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THE COEFFICIENT ESTIMATES FOR A CLASS DEFINED BY HOHLOV OPERATOR USING CONIC DOMAINS

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Abstract. Exploiting this article, we provide the coefficient estimate with $m-$th root transform for a class defined by Hohlov operator using quasi-subordination for conic domains. The authors sincerely hope this article will revive this concept and encourage the other researchers to work in this quasi subordination in the near future in the area of complex function theory.

Keywords: analytic functions, subordination, quasi-subordination, Fekete-Szegö inequality.

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1. Introduction

Let $\mathcal{A}$ denote the class of all analytic function $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

in the open disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0$ and $f'(0) = 1$ and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathcal{U}$. Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk $\mathcal{U}$ onto the region starlike with respect to 1, which is symmetric with respect to $x-$axis. The bounds for the coefficients give information about the geometric properties of these functions. For example, the bound for the second coefficient $|a_2|$ of normalized univalent functions readily yields the growth and distortion bounds for univalent functions. The Fekete-Szegö coefficient functional $|a_3 - \mu a_2^2|$ also naturally arises in the investigation of univalency of analytic functions. In fact, in recent years, the study of the Fekete-Szegö problem was revived by (and has gained momentum) due mainly to the pioneering work of Srivastava et al. [39] (also see [21], [41]). Many other authors have investigated the bounds for the Fekete-Szegö functional for functions in various subclasses of $\mathcal{S}$ for example see the related works in [1], [3], [6], [7], [22] and [42].

A function $f(z)$ is subordinate to a function $g(z)$, written as $f(z) \prec g(z)$, provided that there is a function $w(z)$, analytic Δ, with $w(0) = 0$ such that $|w(z)| < 1$ and $f(z) = g[w(z)]$ for $z \in \mathcal{U}$. In particular if the function $g(z)$ is univalent in $\mathcal{U}$ then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\Delta) \subset g(\mathcal{U})$. In [34], Robertson introduced the concept of quasi-subordination. An analytic function $f(z)$ is quasi-subordinate to an analytic function $g(z)$, in the open unit disk if there exist analytic functions φ and w, with $w(0) = 0$ such that $|\varphi(z)| \le 1$, $|w(z)| < 1$ and $f(z) = \varphi(z)g[w(z)]$. Then we write $f(z) \prec_q g(z)$. If $\varphi(z) = 1$, then the quasi-subordination

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reduces to the subordination. Also, if $w(z) = z$ then $f(z) = \varphi(z)g(z)$ and in this case we say that $f(z)$ is majorized by $g(z)$ and it is written as $f(z) \prec\prec g(z)$ in $\mathcal{U}$. Hence, it is obvious that quasi-subordination is the generalization of subordination as well as majorization. It is unfortunate that the concept quasi-subordination is so for an underlying concept in the area of complex function theory although it deserves much attention as it unifies the concept of both subordination and majorization. Further, we refer to [4, 12, 23, 33] for works related to quasi-subordination.

For fixed k $(0 \leq k < \infty)$, let $k-UCV$ and $k-SP$ be the subclasses of $\mathcal{S}$ consisting, respectively, of functions which are $k-$uniformly convex and $k-$parabolic starlike in $\mathcal{U}$. Thus

$$k-UCV := \left\{ f \in \mathcal{S} : \mathcal{R}\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right|,\ z \in \mathcal{U}\right\},$$

and

$$k-SP := \left\{ f \in \mathcal{S} : \mathcal{R}\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|,\ z \in \mathcal{U}\right\}.$$

This interesting unification of the concepts of univalent convex functions [8] and uniformily convex functions [11] is due to Kanas and Wisniowska [17].

The class $k-SP$, consisting of $k-$parabolic starlike functons is defined from $k-UCV$ via the Alexandar's transforms [18]; that is,

$$f \in k-UCV \Longleftrightarrow g \in k-SP,$$

where $g(z) = zf'(z)$ $(z \in \mathcal{U})$.

The one variable characterization theorem [17] of the class $k-UCV$ gives that $f \in k-UCV$ (respectively $f \in k-SP$) if and only if the values of $p(z) = 1 + \frac{zf''(z)}{f'(z)}$ $\left(\text{respectively } p(z) = \frac{zf'(z)}{f(z)}\right)$ $(z \in \mathcal{U})$ lie in the conic region Ω_k in the $w-$plane, where

$$\Omega_k = \left\{ w = u + iv \in \mathbb{C} : u^2 > k^2(u-1)^2 + k^2v^2,\ u > 0,\ 0 \leq k < \infty \right\}.$$

This characterization enables us to designate precisely the domain Ω_k, as a convex domain contained in the right half-plane. Moreover, Ω_k is an elliptic region for $k > 1$, parabolic for $k = 1$, hyperbolic for $0 < k < 1$ and finally Ω_0 is the whole right half-plane.

Let m be a positive integer. A domain D is said to be $m-$fold symmetric if a rotation of D about the origin through an angle $\frac{2\pi}{m}$ carries D to itself. A function $f(z)$ is said to be $m-$fold symmetric in $\mathcal{U}$ if for every z in $\mathcal{U}$

$$f\left(e^{\frac{2\pi i}{m}} z\right) = e^{\frac{2\pi i}{m}} f(z).$$

In 1916, Gronwall shows that if $f(z)$ is regular and m-fold symmetric in $\mathcal{U}$, then it has a power series expansion of the form

$$f(z) = b_1 z + b_{m+1} z^{m+1} + b_{2m+1} z^{2m+1} + \cdots = \sum_{n=0}^{\infty} b_{nm+1} z^{nm+1}. \tag{2}$$

Conversely, if $f(z)$ is given by the power series (2), then $f(z)$ is $m-$fold symmetric inside the circle of convergence of the series. For a univalent function $f(z)$ of the form in (1), the $m-$th root transform is defined by

$$F(z) = [f(z^m)]^{1/m} = z + \sum_{n=1}^{\infty} b_{mn+1} z^{mn+1},\ z \in \mathcal{U}. \tag{3}$$

The convolution or the Hadamard product of two functions $f, g \in \mathcal{A}$ is denoted by $f * g$ and is defined as follows:

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z),$$

where $f(z)$ is given by (1) and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n.$$

In terms of the Hadamard product (or convolution), the Dziok-Srivastava linear convolution operator involving the generalized hypergeometric function was introduced and studied systematically by Dziok and Srivastava [9], [10]. In fact, in Geometric Function Theory, there are other families of general convolution operators including (for example) the generalized fractional calculus operator and the Srivastava-Wright operator (see [20], [38]). Here, in our present investigation, we recall a much simpler convolution operator $\mathcal{H}_c^{a,b}$ due to Hohlov [13], [14], which indeed is a very specialized case of the widely- (and extensively-) investigated Dziok-Srivastava operator.

