TWMS J. Pure Appl. Math., V.11, N.2, 2020, pp.238-258

ANALYTICAL APPROACH TO A CLASS OF BAGLEY-TORVIK EQUATIONS

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ABSTRACT. Multi-term fractional differential equations have been studied because of their applications in modelling, and solved using miscellaneous mathematical methods. We present explicit analytical solutions for several families of generalized multidimensional Bagley-Torvik equations with permutable matrices and two various fractional orders which are satisfying $\alpha \in (1,2], \beta \in (0,1]$ and $\alpha \in (1,2], \beta \in (1,2]$, both homogeneous and inhomogeneous cases. The results are obtained by means of Mittag-Leffler type matrix functions with double infinite series. In addition, we acquire general solutions of the Bagley-Torvik scalar equations with $\frac{1}{2}$-order and $\frac{3}{2}$-order derivatives. At the end, we present different examples to verify the efficiency our main results.

Keywords: Bagley-Torvik equations, Caputo fractional differentiation, Mittag-Leffler type functions, Fox-Wright functions, double infinite series.

AMS Subject Classification: 34A08, 26A33, 34A25,44A15.

1. Introduction

Fractional differential equations (FDEs), have been attracted growing attention because of their extensive applications in miscellaneous problems in science and engineering like to anomalous diffusion [8], vibration theory [16], electrical circuits [19], stability analysis [9], control theory [24], stochastic analysis [2], time-delay systems [17] and bio-engineering [7].

FDEs containing not only one fractional derivative [3]-[5] but also more than one fractional derivative are intensively studied in many complex systems. Recently, the authors illustrate the physical processes with two essential mathematical ways : multi-term equations [6],[23],[36] and multi-order systems [18],[19].

Multi-term fractional differential equations (FDEs) have been studied due to their applications in modelling and solved using various mathematical methods. Luchko and Gorenflo in [23], solved the multi-term FDEs with constant coefficients and with the Liouville-Caputo fractional derivatives by using the method of operational calculus. Furthermore, in [6], Bazhlekova have considered the multi-term fractional relaxation equations with Liouville-Caputo fractional derivatives by using Laplace transform technique and studied the fundamental and impulse-response solutions of the initial value problem (IVP). Extension of the multi-order FDEs with variable coefficients in terms of generalized fractional derivatives have investigated by Restrepo et al. in [36].

In mechanics, fractional-order differentiation operators are more productive and effective tool in learning the mechanical properties of real materials. Bagley-Torvik equations with $\frac{1}{2}$-order derivative or $\frac{3}{2}$-order derivative illustrates the motion of real physical systems in a Newtonian

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Manuscript received June 2020.

fluid. In 1984, Bagley and Torvik [43] have investigated the following differential equations with fractional-order

$$u''(r) + \frac{2s\sqrt{\mu\rho}}{m}\left({}^C D^{\alpha}_{0+}u\right)(r) + \frac{k}{m}u(r) = \frac{1}{m}f(r), \quad r > 0, \tag{1}$$

with zero initial conditions

$$u(0) = 0, \quad u'(0) = 0, \tag{2}$$

where s - an area of the plate, μ - viscosity, ρ - fluid density, m- mass, k - spring of stiffness and $f(\cdot)$ - an external force. An analytical solution have introduced by Podlubny [32] in the form :

$$u(r) = \int\limits_0^r G(r-s)f(s)\mathrm{d}s, \quad r > 0,$$

with

$$G(r) = \frac{1}{m}\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\left(\frac{k}{m}\right)^p r^{2p+1}\mathcal{E}^{(p)}_{\frac{1}{2},2+\frac{3p}{2}}\left(-\frac{2s\sqrt{\mu\rho}}{m}\sqrt{r}\right),$$

where $\mathcal{E}^{(p)}_{\alpha,\beta}(\cdot)$ is the p-derivative of a two parameter Mittag-Leffler functions.

As one of the important special cases of multi-term FDEs, Bagley-Torvik equations have been discussed by means of analytical methods in [13],[31],[44],[47]. In [31], Pang et al. have studied the general analytical solutions of the generalized Bagley-Torvik equations An analytical approach on defense expenditure and economic growth with Liouville-Caputo fractional differentiation operator of order $0 < \alpha < 2$ by means of three-parameter Mittag-Leffler functions using the Laplace integral transform method. In [44], Z. Wang and X. Wang have changed the following Bagley-Torvik equation

$$u''(r) + \mu\left({}^C D^{\alpha}_{0+}u\right)(r) + u(r) = 0, \quad \mu > 0, \tag{3}$$

where $\alpha = \frac{1}{2}$ or $\alpha = \frac{3}{2}$, to the sequential FDEs and introduced a general solution of (3) by using the approach related to characteristic roots. In [13], Fazli and Nieto have investigated existence and uniqueness results and also approximations of the solutions to the Cauchy problem for the following Bagley-Torvik equation with fractional order of $\alpha \in (1,2)$:

$$\begin{cases} \left(D^2 + AD^{\alpha} + B\right)u(r) = f(r), \quad r \in (0,1], \\ u(0) = a, \quad u'(0) = b, \end{cases} \tag{4}$$

where D^{α} is the Liouville-Caputo fractional derivative, $f : [0,1] \to \mathbb{R}$ is a given continuous function, a, b, A, B are real numbers. In [47], Zafar et al. have considered a general form of the Bagley-Torvik equation as follows:

$$\begin{cases} \lambda_2\left({}^C D^{\alpha}_{0+}u\right)(r) + \lambda_1\left({}^C D^{\beta}_{0+}u\right)(r) + \lambda_0 u(r) = f(r), \quad r > 0, \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases} \tag{5}$$

and introduced analytical representation of solutions in terms of "generalized G-function" by using Laplace transform technique.

In the view of numerical methods have been applied to get approximate analytical solutions of the Bagley-Torvik equations like Adomian decomposition method [35], hybrid functions method [26], wavelet technique [41] and others. In [10], Diethelm and Ford have considered the discretization of Bagley-Torvik equations by means of fractional linear multistep methods and Adams type predictor-corrector pairs. In [42], Srivastava et al. have investigated the solutions of a fractionally-damped generalized Bagley– Torvik equation whose damping characteristics are well-defined by means of the Riemann–Liouville and Liouville–Caputo types fractional differentiation operators via the homotopy analysis method (HAM) which is implemented for computing the dynamic response analysis.

