, b, c)$, we conclude that $T^* = T_3^*$.

(4) If $f_1(M) > 0$, $f_2(M) \leq 0$ and $f(S) > 0$, with together the above Lemma $(1, 4, 5, 7)$,

$$Z_1(M) = Z_2(M) \qquad \text{and} \qquad Z_2(S) = Z_3(S),$$

and the equations $(12)(a, b, c)$ and $29(a, b, c)$, we conclude that T^* is T_1^* or T_2^* associated with the least cost.

(5) If $f_1(M) > 0$, $f_2(M) \leq 0$ and $f(S) \leq 0$, together with the above Lemma $(1, 4, 6, 8)$,

$$Z_1(M) = Z_2(M) \qquad \text{and} \qquad Z_2(S) = Z_3(S),$$

and the equations $(12)(a, b, c)$ and $(29)(a, b, c)$, we conclude that T^* is T_1^* or T_3^* associated with the least cost.

Incorporating the above arguments, we have completed the proof of the Theorem.

5. Illustrative numerical examples

This section provides numerical examples to illustrate the theoretical results derived in the presented paper. The sensitivity analysis of major parameters on the optimal will also carried out.

Example 5.1. Given the maximum lifetime of the deteriorating items is 1 year ($m = 1$), I_p = \$0.15/\$/year, I_e = \$0.12/\$/ year, and $M = 0.15$ year, the above Theorem is applied to obtain the optimal solution (see Table 1 below).

Table 1. The optimal replenishment policy used in the Theorem

	$f_1(M)$	$f_2(M)$	$f(S)$	D	o	c	p	h	S	T^*	$Z^*(T^*)$
(1)	> 0	> 0	> 0	500	10	10	15	1	0.2270	T_1^*=0.0704	5258.9
(2)	≤ 0	≤ 0	> 0	500	50	10	15	1	0.2270	T_2^*=0.1661	5611.0
(3)	≤ 0	≤ 0	≤ 0	500	100	10	15	1	0.2270	T_3^*=0.2300	5862.9
(4)	> 0	≤ 0	> 0	500	45	10	15	1	0.2270	T_2^*=0.1580	5580.1
(5)	>0	≤ 0	≤ 0	200	3	0.5	8	1.5	2.4216	T_1^*=0.1040	130.8002

Next, we further study the effects of changes of parameters of m, c and o on the optimal solutions. Numerical results for different values of m, c and o are provided in Table 2.

Example 5.2. Given D = 250 units/year, h = \$1/unit/year, p = \$8/unit/year, I_p = 0.15\$/year, I_e = \$0.12/\$/year, and M = 0.15/year, we obtain the optimal replenishment cycle time for different choices of the parameters (see Table 2 below).

Table 2. Sensitivity analysis of the involved parameters.

m	c	o	$f_1(M)$	$f_2(M)$	$f(S)$	Theorem	T^*	$Z(T^*)$
1	0.5	3	>0	≤ 0	≤ 0	(5)	T_1^*=0.1030	149.6676
		6	>0	≤ 0	≤ 0	(5)	T_2^*=0.2880	163.9092
		9	≤ 0	≤ 0	≤ 0	(3)	T_2^*=0.3410	173.4226
	1.0	3	>0	≤ 0	>0	(4)	T_1^*=0.0970	280.7132
		6	>0	≤ 0	>0	(4)	T_2^*=0.2250	301.9262
		9	≤ 0	≤ 0	>0	(2)	T_2^*=0.2700	314.0218
	1.5	3	>0	>0	>0	(1)	T_1^*=0.0930	411.5898
		6	>0	≤ 0	>0	(4)	T_2^*=0.1910	437.5353
		9	≤ 0	≤ 0	>0	(2)	T_2^*=0.3710	169.9408
m	c	o	$f_1(M)$	$f_2(M)$	$f(S)$	Theorem	T^*	$Z(T^*)$
1.5	0.5	3	>0	≤ 0	>0	(4)	T_1^*=0.1040	148.9303
		6	>0	≤ 0	>0	(4)	T_2^*=0.3120	161.1727
		9	≤ 0	≤ 0	>0	(2)	T_2^*=0.3710	169.9408
	1.0	3	>0	≤ 0	>0	(4)	T_1^*=0.1000	279.3623
		6	>0	≤ 0	>0	(4)	T_2^*=0.2440	298.2583
		9	≤ 0	≤ 0	>0	(2)	T_2^*=0.2930	309.4140
	1.5	3	>0	>0	>0	(1)	T_1^*=0.0960	409.6808
		6	>0	≤ 0	>0	(4)	T_2^*=0.2270	433.1344
		9	≤ 0	≤ 0	>0	(2)	T_2^*=0.2490	446.2862
m	c	o	$f_1(M)$	$f_2(M)$	$f(S)$	Theorem	T^*	$Z(T^*)$
2	0.5	3	>0	≤ 0	>0	(4)	T_1^*=0.1050	148.4381
		6	>0	≤ 0	>0	(4)	T_2^*=0.3320	159.2410
		9	≤ 0	≤ 0	>0	(2)	T_2^*=0.3950	167.4840
	1.0	3	>0	≤ 0	>0	(4)	T_1^*=0.1010	278.4523
		6	>0	≤ 0	>0	(4)	T_2^*=0.2590	295.6722
		9	≤ 0	≤ 0	>0	(2)	T_2^*=0.3120	306.1680
	1.5	3	>0	≤ 0	>0	(4)	T_1^*=0.0980	408.3849
		6	>0	≤ 0	>0	(4)	T_2^*=0.2190	430.0299
		9	≤ 0	≤ 0	>0	(2)	T_2^*=0.2650	442.4114

Based on the numerical results shown in Table 2, we can obtain the following insight.