For complex numbers a, b and c $(c \neq 0, -1, -2, \cdots)$ the Gaussian hypergeometric function ${}_2F_1(z)$ is defined by

$${}_2F_1(z) = {}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n = 1 + \frac{ab}{c} z + \frac{a(a+1)\, b(b+1)}{c(c+1)} \frac{z^2}{2!} + \cdots \tag{4}$$

where $(\lambda)_n$ is the Pochhamer symbol or shifted factorial, written in terms of the gamma function Γ, by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1, & n = 0 \\ \lambda(\lambda+1) \ldots (\lambda + n - 1), & n \in \mathbb{N} \end{cases}.$$

Note that ${}_2F_1(z)$ is symmetric in a and b and that the series (4) terminates if at least one of the numerator parameters a and b is zero or a negative integer. Gaussian hypergeometric series Hohlov [13], [14] introduced and studied the linear operator $H_c^{a,b} : \mathcal{A} \to \mathcal{A}$ defined by

$$\left(\mathcal{H}_c^{a,b}(f)\right)(z) = z\,{}_2F_1(a, b, c; z) * f(z), \; (f \in \mathcal{A}, \; z \in \mathcal{U}).$$

Observe that for the function f of the form (1), we have

$$\left(\mathcal{H}_c^{a,b}(f)\right)(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (1)_{n-1}} a_n z^n, \; z \in \mathcal{U}. \tag{5}$$

The Hohlov operator $H_c^{a,b}$ unifies several previously well studied operators. Namely

- $H_1^{2,1}(f) = zf'(z) = A(f)$ is the Alexandar transformation, where as $H_2^{1,1}(f) = \int\limits_0^z \frac{f(t)}{t} dt$ is its inverse transform [8];
- $H_3^{1,2}(f) = L(f)$ is the Libera integral operator [40];
- $H_{\gamma+2}^{1,\gamma+1}(f) = B(f)$ is the Bernardi integral operator [40];
- $H_{n+1}^{1,2}(f) = H_n(f)$ is the Noor integral operator of order n [29]-[31];
- $H_1^{1,n+1}(f) = D^n(f)$ $(n > -1)$ is the Ruscheweyh derivative of f order n [36], [37];

- $H_c^{\alpha,1}(f) = L(a,c)(f)$ is the Carlson-Shaffer operator [40];

- $H_{2-\lambda}^{2,1}(f) = \Omega^{\lambda}(f)$ is the Owa-Srivastava operator [32].

Definition 1.1. *Let the class* $k-SP_c^{a,b}$ $(0 \le k < \infty,\ a,b,c \in \mathbb{R},\ c \ne 0,-1,-2...)$ *consist of functions* $f \in \mathcal{A}$ *satisfying the following condition*

$$\mathcal{R}\left(\frac{z\left(\mathcal{H}_c^{a,b}(f)\right)'(z)}{\left(\mathcal{H}_c^{a,b}(f)\right)(z)}\right) > k\left|\frac{z\left(\mathcal{H}_c^{a,b}(f)\right)'(z)}{\left(\mathcal{H}_c^{a,b}(f)\right)(z)} - 1\right|,\ z \in \mathcal{U}. \tag{6}$$

In particular case $k = 1$, we denote by $SP_c^{a,b}$ the class $1 - SP_c^{a,b}$. We obtain the following subclasses studied by various authors.

- for $k = 1$, $a = 2$, $b = 1$, $c = 1$, $1 - SP_1^{2,1} = UCV$, the class of uniformly convex functions has been studied by Goodman [11] and Ma and Minda [25].
- for $k = 1$, $a = 1$, $b = 1$, $c = 2$, $1 - SP_2^{1,1} = SP$, the class of parabolic starlike functions has been studied by Ronning [35].
- for $k = 1$, $a = 2$, $b = 1$, $c = 2 - \lambda$ $(0 \le \lambda \le 1)$, the class $1 - SP_{2-\lambda}^{2,1} = SP_{\lambda}$ has been studied by Srivastava and Mishra [40].
- for $a = 2$, $b = 1$, $c = 2 - \lambda$ $(0 \le \lambda \le 1)$, the class $k - SP_{2-\lambda}^{2,1} = k - SP_{\lambda}$ has been studied by Mishra and Gochhayat [26].
- for $a = 2$, $b = 1$, $c = n + 1$, the class k - $SP_{n+1}^{2,1} = k - UCV_n$ has been studied by Mishra and Gochhayat [27].

In the particular cases $k = 0$, $a = 2$, $b = 1$, $c = 1$, we get $0 - SP_1^{2,1} = CV$, the class of univalent convex functions [8]. Similarly, taking $k = 0$, $a = 1$, $b = 1$, $c = 2$, we get $0 - SP_2^{1,1} = \mathcal{S}^*$, the class of univalent starlike functions [8].

Definition 1.2. *Let the class* $k - SP_c^{a,b}$ $(0 \le k < \infty,\ a,b,c \in \mathbb{R},\ c \ne 0,-1,-2...)$ *consist of functions* $f \in \mathcal{A}$ *satisfying the following condition the quasi-subordination*

$$\frac{z\left(\mathcal{H}_c^{a,b}(f)\right)'(z)}{\left(\mathcal{H}_c^{a,b}(f)\right)(z)} - 1 \prec_q q_k(w(z)) - 1.$$

Note that in 2000, Kanas and Srivastava [16] found conditions on the parameters a, b, c and k, for which Hohlov Operator maps the classes of starlike and univalent functions onto $k - UCV$ and $k - SP$.

In this paper, we obtain the coefficient estimates for a class defined by Hohlov Operator using conic domains.

2. Preliminary results

We need the following lemmas to prove our results.

Let Ω be the class of analytic functions w, normalized by $w(0) = 0$ and satisfying the condition $|w(z)| < 1$.

Lemma 2.1. [19] *If* $w \in \Omega$ *and* $w(z) = w_1 z + w_2 z^2 + \dots$ $(z \in \mathcal{U})$, *then*

$$|w_2 - t w_1^2| \le \max\{1, |t|\}$$

for any complex number t. *The result is sharp for the function* $w(z) = z^2$ *or* $w(z) = z$.