In the same vein as the above articles, we study an IVP of inhomogeneous generalized multidimensional Bagley-Torvik equations involving two fractional orders in various intervals as below:

$$\begin{cases} \left({}^{C}D^{\alpha}_{0+}u\right)(r) - B\left({}^{C}D^{\beta}_{0+}u\right)(r) - Au(r) = g(r), \quad r \in (0,T], \\ u(0) = u_0, \quad u'(0) = u'_0, \quad u_0, u'_0 \in \mathbb{R}^n, \end{cases} \tag{6}$$

where ${}^{C}D^{\alpha}_{0+}$ and ${}^{C}D^{\beta}_{0+}$ are Liouville-Caputo type fractional differentiation of orders α and β in different intervals $1 < \alpha \leq 2$, $0 < \beta \leq 1$ and $1 < \alpha \leq 2$, $1 < \beta \leq 2$, $g : [0,T] \to \mathbb{R}^n$ is a continuous function and $A, B \in \mathbb{R}^{n\times n}$ denote constant matrices which are permutable matrices, that is, $AB = BA$.

The structure of the paper is systemized as below. Section 2 is a preliminary section where we recall important definitions, results and necessary lemmas from fractional calculus, special functions and FDEs. In Section 3 and Section 4, we establish exact analytical solutions in terms of Mittag-Leffler type matrix functions for homogeneous generalized multidimensional Bagley-Torvik equations of two fractional orders which satisfying $\alpha \in (1,2], \beta \in (0,1]$ and $\alpha \in (1,2], \beta \in (1,2]$ with inhomogeneous initial conditions and inhomogeneous generalized multidimensional Bagley-Torvik equations $\alpha \in (1,2], \beta \in (0,1]$ and $\alpha \in (1,2], \beta \in (1,2]$ with zero initial conditions, respectively. To verify our results we make use of substitution method firstly for homogeneous case with inhomogeneous initial conditions and then for inhomogeneous case under zero initial conditions with the help of fractional analogue of variation of constants formula. Then, we prove that our solutions in terms of M-L type matrix functions coincide with the results in terms of generalized Wright matrix functions in [20]. Afterwards, with the help of superposition principle, we obtain the explicit analytical solution for the Cauchy problem for a class of inhomogeneous multidimensional Bagley-Torvik equations with matrix coefficients. Furthermore, we provide exact analytical solutions to scalar Bagley-Torvik equations with $\frac{1}{2}$-order derivative or $\frac{3}{2}$-order derivative, respectively. To verify our main results obtained in Section 3 and Section 4, we provide examples in Section 5 and Section 6 is for the conclusion and future work.

2. Preliminaries

We embark on this section by briefly introducing the essential structure of fractional calculus and fractional differential operators (for the more salient details on the matter, see the textbooks [32], [20]-[11]). We begin by defining the basic gamma and beta functions which are fundamental for fractional calculus.

Definition 2.1 ([46, 34])**.** *The gamma function* $\Gamma : (0,\infty) \to \mathbb{R}$ *is defined by:*

$$\Gamma(\alpha) = \int_0^\infty \tau^{\alpha-1}e^{-\tau}\,\mathrm{d}\tau, \quad \alpha > 0.$$

The beta function is determined as:

$$\mathbf{B}(c,d) = \int_0^1 \tau^{c-1}(1-\tau)^{d-1}\,\mathrm{d}\tau, \quad \text{for} \quad c,d > 0. \tag{7}$$

Furthermore, the beta function can be expressed with the aid of gamma function [46] as below:

$$\mathbf{B}(c,d) = \frac{\Gamma(c)\Gamma(d)}{\Gamma(c+d)}, \quad \text{for} \quad c,d > 0.$$

Definition 2.2 ([28, 37, 30]). *The Riemann-Liouville (R-L) integral operator of fractional order $\alpha > 0$ is given by*

$$I_{0+}^{\alpha} g(\tau) = \frac{1}{\Gamma(\alpha)} \int_0^{\tau} (\tau - r)^{\alpha-1} g(r)\, \mathrm{d}r, \quad \textit{for} \quad \tau > 0. \tag{8}$$

The R-L derivative operator of fractional order $\alpha > 0$ is defined by:

$$D_{0+}^{\alpha} g(\tau) = \frac{\mathrm{d}^n}{\mathrm{d}\tau^n} \left(I_{0+}^{n-\alpha} g(\tau) \right), \quad \textit{where} \quad n-1 < \alpha \leq n.$$

Definition 2.3 ([11]). *Suppose that $\alpha > 0$, $\tau > 0$. The Liouville-Caputo derivative operator of fractional order α is given by:*

$${}^{C}D_{0+}^{\alpha} g(\tau) = I_{0+}^{n-\alpha} \left(\frac{\mathrm{d}^n}{\mathrm{d}\tau^n} g(\tau) \right), \quad \textit{where} \quad n-1 < \alpha \leq n.$$

Note that here we use the constant of integration 0 *for the lower limit of the integral.*

The R-L fractional integral operator and the Liouville-Caputo fractional derivative have the following properties whenever $\alpha > 0$ [32]:

$$I_{0+}^{\alpha}({}^{C}D_{0+}^{\alpha} g(\tau)) = g(\tau) - \sum_{i=0}^{n-1} \frac{\tau^i g^{(i)}(0)}{\Gamma(i+1)},$$

$${}^{C}D_{0+}^{\alpha} \left(I_{0+}^{\alpha} g(\tau) \right) = g(\tau).$$

The relationship between the R-L and Liouville-Caputo fractional derivatives are as follows [28]:

$${}^{C}D_{0+}^{\alpha} g(\tau) = D_{0+}^{\alpha} g(\tau) - \sum_{i=0}^{n-1} \frac{\tau^{i-\alpha} g^{(i)}(0)}{\Gamma(i-\alpha+1)}, \quad \alpha > 0. \tag{9}$$

The Mittag-Leffler function (M-L) is a generalization of the exponential function, first proposed as a single parameter function of one variable by using an infinite series [27]. Extensions to two or three parameters are well known and thoroughly studied in textbooks such as [15], but these still involve single power series in one variable. Extensions to two or several variables have been studied more recently [23, 18, 38].

Definition 2.4 ([27]). *The classical M-L function is defined by:*

$$E_{\alpha}(u) = \sum_{i=0}^{\infty} \frac{u^i}{\Gamma(i\alpha+1)}, \quad \alpha > 0, u \in \mathbb{R}. \tag{10}$$

The two-parameter M-L function [15] *is given by:*

$$E_{\alpha,\beta}(u) = \sum_{i=0}^{\infty} \frac{u^i}{\Gamma(i\alpha+\beta)}, \quad \alpha > 0, \beta \in \mathbb{R}, u \in \mathbb{R}. \tag{11}$$

The three-parameter M-L function [33] *is determined by:*

$$E_{\alpha,\beta}^{\gamma}(u) = \sum_{i=0}^{\infty} \frac{(\gamma)_i}{\Gamma(i\alpha+\beta)} \frac{u^i}{i!}, \quad \alpha > 0, \beta, \gamma \in \mathbb{R}, u \in \mathbb{R}, \tag{12}$$

where $(\gamma)_i$ *is the Pochhammer symbol denoting* $\frac{\Gamma(\gamma+i)}{\Gamma(\gamma)}$. *These series are convergent, locally uniformly in u , provided the* $\alpha > 0$ *condition is satisfied. Note that*

$$E^1_{\alpha,\beta}(u) = E_{\alpha,\beta}(u), \quad E_{\alpha,1}(u) = E_\alpha(u), \quad E_1(u) = e^u.$$

Definition 2.5. [18] *The M-L type function is defined by double infinite series as below:*

$$\tau^{\gamma-1}E_{\alpha,\beta,\gamma}(\lambda\tau^\alpha, \mu\tau^\beta) = \sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{\lambda^l\mu^p}{\Gamma(l\alpha+p\beta+\gamma)}\tau^{l\alpha+p\beta+\gamma-1}, \quad \alpha,\beta>0, \gamma\in\mathbb{R}, \tag{13}$$

The following lemma will be of significance for our results in the main theory of Section 3 and Section 4.