(1) For fixed values of the parameters c and o, any increase in the value of the parameter m will result in a significant increase in the value of the optimal replenishment cycle time and decrease in the optimal annual total cost. For example, for $c = 0.5$ and $o = 9$, the increase in the parameter m from 1 to 1.5 results in a 8.7% $\left(\dfrac{0.3710 - 0.3410}{0.3410} \times 100\% = 8.7\%\right)$ increase in the value of optimal replenishment cycle time and a 2% $\left(\dfrac{173.4226 - 169.9408}{173.4226} \times 100\% = 2.0\%\right)$ decrease in the annual total relevant cost.

(2) For fixed values of the parameters m and o, any increase in the value of the parameter c will result in a significant decrease in the value of the optimal replenishment cycle time and increase in the optimal annual total cost. For example, for $m = 2$ and $o = 9$, the increase of the parameter c from 0.5 to 1 results in a 21.01% $\left(\dfrac{0.3950 - 0.3120}{0.3950} \times 100\% = 21.01\%\right)$ decrease in the value of optimal replenishment cycle time and a 82.8% $\left(\dfrac{306.1680 - 167.4840}{167.4840} \times 100\% = 82.8\%\right)$ increase in the annual total relevant cost.

(3) For fixed values of the parameters m and c, any increase in the value of the parameter o will result in a significant increase in the value of the optimal replenishment cycle

time and increase in the optimal annual total cost. For example, for $c = 1.0$ and $m = 1$, increasing o from 3 to 6 results in a 131.96% $\left(\dfrac{0.225 - 0.097}{0.097} \times 100\% = 131.96\%\right)$ increase in the value of optimal replenishment cycle time and a 7.56% $\left(\dfrac{301.9262 - 280.7132}{280.7132} \times 100\% = 7.56\%\right)$ increase in the annual total relevant cost.

(4) An increase in the ordering cost o leads to an increase in the optimal replenishment cycle time T^* and the total cost Z^*.

(5) An increase in the maximum lifetime m of the deteriorating item leads to an increase in the optimal replenishment cycle time T^*, but a decrease in the total cost Z^*.

(6) The increment in the purchasing cost per unit c causes a positive change in the total cost Z^*, but a negative change in the optimal replenishment cycle time T^*. A simple economic interpretation is that, if the purchasing cost increases, then the retailer receives a higher value of the cost from the purchasing cost and it shortens the replenishment cycle time.

6. Concluding remarks and further observations

In our present investigation, we have built an appropriate EOQ model for deterioration items with permissible delay in payments, in which we have made the following assumptions:

(1) The deterioration rate is time dependent and the product has a maximum lifetime, after which point in time it is 100% deteriorated

and
(2) A credit-risk retailer is asked to cover the procurement cost as a collateral deposit at the time of placing and order. Additionally, the credit-risk retailer reduces the financed loan from constant sales and revenue received gradually and he/she still can utilize the sales revenue to earn interest when he/she pays off all accounts.

Mathematical analytic derivations of many different assertions, which are presented as a Lemma and a Theorem, are developed to determine the relevant conditions of existence and uniqueness of the optimal solution. Herein, we have demonstrated that the total annual relevant cost is convex by using mathematical analytic solution procedures without exploring that the total annual relevant cost is pseudo-convex in T. Numerical examples are given in order to illustrate the solution procedures and to provide some managerial insights such as (for example) when the value of the maximum lifetime of the deteriorating items increases, the optimal replenishment cycle time increases and the annual total relevant cost decreases. As the value of the ordering cost increases, the optimal replenishment cycle time and the annual total relevant cost increase. Moreover, as the unit procurement cost increases, the optimal replenishment cycle time and the annual total annual relevant cost decreases and increases, respectively. A future study will incorporate the proposed model into more realistic assumptions, such as an EOQ model for a deteriorating item with linked to order quantity trade-credit financing, demand as a function of the price or the time, shortages with full or partial backlogging, quantity discount and the time-value of the money.

Finally, with a view to providing incentive and motivation for making further advances along the lines of the supply chain management and associated inventory problems which we have discussed in our present investigation, we have choosen to include herein citations of several related recent works including (for example) those by AlArjani et al. [1], Cárdenas-Barrón et al. [5], Chung et al. (see [10], [11], [12] and [13]), Hou et al. [18], Khan et al. [20], Liao et al. [22], Modak et al. [29], Nobil et al. [30], Shaikh and Cárdenas-Barrón [33], Srivastava et al. [35], Taleizadeh et al. [38], Tiwari et al. [41], Udayakumar and Geetha [42], and Wójtowicz [44].

Conflicts of Interest: The authors declare that there are no conflicts of interest.

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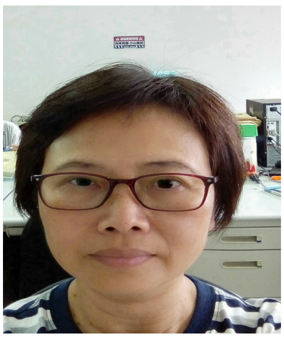

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