Lemma 2.2. [24] *If $w \in \Omega$ and $w(z) = w_1 z + w_2 z^2 + \dots$ $(z \in \mathcal{U})$, then*

$$\left|w_2 - t w_1^2\right| \leq \begin{cases} -t, & if \quad t \leq -1, \\ 1, & if \quad -1 \leq t \leq 1, \\ t, & if \quad t \geq 1. \end{cases}$$

For $t < -1$ or $t > 1$, the equality holds if and only if $w(z) = z$ or one of its rotations. For $-1 < t < 1$, the equality holds if and only if $w(z) = z^2$ or one of its rotations. Equality holds for $t = -1$ if and only if $w(z) = z\frac{\lambda + z}{1 + \lambda z}$ $(0 \leq \lambda \leq 1)$ or one of its rotations, while for $t = 1$, equality holds if and only if $w(z) = -z\frac{\lambda + z}{1 + \lambda z}$ $(0 \leq \lambda \leq 1)$ or one of its rotations.

Lemma 2.3. [19] *Let the function $w \in \Omega$ be given by*

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \dots, \; z \in \mathcal{U}.$$

Then for every complex number μ,

$$\left|w_2 - t w_1^2\right| \leq 1 + (|t| - 1)\left|w_1^2\right|$$

for any complex number t.

Lemma 2.4. [15] *Let $k \in [0, \infty)$ be fixed and $q_k(z)$ be the Riemann map of $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ on to Ω_k satisfying $q_k(0) = 1$, $q_k'(0) > 0$. If*

$$q_k(z) = 1 + Q_1(k) z + Q_2(k) z^2 + Q_3(k) z^3 + \cdots, \; z \in \mathcal{U}, \tag{7}$$

then

$$Q_1 = Q_1(k) = \begin{cases} \dfrac{2A^2}{1 - k^2}, & 0 \leq k < 1, \\ \dfrac{8}{\pi^2}, & k = 1, \\ \dfrac{\pi^2}{4\kappa^2(t)(k^2 - 1)(1 - t)\sqrt{t}}, & k > 1, \end{cases}$$

$$Q_2 = Q_2(k) = D(k) Q_1(k),$$

where

$$D = D(k) = \begin{cases} \dfrac{A^2 + 2}{3}, & 0 \leq k < 1, \\ \dfrac{2}{3}, & k = 1, \\ \dfrac{4\kappa^2(t)\left(t^2 + 6t + 1\right) - \pi^2}{24\kappa^2(t)(1 + t)\sqrt{t}}, & k > 1, \end{cases}$$

$$A = \frac{2}{\pi} \arccos k$$

and $\kappa(t)$ is the complete elliptic integral of first kind for details see ([2], [5] *and* [40]*).*

In this paper $\varphi(z) = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + \dots$ and $|C_n| \leq 1$.

3. Main results

Theorem 3.1. *If* $f \in k - SP_c^{a,b}$ $(0 \leq k < \infty,\ a, b, c \in \mathbb{R},\ a, b, c > 0)$ *and* F *is the* $m-th$ *root transformation of* f *given by* (3), *then*

$$|b_{m+1}| \leq \frac{cQ_1}{mab},$$

$$|b_{2m+1}| \leq \frac{c\,(c+1)}{mab\,(a+1)\,(b+1)}\left[Q_1 + \max\left\{Q_1, \left|\frac{(m-1)\,(a+1)\,(b+1)}{2mab\,(c+1)} + 1\right| Q_1^2 + |Q_2|\right\}\right],$$

and for any complex number μ,

$$|b_{2m+1} - \mu b_{m+1}^2|$$
$$\leq \frac{c\,(c+1)}{mab\,(a+1)\,(b+1)}\left[Q_1 + \max\left\{Q_1, \left|\frac{(m+2\mu-1)\,(a+1)\,(b+1)}{2mab\,(c+1)} + 1\right| Q_1^2 + |Q_2|\right\}\right].$$

Proof. Let $f \in k - SP_c^{a,b}$ $(0 \leq k < \infty, a, b, c \in \mathbb{R}, a, b, c > 0)$, then there exist analytic functions φ and w with $|\varphi(z)| \leq 1$, $w(0) = 0$ and $|w(z)| < 1$ such that

$$\frac{z\left(\mathcal{H}_c^{a,b}\,(f)\right)'(z)}{\left(\mathcal{H}_c^{a,b}\,(f)\right)(z)} - 1 = \varphi(z)[q_k(w(z)) - 1]. \tag{8}$$

By a simple calculations, we get

$$\frac{z\left(\mathcal{H}_c^{a,b}\,(f)\right)'(z)}{\left(\mathcal{H}_c^{a,b}\,(f)\right)(z)} - 1 = \frac{ab}{c}a_2 z + \left(\frac{ab\,(a+1)\,(b+1)}{c\,(c+1)}a_3 - \frac{a^2b^2}{c^2}a_2^2\right)z^2 + \cdots \tag{9}$$

and

$$\varphi(z)[\phi(w(z)) - 1] = Q_1C_0w_1z + [Q_1C_1w_1 + C_0(Q_1w_2 + Q_2w_1^2)]z^2 + \cdots. \tag{10}$$

Using (9) and (10) in (8), we have

$$a_2 = \frac{cQ_1C_0w_1}{ab} \tag{11}$$

and

$$a_3 = \frac{c\,(c+1)}{ab\,(a+1)\,(b+1)}[Q_1C_1w_1 + Q_1w_2C_0 + C_0(Q_2 + Q_1^2C_0)w_1^2]. \tag{12}$$

For a function f given by (1), a computation shows that

$$[f(z^m)]^{1/m} = z + \frac{1}{m}a_2 z^{m+1} + \left[\frac{1}{m}a_3 - \frac{1}{2}\left(\frac{m-1}{m^2}\right){a_2}^2\right]z^{2m+1} + \cdots. \tag{13}$$

Upon equating the coefficients of z^{m+1} and z^{2m+1} in view of (2) and (13), we get

$$b_{m+1} = \frac{1}{m}a_2 \quad \text{and} \quad b_{2m+1} = \frac{a_3}{m} - \frac{1}{2}\left(\frac{m-1}{k^2}\right){a_2}^2. \tag{14}$$