Lemma 2.1 ([18]). *For any* $\tau \in \mathbb{R}$ *and any parameters* $\alpha, \beta, \gamma, \mu, \lambda \in \mathbb{R}$ *satisfying* $\alpha, \beta > 0$ *and* $\gamma - 1 > \lfloor\alpha\rfloor$, *we have:*

$${}^C D^\alpha_{0+}\Big[\tau^{\gamma-1}E_{\alpha,\beta,\gamma}(\lambda\tau^\alpha, \mu\tau^\beta)\Big] = \tau^{\gamma-\alpha-1}E_{\alpha,\beta,\gamma-\alpha}(\lambda\tau^\alpha, \mu\tau^\beta).$$

Proof. We have

$${}^C D^\nu_{0+}\left(\frac{\tau^\mu}{\Gamma(\mu+1)}\right) = \begin{cases} \frac{\tau^{\mu-\nu}}{\Gamma(\mu-\nu+1)}, & \mu > \lfloor\nu\rfloor, \\ 0, & \text{otherwise}. \end{cases}$$

Therefore, given the condition $\gamma - 1 > \lfloor\alpha\rfloor$, we can obtain that

$$\begin{aligned} {}^C D^\alpha_{0+}\left[\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\frac{(l+p)!\lambda^l\mu^p\tau^{l\alpha+p\beta+\gamma-1}}{\Gamma(l\alpha+p\beta+\gamma)l!p!}\right] &= \sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\frac{(l+p)!\lambda^l\mu^p}{l!p!}\,{}^C D^\alpha_{0+}\left(\frac{\tau^{l\alpha+p\beta+\gamma-1}}{\Gamma(l\alpha+p\beta+\gamma)}\right) \\ &= \sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\frac{(l+p)!\lambda^l\mu^p\tau^{l\alpha+p\beta+\gamma-\alpha-1}}{\Gamma(l\alpha+p\beta+\gamma-\alpha)l!p!} \\ &= \tau^{\gamma-\alpha-1}E_{\alpha,\beta,\gamma-\alpha}(\lambda\tau^\alpha, \mu\tau^\beta). \end{aligned}$$

This completes the proof. □

Definition 2.6. *Let* $\lambda_i, \mu_j \in \mathbb{C}$, $\alpha_i, \beta_j \in \mathbb{R}$, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$. *Generalized Wright function or more appropriately Fox-Wright function* $p\Psi_q(\cdot) : \mathbb{C} \to \mathbb{C}$ *is defined by:*

$$p\Psi_q(u) = p\Psi_q\left[\begin{array}{c} (\lambda_i, \alpha_i)_{1,p} \\ (\mu_j, \beta_j)_{1,q} \end{array}\Big| u\right] = \sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(\lambda_i+\alpha_i k)}{\prod_{j=1}^{q}\Gamma(\mu_j+\beta_j k)}\frac{u^k}{k!}. \tag{14}$$

This Fox-Wright function was established by Fox [14] and Wright [45]. If the following condition is satisfied:

$$\sum_{j=1}^{q}\beta_j - \sum_{i=1}^{p}\alpha_i > -1.$$

then the series in (14) is convergent for arbitrary $u \in \mathbb{C}$.

In terms of Laplace integral transform method, Kilbas et al. [20] have considered the Cauchy problem for FDEs with two fractional orders by using generalized Wright functions, in both homogeneous and in-homogeneous cases. It is necessary to note that our results by means of a M-L type functions with double infinite series are identical with the results in terms of Fox-Wright functions in [20].

3. Analytical representation of solutions to generalized Bagley-Torvik equation: $\alpha \in (1,2]$ and $\beta \in (0,1]$

In this section, firstly, we consider the Cauchy problem for the homogeneous linear generalized multidimensional Bagley-Torvik equation in the form of:

$$\begin{cases} \left({}^C D^{\alpha}_{0+} u\right)(r) - B\left({}^C D^{\beta}_{0+} u\right)(r) - Au(r) = 0, \quad r \in (0,T], \\ u(0) = u_0, \quad u'(0) = u'_0, \end{cases} \tag{15}$$

where $\alpha \in (1,2]$ and $\beta \in (0,1]$.

Theorem 3.1. *The analytical solution $u \in C^2([0,T],\mathbb{R}^n)$ to the IVP for homogeneous multidimensional Bagley-Torvik equation (15) is represented by:*

$$u(r) = \left(1 + r^{\alpha} A E_{\alpha,\alpha-\beta,\alpha+1}(Ar^{\alpha}, Br^{\alpha-\beta})\right) u_0 + r E_{\alpha,\alpha-\beta,2}(Ar^{\alpha}, Br^{\alpha-\beta}) u'_0, \quad r > 0.$$

Proof. Having found explicit form for $u(r)$, it remains to verify that this is a solution of (15) indeed. Starting from the left-hand-side by using Lemma 2.1 and the following Pascal identity for binomial coefficients:

$$\binom{l+p}{p} = \binom{l+p-1}{p} + \binom{l+p-1}{p-1}, \quad l,p \geq 1,$$

we arrive at

$$\begin{aligned}
\left({}^C D^{\alpha}_{0+} u\right)(r) &= A\, {}^C D^{\alpha}_{0+}\left(r^{\alpha} E_{\alpha,\alpha-\beta,\alpha+1}(Ar^{\alpha}, Br^{\alpha-\beta})\right) u_0 + {}^C D^{\alpha}_{0+}\left(r E_{\alpha,\alpha-\beta,2}(Ar^{\alpha}, Br^{\alpha-\beta})\right) u'_0 \\
&= {}^C D^{\alpha}_{0+}\Big(A\frac{r^{\alpha}}{\Gamma(\alpha+1)} + \sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\binom{l+p-1}{p}\frac{A^{l+1}B^p r^{l\alpha+p(\alpha-\beta)+\alpha}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha+1)} \\
&+ \sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p-1}{p-1}\frac{A^{l+1}B^p r^{l\alpha+p(\alpha-\beta)+\alpha}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha+1)}\Big)u_0 \\
&+ {}^C D^{\alpha}_{0+}\Big(Ar + \sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\binom{l+p-1}{p}\frac{A^{l+1}B^p r^{l\alpha+p(\alpha-\beta)+1}}{\Gamma(l\alpha+p(\alpha-\beta)+2)} \\
&+ \sum_{l=0}^{\infty}\sum_{p=1}^{\infty}\binom{l+p-1}{p-1}\frac{A^{l+1}B^p r^{l\alpha+p(\alpha-\beta)+1}}{\Gamma(l\alpha+p(\alpha-\beta)+2)}\Big)u'_0 \\
&= \Big(A + \sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\binom{l+p-1}{p}\frac{A^{l+1}B^p r^{l\alpha+p(\alpha-\beta)}}{\Gamma(l\alpha+p(\alpha-\beta)+1)} \\
&+ \sum_{l=0}^{\infty}\sum_{p=1}^{\infty}\binom{l+p-1}{p-1}\frac{A^{l+1}B^p r^{l\alpha+p(\alpha-\beta)}}{\Gamma(l\alpha+p(\alpha-\beta)+1)}\Big)u_0 \\
&+ \Big(\sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\binom{l+p-1}{p}\frac{A^{l}B^p r^{l\alpha+p(\alpha-\beta)-\alpha+1}}{\Gamma(l\alpha+p(\alpha-\beta)-\alpha+2)} \\
&+ \sum_{l=0}^{\infty}\sum_{p=1}^{\infty}\binom{l+p-1}{p-1}\frac{A^{l}B^p r^{l\alpha+p(\alpha-\beta)-\alpha+1}}{\Gamma(l\alpha+p(\alpha-\beta)-\alpha+2)}\Big)u'_0 \\
&= Au_0 + \sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{A^{l+2}B^p r^{(l+1)\alpha+p(\alpha-\beta)}}{\Gamma((l+1)\alpha+p(\alpha-\beta)+1)}u_0
\end{aligned}$$

$$+\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{A^{l+1}B^{p+1}r^{l\alpha+(p+1)(\alpha-\beta)}}{\Gamma(l\alpha+(p+1)(\alpha-\beta)+1)}u_0$$
$$+\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{A^{l+1}B^{p}r^{(l+1)\alpha+p(\alpha-\beta)-\alpha+1}}{\Gamma((l+1)\alpha+p(\alpha-\beta)-\alpha+2)}u_0'$$
$$+\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{A^{l}B^{p+1}r^{l\alpha+(p+1)(\alpha-\beta)-\alpha+1}}{\Gamma(l\alpha+(p+1)(\alpha-\beta)-\alpha+2)}\Big)u_0'$$
$$= Au_0 + A^2 r^{\alpha}E_{\alpha,\alpha-\beta,\alpha+1}(Ar^{\alpha}, Br^{\alpha-\beta})u_0$$
$$+ ABr^{\alpha-\beta}E_{\alpha,\alpha-\beta,\alpha-\beta+1}(Ar^{\alpha}, Br^{\alpha-\beta})u_0$$
$$+ ArE_{\alpha,\alpha-\beta,2}(Ar^{\alpha}, Br^{\alpha-\beta})u_0'$$
$$+ Br^{1-\beta}E_{\alpha,\alpha-\beta,2-\beta}(Ar^{\alpha}, Br^{\alpha-\beta})u_0'.$$

In a similar way, we get:

$$\begin{aligned} B\left({}^{C}D_{0^+}^{\beta}u\right)(r) &= AB\,{}^{C}D_{0^+}^{\beta}\left(r^{\alpha}E_{\alpha,\alpha-\beta,\alpha+1}(Ar^{\alpha}, Br^{\alpha-\beta})\right)u_0 \\ &+ B\,{}^{C}D_{0^+}^{\beta}\left(rE_{\alpha,\alpha-\beta,2}(Ar^{\alpha}, Br^{\alpha-\beta})\right)u_0' \\ &= ABr^{\alpha-\beta}E_{\alpha,\alpha-\beta,\alpha-\beta+1}(Ar^{\alpha}, Br^{\alpha-\beta})u_0 \\ &+ Br^{1-\beta}E_{\alpha,\alpha-\beta,2-\beta}(Ar^{\alpha}, Br^{\alpha-\beta})u_0'. \end{aligned}$$

and

$$Au(r) = Au_0 + A^2 r^{\alpha}E_{\alpha,\alpha-\beta,\alpha+1}(Ar^{\alpha}, Br^{\alpha-\beta})u_0 + ArE_{\alpha,\alpha-\beta,2}(Ar^{\alpha}, Br^{\alpha-\beta})u_0'.$$

Taking a linear combination of above results, we attain the desired result. □

Secondly, we consider the Cauchy problem for the inhomogeneous linear generalized multidimensional Bagley-Torvik equation in the form of:

$$\begin{cases} \left({}^{C}D_{0^+}^{\alpha}u\right)(r) - B\left({}^{C}D_{0^+}^{\beta}u\right)(r) - Au(r) = g(r), \quad r \in (0,T], \\ u(0) = u_0, \quad u'(0) = u_0', \end{cases} \tag{16}$$

where $\alpha \in (1,2]$ and $\beta \in (0,1]$.

Theorem 3.2. *The inhomogeneous generalized multidimensional Bagley-Torvik equation* (16) *with zero initial conditions has the following solution:*

$$u(r) = \int_0^r (r-s)^{\alpha-1}E_{\alpha,\alpha-\beta,\alpha}(A(r-s)^{\alpha}, B(r-s)^{\alpha-\beta})g(s)\mathrm{d}s, \quad r > 0.$$

Proof. By using the method of variation of constants, any solution of inhomogeneous system $u(r)$ should be satisfied in the form:

$$u(r) = \int_0^r (r-s)^{\alpha-1}E_{\alpha,\alpha-\beta,\alpha}(A(r-s)^{\alpha}, B(r-s)^{\alpha-\beta})h(s)\mathrm{d}s, \quad r > 0.$$

where $h(s), \quad 0 \le s \le r$ is an unknown vector function and $u(0) = 0$. On the other hand, in this case, the R-L and Liouville-Caputo type fractional derivatives are equal in accordance with (9). Therefore, in the work below we shall apply R-L derivative instead of Liouville-Caputo one to verify the solution of differential equation. So, having Liouville-Caputo fractional differentiation on both sides of (3), we attain the following results:

$$\left({}^C D^\alpha_{0^+} u\right)(r) = \left(D^\alpha_{0_+} u\right)(r) = D^\alpha_{0_+}\left(\int_0^r (r-s)^{\alpha-1} E_{\alpha,\alpha-\beta,\alpha}(A(r-s)^\alpha, B(r-s)^{\alpha-\beta}) g(s) \mathrm{d}s\right)$$