Further, from (11) to (15) , we have

$$b_{m+1} = \frac{cQ_1C_0w_1}{mab} \tag{15}$$

and

$$b_{2m+1} \tag{16}$$
$$= \frac{2abc\,(c+1)\,[Q_1C_1w_1 + Q_1w_2C_0 + C_0(Q_2 + Q_1^2C_0)w_1^2] - \dfrac{(m-1)\,(a+1)\,(b+1)\,c^2Q_1^2C_0^2w_1^2}{m}}{2ma^2b^2\,(a+1)\,(b+1)}.$$

Also, for any complex number μ,

$$b_{2m+1}-\mu b_{m+1}^{2}=\frac{c\left(c+1\right)Q_{1}}{mab\left(a+1\right)\left(b+1\right)}\left\{C_{1}w_{1}+C_{0}\left[w_{2}+\frac{Q_{2}}{Q_{1}}w_{1}^{2}+Q_{1}C_{0}w_{1}^{2}-A_{1}w_{1}^{2}\right]\right\}, \quad (17)$$

where

$$A_{1}=\frac{\left(m+2\mu-1\right)\left(a+1\right)\left(b+1\right)cQ_{1}C_{0}}{2mab\left(c+1\right)}.$$

Since $D=\dfrac{Q_{2}}{Q_{1}}$, we get

$$b_{2m+1}-\mu b_{m+1}^{2}=\frac{c\left(c+1\right)Q_{1}}{mab\left(a+1\right)\left(b+1\right)}\left\{C_{1}w_{1}+C_{0}\left[w_{2}+\left(D+Q_{1}C_{0}-A_{1}\right)w_{1}^{2}\right]\right\}.$$

Using the inequalities $|C_n|\leq 1$, $|w_n(z)|\leq 1$, we get $|b_{m+1}|\leq \dfrac{cQ_1}{mab}$
and

$$\left|b_{2m+1}-\mu b_{m+1}^{2}\right|\leq\frac{c\left(c+1\right)Q_{1}}{mab\left(a+1\right)\left(b+1\right)}\left[1+\left|w_{2}+\left(D+Q_{1}C_{0}-A_{1}\right)w_{1}^{2}\right|\right].$$

An application of Lemma 2.1 to $\left|w_{2}-\left(A_{1}-D-Q_{1}C_{0}\right)w_{1}^{2}\right|$, yields

$$\left|b_{2m+1}-\mu b_{m+1}^{2}\right|\leq\frac{c\left(c+1\right)Q_{1}}{mab\left(a+1\right)\left(b+1\right)}\left[1+\max\left\{1,\left|A_{1}-D-Q_{1}C_{0}\right|\right\}\right].$$

Since,

$$\left|A_{1}-D-Q_{1}C_{0}\right|\leq\left|\frac{\left(m+2\mu-1\right)\left(a+1\right)\left(b+1\right)Q_{1}}{2mab\left(c+1\right)}\right|+|D|+|Q_{1}|$$

and hence conclude that

$$\begin{aligned}&\left|b_{2m+1}-\mu b_{m+1}^{2}\right|\\ \leq\ &\frac{c\left(c+1\right)}{mab\left(a+1\right)\left(b+1\right)}\left[Q_{1}+\max\left\{Q_{1},\left|\frac{\left(m+2\mu-1\right)\left(a+1\right)\left(b+1\right)}{2mab\left(c+1\right)}+1\right|Q_{1}^{2}+|Q_{2}|\right\}\right].\end{aligned}$$

For $\mu=0$, we get

$$\left|b_{2m+1}\right|\leq\frac{c\left(c+1\right)}{mab\left(a+1\right)\left(b+1\right)}\left[Q_{1}+\max\left\{Q_{1},\left|\frac{\left(m-1\right)\left(a+1\right)\left(b+1\right)}{2mab\left(c+1\right)}+1\right|Q_{1}^{2}+|Q_{2}|\right\}\right].$$

This essentially completes the proof of Theorem 3.1. □

If $m=1$ and $\mu=0$ in Theorem 3.1 then we have the following corollary.

Corollary 3.1. *If* $f\in k-SP_{c}^{a,b}$ $(0\leq k<\infty,\ a,b,c\in\mathbb{R},\ a,b,c>0)$ *and* F *is the* $m-th$ *root transformation of* f *given by* (3). *Then*

$$\left|b_{3}-\mu b_{2}^{2}\right|\leq\frac{c\left(c+1\right)}{ab\left(a+1\right)\left(b+1\right)}\left[Q_{1}+\max\left\{Q_{1},Q_{1}^{2}+|Q_{2}|\right\}\right].$$

Putting the values of $Q_1=Q_1(k)$ and $D=D(k)$ from Lemma 2.4 in Theorem 3.1 for $0\leq k<1$, $k=1$ and $k>1$ respectively, we get the following corollaries.

Corollary 3.2. *If* $f\in k-SP_{c}^{a,b}$ $(0\leq k<\infty,\ a,b,c\in\mathbb{R},\ a,b,c>0)$ *and* F *is the* $m-th$ *root transformation of* f *given by* (3) *and* $0\leq k<1$. *Then*

$$\left|b_{m+1}\right|\leq\frac{c}{mab}\left(\frac{2A^{2}}{1-k^{2}}\right),$$

$$\left|b_{2m+1}\right|\leq\frac{c\left(c+1\right)}{mab\left(a+1\right)\left(b+1\right)}\left[+\max\left\{\frac{2A^{2}}{1-k^{2}},A_{2}+\left|\left(\frac{2A^{2}}{1-k^{2}}\right)\left(\frac{A^{2}+2}{3}\right)\right|\right\}\right],$$

and for any complex number μ,

$$\begin{aligned}&|b_{2m+1}-\mu b_{m+1}^2|\\ \leq\ & \frac{c(c+1)}{mab(a+1)(b+1)}\left[\frac{2A^2}{1-k^2}+\max\left\{\frac{2A^2}{1-k^2}, A_3+\left|\left(\frac{2A^2}{1-k^2}\right)\left(\frac{A^2+2}{3}\right)\right|\right\}\right],\end{aligned}$$

where

$$A_2=\left|\frac{(m-1)(a+1)(b+1)}{2mab(c+1)}+1\right|\left(\frac{2A^2}{1-k^2}\right)^2$$

and

$$A_3=\left|\frac{(m+2\mu-1)(a+1)(b+1)}{2mab(c+1)}+1\right|\left(\frac{2A^2}{1-k^2}\right)^2.$$

Corollary 3.3. *If* $f\in k-SP_c^{a,b}$ $(0\leq k<\infty, a,b,c\in\mathbb{R}, a,b,c>0)$ *and* F *is the* $m-th$ *root transformation of f given by* (3) *and* $k=1$. *Then*

$$|b_{m+1}|\leq\frac{8c}{mab\pi^2},$$

$$|b_{2m+1}|\leq\frac{c(c+1)}{mab(a+1)(b+1)}\left[\frac{8}{\pi^2}+\max\left\{\frac{8}{\pi^2},\frac{64}{\pi^4}\left|\frac{(m-1)(a+1)(b+1)}{2mab(c+1)}+1\right|+\frac{16}{3\pi^2}\right\}\right],$$

and for any complex number μ,

$$\begin{aligned}&\left|b_{2m+1}-\mu b_{m+1}^2\right|\\ \leq\ & \frac{c(c+1)}{mab(a+1)(b+1)}\left[\frac{8}{\pi^2}+\max\left\{\frac{8}{\pi^2},\frac{64}{\pi^4}\left|\frac{(m+2\mu-1)(a+1)(b+1)}{2mab(c+1)}+1\right|+\frac{16}{3\pi^2}\right\}\right].\end{aligned}$$

Corollary 3.4. *If* $f\in k-SP_c^{a,b}$ $(0\leq k<\infty,\ a,b,c\in\mathbb{R},\ a,b,c>0)$ *and* F *is the* $m-th$ *root transformation of f given by* (3) *and* $k>1$. *Then*

$$|b_{m+1}|\leq\frac{c}{mab}A_4,$$

$$|b_{2m+1}|\leq\frac{c(c+1)}{mab(a+1)(b+1)}\left[A_4+\max\left\{A_4,\ A_5+A_6\right\}\right],$$

and for any complex number μ,

$$|b_{2m+1}-\mu b_{m+1}^2|\leq\frac{c(c+1)}{mab(a+1)(b+1)}\left[A_4+\max\left\{A_4,\ A_7+A_5\right\}\right],$$

$$\begin{aligned}A_4 &= \frac{\pi^2}{4\kappa^2(t)(k^2-1)(1-t)\sqrt{t}},\\ A_5 &= \left|\frac{(m-1)(a+1)(b+1)}{2mab(c+1)}+1\right|(A_4)^2\end{aligned}$$

and

$$\begin{aligned}A_6 &= \left|\left(\frac{4\kappa^2(t)\left(t^2+6t+1\right)-\pi^2}{24\kappa^2(t)(1+t)\sqrt{t}}\right)A_4\right|,\\ A_7 &= \left|\frac{(m+2\mu-1)(a+1)(b+1)}{2mab(c+1)}+1\right|(A_4)^2.\end{aligned}$$

Theorem 3.2. *If F is the $m-th$ root transformation of f given by* (3) *and $f \in \mathcal{A}$ satisfies*

$$\frac{z\left(\mathcal{H}_c^{a,b}(f)\right)'(z)}{\left(\mathcal{H}_c^{a,b}(f)\right)(z)} - 1 \prec\prec q_k(z) - 1,$$

then the following inequalities hold:

$$|b_{m+1}| \leq \frac{cQ_1}{mab},$$

$$|b_{2m+1}| \leq \frac{c(c+1)}{mab(a+1)(b+1)}\left[Q_1 + |Q_2| + \left|1 - \frac{(m-1)(a+1)(b+1)}{2mab(c+1)}\right| Q_1^2\right],$$

and for any complex number μ,

$$|b_{2m+1} - \mu b_{m+1}^2| \leq \frac{c(c+1)}{mab(a+1)(b+1)}\left[Q_1 + |Q_2| + \left|1 - \frac{(m+2\mu-1)(a+1)(b+1)}{2mab(c+1)}\right| Q_1^2\right].$$

Proof. The result follows by taking $w(z) = z$ in the proof of Theorem 3.1. □

Theorem 3.3. *If $f \in k - SP_c^{a,b}$ $(0 \leq k < \infty,\ a,b,c \in \mathbb{R},\ a,b,c > 0)$ and F is the $m-th$ root transformation of f given by* (3)*, then*

$$|b_{2m+1} - \mu b_{m+1}^2| \leq \begin{cases} \frac{c(c+1)Q_1}{mab(a+1)(b+1)}(1 + D + Q_1C_0 - A_1), & if \quad \mu \leq \sigma_1, \\ \frac{2c(c+1)Q_1}{mab(a+1)(b+1)}, & if \quad \sigma_1 \leq \mu \leq \sigma_2, \\ \frac{c(c+1)Q_1}{mab(a+1)(b+1)}(1 + A_1 - D - Q_1C_0), & if \quad \mu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 = \frac{mab(c+1)}{(a+1)(b+1)cQ_1C_0}\left(D + Q_1C_0 - \frac{(m-1)(a+1)(b+1)cQ_1C_0}{2mab(c+1)} - 1\right),$$

$$\sigma_2 = \frac{mab(c+1)}{(a+1)(b+1)cQ_1C_0}\left(D + Q_1C_0 - \frac{(m-1)(a+1)(b+1)cQ_1C_0}{2mab(c+1)} + 1\right).$$

Proof. From the (17), we have

$$b_{2m+1} - \mu b_{m+1}^2 = \frac{c(c+1)Q_1C_1w_1}{mab(a+1)(b+1)} + \frac{c(c+1)Q_1C_0}{mab(a+1)(b+1)}\left[w_2 - (A_1 - D - Q_1C_0)w_1^2\right].$$

Using the inequalities $|C_n| \leq 1$, $|w_n(z)| \leq 1$, we get

$$b_{2m+1} - \mu b_{m+1}^2 \leq \frac{c(c+1)Q_1}{mab(a+1)(b+1)} + \frac{c(c+1)Q_1}{mab(a+1)(b+1)}\left[\left|w_2 - (A_1 - D - Q_1C_0)w_1^2\right|\right].$$

The second result is established by an application of Lemma 2.2.
If $A_1 - D - Q_1C_0 \leq -1$, then

$$\mu \leq \frac{mab(c+1)}{(a+1)(b+1)cQ_1C_0}\left(D + Q_1C_0 - \frac{(m-1)(a+1)(b+1)cQ_1C_0}{2mab(c+1)} - 1\right),$$

which implies that $\mu \leq \sigma_1$. Where

$$\sigma_1 = \frac{mab(c+1)}{(a+1)(b+1)cQ_1C_0}\left(D + Q_1C_0 - \frac{(m-1)(a+1)(b+1)cQ_1C_0}{2mab(c+1)} - 1\right).$$