$$= \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dr^2} \int_0^r (r-s)^{1-\alpha} \mathrm{d}s \int_0^s (s-\tau)^{\alpha-1} E_{\alpha,\alpha-\beta,\alpha}(A(s-\tau)^\alpha, B(s-\tau)^{\alpha-\beta}) g(\tau) \mathrm{d}\tau$$

$$= \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dr^2} \int_0^r \int_0^s (r-s)^{1-\alpha}(s-\tau)^{\alpha-1} E_{\alpha,\alpha-\beta,\alpha}(A(s-\tau)^\alpha, B(s-\tau)^{\alpha-\beta}) g(\tau) \mathrm{d}\tau \mathrm{d}s$$

$$= \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dr^2} \int_0^r \int_\tau^r (r-s)^{1-\alpha}(s-\tau)^{\alpha-1} E_{\alpha,\alpha-\beta,\alpha}(A(s-\tau)^\alpha, B(s-\tau)^{\alpha-\beta}) g(\tau) \mathrm{d}s \mathrm{d}\tau$$

$$= \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dr^2} \int_0^r \int_\tau^r (r-s)^{1-\alpha}(s-\tau)^{\alpha-1} \sum_{l=0}^\infty \sum_{p=0}^\infty \binom{l+p}{p} \frac{(A(s-\tau)^\alpha)^l (B(s-\tau)^{\alpha-\beta})^p}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha)} g(\tau) \mathrm{d}s \mathrm{d}\tau$$

$$= \sum_{l=0}^\infty \sum_{p=0}^\infty \binom{l+p}{p} \frac{A^l B^p}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha)} \frac{d^2}{dr^2} \int_0^r g(\tau)\mathrm{d}\tau \int_\tau^r (r-s)^{1-\alpha}(s-\tau)^{l\alpha+p(\alpha-\beta)+\alpha-1} \mathrm{d}s$$

$$= \sum_{l=0}^\infty \sum_{p=0}^\infty \binom{l+p}{p} \frac{A^l B^p}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha)} \frac{d^2}{dr^2} \int_0^r (r-\tau)^{l\alpha+p(\alpha-\beta)+1}$$

$$\times \mathbf{B}(2-\alpha, l\alpha+p(\alpha-\beta)+\alpha) g(\tau) \mathrm{d}\tau$$

$$= \sum_{l=0}^\infty \sum_{p=0}^\infty \binom{l+p}{p} \frac{A^l B^p}{\Gamma(l\alpha+p(\alpha-\beta)+2)} \frac{d^2}{dr^2} \int_0^r (r-\tau)^{l\alpha+p(\alpha-\beta)+1} g(\tau) \mathrm{d}\tau$$

$$= \frac{d^2}{dr^2} \int_0^r (r-\tau) g(\tau) \mathrm{d}\tau$$

$$+ \frac{d^2}{dr^2} \int_0^r (r-\tau) \sum_{l=1}^\infty \sum_{p=0}^\infty \binom{l+p-1}{p} \frac{(A(r-\tau)^\alpha)^l (B(r-\tau)^{\alpha-\beta})^p}{\Gamma(l\alpha+p(\alpha-\beta)+2)} g(\tau) \mathrm{d}\tau$$

$$+ \frac{d^2}{dr^2} \int_0^r (r-\tau) \sum_{l=0}^\infty \sum_{p=1}^\infty \binom{l+p-1}{p-1} \frac{(A(r-\tau)^\alpha)^l (B(r-\tau)^{\alpha-\beta})^p}{\Gamma(l\alpha+p(\alpha-\beta)+2)} g(\tau) \mathrm{d}\tau$$

$$= g(r) + \int_0^r (r-\tau)^{-1} \sum_{l=1}^\infty \sum_{p=0}^\infty \binom{l+p-1}{p} \frac{(A(r-\tau)^\alpha)^l (B(r-\tau)^{\alpha-\beta})^p}{\Gamma(l\alpha+p(\alpha-\beta))} g(\tau) \mathrm{d}\tau$$

$$+ \int_0^r (r-\tau)^{-1} \sum_{l=0}^\infty \sum_{p=1}^\infty \binom{l+p-1}{p-1} \frac{(A(r-\tau)^\alpha)^l (B(r-\tau)^{\alpha-\beta})^p}{\Gamma(l\alpha+p(\alpha-\beta))} g(\tau) \mathrm{d}\tau$$

$$= g(r) + A \int_0^r (r-\tau)^{\alpha-1} \sum_{l=0}^\infty \sum_{p=0}^\infty \binom{l+p}{p} \frac{(A(r-\tau)^\alpha)^l (B(r-\tau)^{\alpha-\beta})^p}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha)} g(\tau) \mathrm{d}\tau$$

$$+ B\int\limits_0^r (r-\tau)^{\alpha-\beta-1}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{(A(r-\tau)^{\alpha})^l(B(r-\tau)^{\alpha-\beta})^p}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha-\beta)}g(\tau)\mathrm{d}\tau$$

$$= g(r) + A\int\limits_0^r (r-\tau)^{\alpha-1}E_{\alpha,\alpha-\beta,\alpha}(A(r-\tau)^{\alpha},B(r-\tau)^{\alpha-\beta})g(\tau)\mathrm{d}\tau$$

$$+ B\int\limits_0^r (r-\tau)^{\alpha-\beta-1}E_{\alpha,\alpha-\beta,\alpha-\beta}(A(r-\tau)^{\alpha},B(r-\tau)^{\alpha-\beta})g(\tau)\mathrm{d}\tau,$$

Similarly, we have:

$$\left({}^C D^{\beta}_{0^+}u\right)(r) = \left(D^{\beta}_{0_+}u\right)(r) = D^{\beta}_{0_+}\left(\int\limits_0^r (r-s)^{\alpha-1}E_{\alpha,\alpha-\beta,\alpha}(A(r-s)^{\alpha},B(r-s)^{\alpha-\beta})g(s)\mathrm{d}s\right)$$

$$= \frac{1}{\Gamma(1-\beta)}\frac{d}{dr}\int\limits_0^r (r-s)^{-\beta}\mathrm{d}s\int\limits_0^s (s-\tau)^{\alpha-1}E_{\alpha,\alpha-\beta,\alpha}(A(s-\tau)^{\alpha},B(s-\tau)^{\alpha-\beta})g(\tau)\mathrm{d}\tau$$

$$= \frac{1}{\Gamma(1-\beta)}\frac{d}{dr}\int\limits_0^r\int\limits_0^s (r-s)^{-\beta}(s-\tau)^{\alpha-1}E_{\alpha,\alpha-\beta,\alpha}(A(s-\tau)^{\alpha},B(s-\tau)^{\alpha-\beta})g(\tau)\mathrm{d}\tau\mathrm{d}s$$