Hence, we have

$$b_{2m+1} - \mu b_{m+1}^2 \leq \frac{c(c+1)Q_1}{mab(a+1)(b+1)}\left[1 + D + Q_1C_0 - A_1\right],$$

which is the first inequality of the Theorem 3.3. If $A_1 - D - Q_1C_0 \geq 1$, then

$$\mu \geq \frac{mab\,(c+1)}{(a+1)\,(b+1)\,cQ_1C_0}\left(D + Q_1C_0 - \frac{(m-1)\,(a+1)\,(b+1)\,cQ_1C_0}{2mab\,(c+1)} + 1\right),$$

which implies that $\mu \geq \sigma_2$. Where

$$\sigma_2 = \frac{mab\,(c+1)}{(a+1)\,(b+1)\,cQ_1C_0}\left(D + Q_1C_0 - \frac{(m-1)\,(a+1)\,(b+1)\,cQ_1C_0}{2mab\,(c+1)} + 1\right).$$

Hence, we have

$$b_{2m+1} - \mu b_{m+1}^2 \leq \frac{c\,(c+1)Q_1}{mab\,(a+1)\,(b+1)}\left[1 + A_1 - D - Q_1C_0\right],$$

which is the third inequality of the Theorem 3.3. If $1 \leq A_1 - D - Q_1C_0 \leq -1$, then

$$b_{2m+1} - \mu b_{m+1}^2 \leq \frac{2c\,(c+1)Q_1}{mab\,(a+1)\,(b+1)},$$

which is the middle inequality of the Theorem 3.3. This essentially completes the proof of Theorem 3.3. □

Remark 3.1. *For $\varphi(z) = 1$ and $m = 1$, Theorem 3.3 reduces the Theorem 1 in Mishra and Panigrahi(*[28]*).*

Theorem 3.4. *If $f \in k - SP_c^{a,b}$ $(0 \leq k < \infty,\ a, b, c \in \mathbb{R},\ a, b, c > 0)$ and F is the $m-$th root transformation of given by* (3)*, then*

$$\left|b_{2m+1} - \mu b_{m+1}^2\right| \leq \begin{cases} \dfrac{c\,(c+1)\,Q_1}{mab\,(a+1)\,(b+1)}\,(1 + D + Q_1C_0 - A_1), & if \quad \mu \leq \alpha_2, \\ \dfrac{2c\,(c+1)\,Q_1}{mab\,(a+1)\,(b+1)}, & if \quad \alpha_2 \leq \mu \leq \alpha_1, \\ \dfrac{c\,(c+1)\,Q_1}{mab\,(a+1)\,(b+1)}\,(1 + A_1 - D - Q_1C_0), & if \quad \mu \geq \alpha_1, \end{cases}$$

where

$$\alpha_1 = \frac{mab\,(c+1)}{(a+1)\,(b+1)\,cQ_1C_0}\left(D + Q_1C_0 - \frac{(m-1)\,(a+1)\,(b+1)\,cQ_1C_0}{2mab\,(c+1)} + 1\right),$$

$$\alpha_2 = \frac{mab\,(c+1)}{(a+1)\,(b+1)\,cQ_1C_0}\left(D + Q_1C_0 - \frac{(m-1)\,(a+1)\,(b+1)\,cQ_1C_0}{2mab\,(c+1)} - 1\right).$$

Proof. From the (17), we have

$$b_{2m+1} - \mu b_{m+1}^2 = \frac{c\,(c+1)Q_1C_1w_1}{mab\,(a+1)\,(b+1)} + \frac{c\,(c+1)Q_1C_0}{mab\,(a+1)\,(b+1)}\left[w_2 + (D + Q_1C_0 - A_1)\,w_1^2\right], \quad (18)$$

which implies that

$$b_{2m+1} - \mu b_{m+1}^2 = \frac{c\,(c+1)Q_1C_1w_1}{mab\,(a+1)\,(b+1)} + \frac{c\,(c+1)Q_1C_0}{mab\,(a+1)\,(b+1)}\left[w_2 - w_1^2 + (1 + D + Q_1C_0 - A_1)\,w_1^2\right].$$

Using the inequalities $|C_n| \leq 1$, $|w_n(z)| \leq 1$, we get

$$\begin{aligned} &\left|b_{2m+1} - \mu b_{m+1}^2\right| \qquad (19) \\ \leq\ & \frac{c\,(c+1)Q_1}{mab\,(a+1)\,(b+1)} + \frac{c\,(c+1)Q_1}{mab\,(a+1)\,(b+1)}\left[\left|w_2 - w_1^2\right| + (1 + D + Q_1C_0 - A_1)\,|w_1|^2\right]. \end{aligned}$$

Suppose that $\mu \geq \alpha_1$, the expression inside the second modulus symbol on the right hand side of (19) is non negative. Then, using the estimate $\left|w_2 - w_1^2\right| \leq 1$ from Lemma 2.3, we get

$$\left|b_{2m+1} - \mu b_{m+1}^2\right| \leq \frac{c\,(c+1)Q_1}{mab\,(a+1)\,(b+1)} + \frac{c\,(c+1)Q_1}{mab\,(a+1)\,(b+1)}\left[1 + (A_1 - 1 - D - Q_1C_0)\right],$$

which implies that

$$b_{2m+1} - \mu b_{m+1}^2 \leq \frac{c\,(c+1)Q_1}{mab\,(a+1)\,(b+1)}\left[1 + A_1 - D - Q_1C_0\right].$$

This is precisely the last inequality in Theorem 3.4.
On the other hand $\mu \leq \alpha_2$, then (18) gives

$$\left|b_{2m+1} - \mu b_{m+1}^2\right| \leq \frac{c\,(c+1)Q_1}{mab\,(a+1)\,(b+1)} + \frac{c\,(c+1)Q_1}{mab\,(a+1)\,(b+1)}\left[|w_2| + (D + Q_1C_0 - A_1)\,|w_1|^2\right].$$

Applying estimate $|w_2| \leq 1 - |w_1|^2$ of Lemma 2.3 and $|w_1| \leq 1$, we have

$$\begin{aligned} &\left|b_{2m+1} - \mu b_{m+1}^2\right| \\ \leq\; &\frac{c\,(c+1)Q_1}{mab\,(a+1)\,(b+1)} + \frac{c\,(c+1)Q_1}{mab\,(a+1)\,(b+1)}\left[1 - |w_1|^2 + (D + Q_1C_0 - A_1)\,|w_1|^2\right], \end{aligned}$$

which implies that

$$\left|b_{2m+1} - \mu b_{m+1}^2\right| \leq \frac{c\,(c+1)Q_1}{mab\,(a+1)\,(b+1)}\left[1 + D + Q_1C_0 - A_1\right].$$

This is the first inequality in Theorem 3.4.
Lastly, if $\alpha_2 \leq \mu \leq \alpha_1$, then $|D + Q_1C_0 - A_1| \leq 1$.
Therefore, (18) yields

$$b_{2m+1} - \mu b_{m+1}^2 \leq \frac{2c\,(c+1)Q_1}{mab\,(a+1)\,(b+1)},$$

which is the middle inequality of the Theorem 3.4. This essentially completes the proof of Theorem 3.4. □

Remark 3.2. *For $\varphi(z) = 1$ and $m = 1$, Theorem 3.4 reduces the Theorem 1 in Mishra and Panigrahi(*[28]*).*

Remark 3.3. *Applying Lemma 2.2 and Lemma 2.3 to (17), we get the same results as in Theorem 3.3 and Theorem 3.4.*

Putting the values of $Q_1 = Q_1(k)$ and $D = D(k)$ from Lemma 2.4 in Theorem 3.4 for $0 \leq k < 1$, $k = 1$ and $k > 1$ respectively, we get the following corollaries.

Corollary 3.5. *If $f \in k - SP_c^{a,b}$ $(0 \leq k < \infty,\ a, b, c \in \mathbb{R},\ a, b, c > 0)$ and F is the $m-th$ root transformation of f given by* (3) *and $0 \leq k < 1$. Then*

$$\left|b_{2m+1} - \mu b_{m+1}^2\right| \leq \begin{cases} \dfrac{2A^2c\,(c+1)}{mab\,(a+1)\,(b+1)\,(1-k^2)}\left(1 + \dfrac{A^2+2}{3} + \dfrac{2A^2C_0}{1-k^2} - A_8\right), & if \quad \mu \leq \beta_2, \\ \dfrac{4A^2c\,(c+1)}{mab\,(a+1)\,(b+1)\,(1-k^2)}, & if \quad \beta_2 \leq \mu \leq \beta_1, \\ \dfrac{2A^2c\,(c+1)}{mab\,(a+1)\,(b+1)\,(1-k^2)}\left(1 + A_8 - \dfrac{A^2+2}{3} - \dfrac{2A^2C_0}{1-k^2}\right), & if \quad \mu \geq \beta_1, \end{cases}$$

where

$$A_8 = \frac{(m + 2\mu - 1)\,(a+1)\,(b+1)\,A^2cC_0}{mab\,(c+1)\,(1-k^2)},$$

$$\beta_1 = \frac{mab\,(c+1)}{(a+1)\,(b+1)\,cC_0}\left(\frac{\left(A^2+2\right)\left(1-k^2\right)}{6A^2} + C_0 - \frac{(m-1)\,(a+1)\,(b+1)\,cC_0}{2mab\,(c+1)} + 1\right)$$

and

$$\beta_2 = \frac{mab\,(c+1)}{(a+1)\,(b+1)\,cC_0}\left(\frac{\left(A^2+2\right)\left(1-k^2\right)}{6A^2} + C_0 - \frac{(m-1)\,(a+1)\,(b+1)\,C_0}{2mab\,(c+1)} - 1\right).$$

Corollary 3.6. *If* $f \in k - SP_c^{a,b}$ $(0 \le k < \infty,\ a,b,c \in \mathbb{R},\ a,b,c > 0)$ *and* F *is the* $m-th$ *root transformation of* f *given by* (3) *and* $k = 1$. *Then*

$$|b_{2m+1} - \mu b_{m+1}^2| \le \begin{cases} \dfrac{8c\,(c+1)}{mab\,(a+1)\,(b+1)\,\pi^2}\left(1 + \dfrac{2}{3} + \dfrac{8C_0}{\pi^2} - A_9\right), & if \quad \mu \le \chi_2, \\ \dfrac{16c\,(c+1)}{mab\,(a+1)\,(b+1)\,\pi^2}, & if \quad \chi_2 \le \mu \le \chi_1, \\ \dfrac{8c\,(c+1)}{mab\,(a+1)\,(b+1)\,\pi^2}\left(1 + A_9 - \dfrac{2}{3} - \dfrac{8C_0}{\pi^2}\right), & if \quad \mu \ge \chi_1, \end{cases}$$

where

$$A_9 = \frac{4\,(m+2\mu-1)\,(a+1)\,(b+1)\,cC_0}{mab\,(c+1)\,\pi^2},$$

$$\chi_1 = \frac{mab\,(c+1)}{(a+1)\,(b+1)\,cC_0}\left(\frac{\pi^2}{12} + C_0 - \frac{(m-1)\,(a+1)\,(b+1)\,cC_0}{2mab\,(c+1)} + 1\right)$$

and

$$\chi_2 = \frac{mab\,(c+1)}{(a+1)\,(b+1)\,cC_0}\left(\frac{\pi^2}{12} + C_0 - \frac{(m-1)\,(a+1)\,(b+1)\,cC_0}{2mab\,(c+1)} - 1\right).$$

Corollary 3.7. *If* $f \in k - SP_c^{a,b}$ $(0 \le k < \infty, a,b,c \in \mathbb{R}, a,b,c > 0)$ *and* F *is the* $m-th$ *root transformation of* f *given by* (3) *and* $k > 1$. *Then*

$$|b_{2m+1} - \mu b_{m+1}^2|$$
$$\le \begin{cases} \dfrac{c\,(c+1)\,\pi^2}{4mab\,(a+1)\,(b+1)\,\kappa^2(t)\,(k^2-1)\,(1-t)\,\sqrt{t}}\,(1 + A_{12} - A_{10}), & if \quad \mu \le \delta_2, \\ \dfrac{c\,(c+1)\,\pi^2}{2mab\,(a+1)\,(b+1)\,\kappa^2(t)\,(k^2-1)\,(1-t)\,\sqrt{t}}, & if \quad \delta_2 \le \mu \le \delta_1, \\ \dfrac{c\,(c+1)\,\pi^2}{4mab\,(a+1)\,(b+1)\,\kappa^2(t)\,(k^2-1)\,(1-t)\,\sqrt{t}}\,(1 + A_{10} - A_{12}), & if \quad \mu \ge \delta_1, \end{cases}$$

where

$$A_{10} = \frac{(m+2\mu-1)\,(a+1)\,(b+1)\,\pi^2 cC_0}{4mab\,(c+1)\,\kappa^2(t)\,(k^2-1)\,(1-t)\,\sqrt{t}},$$