$$= \frac{1}{\Gamma(1-\beta)}\frac{d}{dr}\int\limits_0^r\int\limits_{\tau}^r (r-s)^{-\beta}(s-\tau)^{\alpha-1}E_{\alpha,\alpha-\beta,\alpha}(A(s-\tau)^{\alpha},B(s-\tau)^{\alpha-\beta})g(\tau)\mathrm{d}s\mathrm{d}\tau$$

$$= \frac{1}{\Gamma(1-\beta)}\frac{d}{dr}\int\limits_0^r\int\limits_{\tau}^r (r-s)^{-\beta}(s-\tau)^{\alpha-1}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{(A(s-\tau)^{\alpha})^l(B(s-\tau)^{\alpha-\beta})^p}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha)}g(\tau)\mathrm{d}s\mathrm{d}\tau$$

$$= \sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{A^lB^p}{\Gamma(1-\beta)\Gamma(l\alpha+p(\alpha-\beta)+\alpha)}\frac{d}{dr}\int\limits_0^r g(\tau)\mathrm{d}\tau\int\limits_{\tau}^r (r-s)^{-\beta}(s-\tau)^{l\alpha+p(\alpha-\beta)+\alpha-1}\mathrm{d}s$$

$$= \sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{A^lB^p}{\Gamma(1-\beta)\Gamma(l\alpha+p(\alpha-\beta)+\alpha)}\frac{d}{dr}\int\limits_0^r (r-\tau)^{l\alpha+p(\alpha-\beta)+\alpha-\beta}$$

$$\times \mathbf{B}(1-\beta,l\alpha+p(\alpha-\beta)+\alpha)g(\tau)\mathrm{d}\tau$$

$$= \sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{A^lB^p}{\Gamma(l\alpha+(p+1)(\alpha-\beta)+1)}\frac{d}{dr}\int\limits_0^r (r-\tau)^{l\alpha+(p+1)(\alpha-\beta)}g(\tau)\mathrm{d}\tau$$

$$= \int\limits_0^r (r-\tau)^{\alpha-\beta-1}\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{(A(r-\tau)^{\alpha})^l(B(r-\tau)^{\alpha-\beta})^p}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha-\beta)}g(\tau)\mathrm{d}\tau$$

$$= B\int\limits_0^r (r-\tau)^{\alpha-\beta-1}E_{\alpha,\alpha-\beta,\alpha-\beta}(A(r-\tau)^{\alpha},B(r-\tau)^{\alpha-\beta})g(\tau)\mathrm{d}\tau,$$

and

$$Au(r) = A\int\limits_0^r (r-s)^{\alpha-1}E_{\alpha,\alpha-\beta,\alpha}(A(r-s)^{\alpha},B(r-s)^{\alpha-\beta})g(s)\mathrm{d}s,$$

If we substitute last two expressions into Equation (16), we verify that $h(r) = g(r)$ for $0 \leq r \leq T$. Proof is complete. $\square$

Therefore, we find the explicit formula of solutions to linear inhomogeneous generalized multidimensional Bagley-Torvik equations by applying the classical ideas to find solution of (16).

Theorem 3.3. *A unique solution $u \in C^2([0,T], \mathbb{R}^n)$ of the Cauchy problem for inhomogeneous generalized Bagley-Torvik equation* (16) *has the formula:*

$$u(r) = \Big(1 + r^\alpha A E_{\alpha,\alpha-\beta,\alpha+1}(Ar^\alpha, Br^{\alpha-\beta})\Big)u_0 + rE_{\alpha,\alpha-\beta,2}(Ar^\alpha, Br^{\alpha-\beta})u_0' + \int_0^r (r-s)^{\alpha-1}E_{\alpha,\alpha-\beta,\alpha}(A(r-s)^\alpha, B(r-s)^{\alpha-\beta})g(s)\mathrm{d}s, \quad r > 0. \tag{17}$$

3.1. **Special case: Bagley-Torvik equation of $\frac{1}{2}$-order derivative.** If we choose $\alpha = 2$, $\beta = \frac{1}{2}$ and $A = -\frac{2s\sqrt{\mu\rho}}{m}$, $B = -\frac{k}{m}$, we get classical scalar Bagley-Torvik equation. We consider the Cauchy problem for homogeneous linear Bagley-Torvik equation in the form of:

$$\begin{cases} u''(r) + \frac{2s\sqrt{\mu\rho}}{m}({}^C D_{0+}^{\frac{1}{2}}u)(r) + \frac{k}{m}u(r) = 0, \quad r \in (0,T], \\ u(0) = u_0, \quad u'(0) = u_0', \end{cases} \tag{18}$$

Theorem 3.4. *A unique solution $u \in C^2([0,T], \mathbb{R})$ of the homogeneous Bagley-Torvik equation with the inhomogeneous initial conditions* (18) *has the formula:*

$$u(r) = \Big(1 - \frac{k}{m}r^2 E_{2,\frac{3}{2},3}(-\frac{k}{m}r^2, -\frac{2s\sqrt{\mu\rho}}{m}r^{\frac{3}{2}})\Big)u_0 + rE_{2,\frac{3}{2},2}(-\frac{k}{m}r^2, -\frac{2s\sqrt{\mu\rho}}{m}r^{\frac{3}{2}})u_0', \quad r > 0. \tag{19}$$

The Cauchy problem for inhomogeneous linear Bagley-Torvik equation in the form of :

$$\begin{cases} u''(r) + \frac{2s\sqrt{\mu\rho}}{m}({}^C D_{0+}^{\frac{1}{2}}u)(r) + \frac{k}{m}u(r) = \frac{1}{m}g(r), \quad r \in (0,T], \\ u(0) = u_0, \quad u'(0) = u_0', \end{cases} \tag{20}$$

has the following solution:

Theorem 3.5. *A unique solution $u \in C^2([0,T], \mathbb{R})$ of the inhomogeneous Bagley-Torvik equation with the inhomogeneous initial conditions* (20) *has the formula:*

$$u(r) = \Big(1 - \frac{k}{m}r^2 E_{2,\frac{3}{2},3}(-\frac{k}{m}r^2, -\frac{2s\sqrt{\mu\rho}}{m}r^{\frac{3}{2}})\Big)u_0 + rE_{2,\frac{3}{2},2}(-\frac{k}{m}r^2, -\frac{2s\sqrt{\mu\rho}}{m}r^{\frac{3}{2}})u_0' + \frac{1}{m}\int_0^r (r-s)E_{2,\frac{3}{2},2}(-\frac{k}{m}(r-s)^2, -\frac{2s\sqrt{\mu\rho}}{m}(r-s)^{\frac{3}{2}})g(s)\mathrm{d}s, \quad r > 0. \tag{21}$$