$$A_{11} = \frac{\kappa^2(t)\left(k^2-1\right)(1-t)\sqrt{t}}{\pi^2}\left(\frac{4\kappa^2(t)\left(t^2+6t+1\right)-\pi^2}{24\kappa^2(t)\,(1+t)\sqrt{t}}\right),$$

$$A_{12} = \left(\frac{4\kappa^2(t)\left(t^2+6t+1\right)-\pi^2}{24\kappa^2(t)\,(1+t)\sqrt{t}}\right) + \frac{\pi^2 C_0}{4\kappa^2(t)\,(k^2-1)\,(1-t)\sqrt{t}},$$

$$\delta_1 = \frac{mab\,(c+1)}{(a+1)\,(b+1)\,cC_0}\left(A_{11} + C_0 - \frac{(m-1)\,(a+1)\,(b+1)\,cC_0}{2mab\,(c+1)} + 1\right)$$

and

$$\delta_2 = \frac{mab\,(c+1)}{(a+1)\,(b+1)\,cC_0}\left(A_{11} + C_0 - \frac{(m-1)\,(a+1)\,(b+1)\,cC_0}{2mab\,(c+1)} - 1\right).$$

4. Improvements of the main results

In this section, we discuss the improvements of the second inequality in Theorem 3.4.

Remark 4.1. *The second inequality in Theorem 3.4 can be improved as follows:*

$$|b_{2m+1} - \mu b_{m+1}^2| + |\mu C_0 - |\alpha_2|| \, |b_{m+1}|^2 \leq \frac{2c(c+1) Q_1}{mab(a+1)(b+1)}, \quad if \quad \alpha_2 \leq \mu \leq \alpha_3,$$

and

$$|b_{2m+1} - \mu b_{m+1}^2| + ||\alpha_1| - \mu C_0| \, |b_{m+1}|^2 \leq \frac{2c(c+1) Q_1}{mab(a+1)(b+1)}, \quad if \quad \alpha_3 \leq \mu \leq \alpha_1,$$

where

$$\alpha_3 = \frac{mab(c+1)}{(a+1)(b+1)cQ_1C_0}\left(D + Q_1C_0 - \frac{(m-1)(a+1)(b+1)cQ_1C_0}{2mab(c+1)}\right).$$

Putting the values of $Q_1 = Q_1(k)$ and $D = D(k)$ from Lemma 2.4 in Theorem 3.4 for $0 \leq k < 1$, $k = 1$ and $k > 1$ respectively, we get the following remarks.

Remark 4.2. *Putting the values of $Q_1 = Q_1(k)$ and $D = D(k)$ from Lemma 2.4 in Theorem 3.4 for $0 \leq k < 1$ in Remark 4.1 and the second inequality in Corollary 3.5 can be improved as follows:*

$$|b_{2m+1} - \mu b_{m+1}^2| + |\mu C_0 - |\beta_2|| \, |b_{m+1}|^2 \leq \frac{4A^2c(c+1)}{mab(a+1)(b+1)(1-k^2)}, \quad if \quad \beta_2 \leq \mu \leq \beta_3,$$

and

$$|b_{2m+1} - \mu b_{m+1}^2| + ||\beta_1| - \mu C_0| \, |b_{m+1}|^2 \leq \frac{4A^2c(c+1)}{mab(a+1)(b+1)(1-k^2)}, \quad if \quad \beta_3 \leq \mu \leq \beta_1,$$

where

$$\beta_3 = \frac{mab(c+1)}{(a+1)(b+1)cC_0}\left(\frac{(A^2+2)(1-k^2)}{6A^2} + C_0 - \frac{(m-1)(a+1)(b+1)cC_0}{2mab(c+1)}\right).$$

Remark 4.3. *Putting the values of $Q_1 = Q_1(k)$ and $D = D(k)$ from Lemma 2.4 in Theorem 3.4 for $k = 1$ in Remark 4.1 and the second inequality in Corollary 3.6 can be improved as follows:*

$$|b_{2m+1} - \mu b_{m+1}^2| + |\mu C_0 - |\chi_2|| \, |b_{m+1}|^2 \leq \frac{16c(c+1)}{mab(a+1)(b+1)\pi^2}, \quad if \quad \chi_2 \leq \mu \leq \chi_3,$$

and

$$|b_{2m+1} - \mu b_{m+1}^2| + ||\chi_1| - \mu C_0| \, |b_{m+1}|^2 \leq \frac{16c(c+1)}{mab(a+1)(b+1)\pi^2}, \quad if \quad \chi_3 \leq \mu \leq \chi_1,$$

where

$$\chi_3 = \frac{mab(c+1)}{(a+1)(b+1)cC_0}\left(\frac{\pi^2}{12} + C_0 - \frac{(m-1)(a+1)(b+1)cC_0}{2mab(c+1)}\right).$$

Remark 4.4. *Putting the values of $Q_1 = Q_1(k)$ and $D = D(k)$ from Lemma 2.4 in Theorem 3.4 for $k > 1$ in Remark 4.1 and the second inequality in Corollary 3.7 can be improved as follows:*

If $\delta_2 \leq \mu \leq \delta_3$

$$|b_{2m+1} - \mu b_{m+1}^2| + |\mu C_0 - |\delta_2|| \, |b_{m+1}|^2 \leq \frac{c(c+1)\pi^2}{2mab(a+1)(b+1)\kappa^2(t)(k^2-1)(1-t)\sqrt{t}},$$

and if $\delta_3 \le \mu \le \delta_1$

$$\left|b_{2m+1} - \mu b_{m+1}^2\right| + \left|\left|\delta_1\right| - \mu C_0\right| \left|b_{m+1}\right|^2 \le \frac{c\,(c+1)\,\pi^2}{2mab\,(a+1)\,(b+1)\,\kappa^2\,(t)\,(k^2-1)\,(1-t)\,\sqrt{t}},$$

where

$$\delta_3 = \frac{mab\,(c+1)}{(a+1)\,(b+1)\,cC_0}\left(A_{11} + C_0 - \frac{(m-1)\,(a+1)\,(b+1)\,cC_0}{2mab\,(c+1)}\right).$$

5. Conclusion

In this article, we provide the coefficient estimate with $m-$th root transform for a class defined by Hohlov operator using quasi-subordination for conic domains. The authors sincerely hope this article will revive this concept and encourage the other researchers to work in this quasi subordination in the near future in the area of complex function theory.

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