The exact solution of the Cauchy problem for inhomogeneous linear Bagley-Torvik equation in the form of :

$$\begin{cases} u''(r) + \frac{2s\sqrt{\mu\rho}}{m}({}^C D_{0+}^{\frac{1}{2}}u)(r) + \frac{k}{m}u(r) = \frac{1}{m}g(r), \quad r \in (0,T], \\ u(0) = 0, \quad u'(0) = 0, \end{cases} \tag{22}$$

has the following solution:

Theorem 3.6. *A unique solution $u \in C^2([0,T], \mathbb{R})$ of the inhomogeneous Bagley-Torvik equation with the homogeneous initial conditions* (22) *has the formula:*

$$u(r) = \frac{1}{m}\int_0^r G(r-s)g(s)\mathrm{d}s, \quad r > 0, \tag{23}$$

where

$$G(r)=rE_{2,\frac{3}{2},2}(-\frac{k}{m}r^2,-\frac{2s\sqrt{\mu\rho}}{m}r^{\frac{3}{2}}).$$

Remark 3.1. *The Cauchy problem* (16) *has also a solution in terms of Fox-Wright matrix functions below.*

$$\begin{aligned}u(r)=&\Bigg\{\sum_{l=0}^{\infty}\frac{A^lr^{l\alpha}}{l!}1\Psi_1\left[\begin{array}{c}(l+1,1)\\(l\alpha+1,\alpha-\beta)\end{array}\Big|Br^{\alpha-\beta}\right]\\&-B\sum_{l=0}^{\infty}\frac{A^lr^{l\alpha+\alpha-\beta}}{l!}1\Psi_1\left[\begin{array}{c}(l+1,1)\\(l\alpha+\alpha-\beta+1,\alpha-\beta)\end{array}\Big|Br^{\alpha-\beta}\right]\Bigg\}u_0\\&+\sum_{l=0}^{\infty}\frac{A^lr^{l\alpha+1}}{l!}1\Psi_1\left[\begin{array}{c}(l+1,1)\\(l\alpha+2,\alpha-\beta)\end{array}\Big|Br^{\alpha-\beta}\right]u_0'\\&+\int_0^r(r-s)^{\alpha-1}G_{\alpha,\beta;\lambda,\mu}(r-s)g(s)\mathrm{d}s,\quad r>0,\end{aligned}$$

where

$$G_{\alpha,\beta;\lambda,\mu}(r)\sum_{l=0}^{\infty}\frac{A^lr^{l\alpha}}{l!}1\Psi_1\left[\begin{array}{c}(l+1,1)\\(l\alpha+\alpha,\alpha-\beta)\end{array}\Big|Br^{\alpha-\beta}\right].$$

Proof. Using the definition of Fox-Wright function [14, 45], we arrive at

$$\begin{aligned}u(r)=&\Bigg\{\sum_{l=0}^{\infty}\frac{A^lr^{l\alpha}}{l!}\sum_{p=0}^{\infty}\frac{\Gamma(l+1+p)}{\Gamma(l\alpha+1+p(\alpha-\beta))}\frac{B^pr^{p(\alpha-\beta)}}{p!}\\&-B\sum_{l=0}^{\infty}\frac{A^lr^{l\alpha+\alpha-\beta}}{l!}\sum_{p=0}^{\infty}\frac{\Gamma(l+1+p)}{\Gamma(l\alpha+\alpha-\beta+1+p(\alpha-\beta))}\frac{B^pr^{p(\alpha-\beta)}}{p!}\Bigg\}u_0\\&+\sum_{l=0}^{\infty}\frac{A^lr^{l\alpha+1}}{l!}\sum_{p=0}^{\infty}\frac{\Gamma(l+1+p)}{\Gamma(l\alpha+p(\alpha-\beta)+2)}\frac{B^pr^{p(\alpha-\beta)}}{p!}u_0'\\&+\int_0^r\sum_{l=0}^{\infty}\frac{A^l(r-s)^{l\alpha+\alpha-1}}{l!}1\Psi_1\left[\begin{array}{c}(l+1,1)\\(l\alpha+\alpha,\alpha-\beta)\end{array}\Big|B(r-s)^{\alpha-\beta}\right]g(s)\mathrm{d}s\\=&\Bigg\{\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{A^lB^pr^{l\alpha+p(\alpha-\beta)}}{\Gamma(l\alpha+p(\alpha-\beta)+1)}\\&-\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{A^lB^{p+1}r^{l\alpha+(p+1)(\alpha-\beta)}}{\Gamma(l\alpha+(p+1)(\alpha-\beta)+1)}\Bigg\}u_0\\&+\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{A^lB^pr^{l\alpha+p(\alpha-\beta)+1}}{\Gamma(l\alpha+p(\alpha-\beta)+2)}u_0'\\&+\int_0^r\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{A^lB^pr^{l\alpha+p(\alpha-\beta)+\alpha-1}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha)}g(s)\mathrm{d}s\\=&\Bigg\{1+\sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\binom{l+p-1}{p}\frac{A^lB^pr^{l\alpha+p(\alpha-\beta)}}{\Gamma(l\alpha+p(\alpha-\beta)+1)}\end{aligned}$$

$$+\sum_{l=0}^{\infty}\sum_{p=1}^{\infty}\binom{l+p-1}{p-1}\frac{A^lB^pr^{l\alpha+p(\alpha-\beta)}}{\Gamma(l\alpha+p(\alpha-\beta)+1)}$$
$$\left.-\sum_{l=0}^{\infty}\sum_{p=1}^{\infty}\binom{l+p-1}{p-1}\frac{A^lB^pr^{l\alpha+p(\alpha-\beta)}}{\Gamma(l\alpha+p(\alpha-\beta)+1)}\right\}u_0$$
$$+\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{A^lB^pr^{l\alpha+p(\alpha-\beta)+1}}{\Gamma(l\alpha+p(\alpha-\beta)+2)}u_0'$$
$$+\int_0^r\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{A^lB^pr^{l\alpha+p(\alpha-\beta)+\alpha-1}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha)}g(s)\mathrm{d}s$$
$$=\left\{1+\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{A^lB^pr^{l\alpha+p(\alpha-\beta)+\alpha}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha+1)}\right\}u_0$$
$$+\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{A^lB^pr^{l\alpha+p(\alpha-\beta)+1}}{\Gamma(l\alpha+p(\alpha-\beta)+2)}u_0'$$
$$+\int_0^r\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{A^lB^pr^{l\alpha+p(\alpha-\beta)+\alpha-1}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha)}g(s)\mathrm{d}s$$
$$=\Big(1+r^{\alpha}AE_{\alpha,\alpha-\beta,\alpha+1}(Ar^{\alpha},Br^{\alpha-\beta})\Big)u_0+rE_{\alpha,\alpha-\beta,2}(Ar^{\alpha},Br^{\alpha-\beta})u_0'$$
$$+\int_0^r(r-s)^{\alpha-1}E_{\alpha,\alpha-\beta,\alpha}(A(r-s)^{\alpha},B(r-s)^{\alpha-\beta})g(s)\mathrm{d}s,\quad r>0.$$

Therefore, our solution in terms of M-L type matrix functions coincide with the solution by means of Fox-Wright matrix functions shown in [20]. □

4. Representation of solutions to generalized Bagley-Torvik equations: $\alpha\in(1,2]$, $\beta\in(1,2]$

In this section, first, we consider the homogeneous linear generalized multidimensional Bagley-Torvik equation in the form of:

$$\begin{cases}\left({}^CD_{0^+}^{\alpha}u\right)(r)-B\left({}^CD_{0^+}^{\beta}u\right)(r)-Au(r)=0,\quad r\in(0,T],\\ u(0)=u_0,\quad u'(0)=u_0',\end{cases}\tag{24}$$

where $1<\alpha\leq 2$, $1<\beta<2$.

Theorem 4.1. *The analytical solution $u\in C^2([0,T],\mathbb{R}^n)$ to the IVP for homogeneous generalized multidimensional Bagley-Torvik equation* (24) *is:*

$$u(r)=\Big(1+r^{\alpha}AE_{\alpha,\alpha-\beta,\alpha+1}(Ar^{\alpha},Br^{\alpha-\beta})\Big)u_0+\Big(r+r^{\alpha+1}AE_{\alpha,\alpha-\beta,\alpha+2}(Ar^{\alpha},Br^{\alpha-\beta})\Big)u_0',\quad r>0.$$

Proof. Having found explicit form for $u(r)$, it remains to verify that this is a solution of (24) indeed. Starting from the left-hand-side by using Lemma 2.1 and the following Pascal identity for binomial coefficients:

$$\binom{l+p}{p}=\binom{l+p-1}{p}+\binom{l+p-1}{p-1},\quad l,p\geq 1,$$

we arrive at

$$\begin{aligned}
\left({}^C D_{0+}^{\alpha} u\right)(r) &= A\, {}^C D_{0+}^{\alpha} \left(r^{\alpha} E_{\alpha,\alpha-\beta,\alpha+1}(Ar^{\alpha}, Br^{\alpha-\beta})\right) u_0 \\
&+ A\, {}^C D_{0+}^{\alpha} \left(r^{\alpha+1} E_{\alpha,\alpha-\beta,\alpha+2}(Ar^{\alpha}, Br^{\alpha-\beta})\right) u_0' \\
&= {}^C D_{0+}^{\alpha} \Big(A \frac{r^{\alpha}}{\Gamma(\alpha+1)} + \sum_{l=1}^{\infty} \sum_{p=0}^{\infty} \binom{l+p-1}{p} \frac{A^{l+1} B^p r^{l\alpha+p(\alpha-\beta)+\alpha}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha+1)} \\
&+ \sum_{l=0}^{\infty} \sum_{p=0}^{\infty} \binom{l+p-1}{p-1} \frac{A^{l+1} B^p r^{l\alpha+p(\alpha-\beta)+\alpha}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha+1)} \Big) u_0 \\
&+ {}^C D_{0+}^{\alpha} \Big(A \frac{r^{\alpha+1}}{\Gamma(\alpha+2)} + \sum_{l=1}^{\infty} \sum_{p=0}^{\infty} \binom{l+p-1}{p} \frac{A^{l+1} B^p r^{l\alpha+p(\alpha-\beta)+\alpha+1}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha+2)} \\
&+ \sum_{l=0}^{\infty} \sum_{p=1}^{\infty} \binom{l+p-1}{p-1} \frac{A^{l+1} B^p r^{l\alpha+p(\alpha-\beta)+\alpha+1}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha+2)} \Big) u_0' \\
&= \Big(A + \sum_{l=1}^{\infty} \sum_{p=0}^{\infty} \binom{l+p-1}{p} \frac{A^{l+1} B^p r^{l\alpha+p(\alpha-\beta)}}{\Gamma(l\alpha+p(\alpha-\beta)+1)} \\
&+ \sum_{l=0}^{\infty} \sum_{p=1}^{\infty} \binom{l+p-1}{p-1} \frac{A^{l+1} B^p r^{l\alpha+p(\alpha-\beta)}}{\Gamma(l\alpha+p(\alpha-\beta)+1)} \Big) u_0 \\
&+ \Big(Ar + \sum_{l=1}^{\infty} \sum_{p=0}^{\infty} \binom{l+p-1}{p} \frac{A^{l} B^p r^{l\alpha+p(\alpha-\beta)+1}}{\Gamma(l\alpha+p(\alpha-\beta)+2)} \\
&+ \sum_{l=0}^{\infty} \sum_{p=1}^{\infty} \binom{l+p-1}{p-1} \frac{A^{l} B^p r^{l\alpha+p(\alpha-\beta)+1}}{\Gamma(l\alpha+p(\alpha-\beta)+2)} \Big) u_0' \\
&= Au_0 + \sum_{l=0}^{\infty} \sum_{p=0}^{\infty} \binom{l+p}{p} \frac{A^{l+2} B^p r^{(l+1)\alpha+p(\alpha-\beta)}}{\Gamma((l+1)\alpha+p(\alpha-\beta)+1)} u_0 \\
&+ \sum_{l=0}^{\infty} \sum_{p=0}^{\infty} \binom{l+p}{p} \frac{A^{l+1} B^{p+1} r^{l\alpha+(p+1)(\alpha-\beta)}}{\Gamma(l\alpha+(p+1)(\alpha-\beta)+1)} u_0 \\
&+ Aru_0' + \sum_{l=0}^{\infty} \sum_{p=0}^{\infty} \binom{l+p}{p} \frac{A^{l+1} B^p r^{(l+1)\alpha+p(\alpha-\beta)+1}}{\Gamma((l+1)\alpha+p(\alpha-\beta)+2)} u_0' \\
&+ \sum_{l=0}^{\infty} \sum_{p=0}^{\infty} \binom{l+p}{p} \frac{A^{l} B^{p+1} r^{l\alpha+(p+1)(\alpha-\beta)+1}}{\Gamma(l\alpha+(p+1)(\alpha-\beta)+2)} \Big) u_0' \\
&= Au_0 + A^2 r^{\alpha} E_{\alpha,\alpha-\beta,\alpha+1}(Ar^{\alpha}, Br^{\alpha-\beta}) u_0 + ABr^{\alpha-\beta} E_{\alpha,\alpha-\beta,\alpha-\beta+1}(Ar^{\alpha}, Br^{\alpha-\beta}) u_0 \\
&+ Aru_0' + Ar^{\alpha+1} E_{\alpha,\alpha-\beta,\alpha+2}(Ar^{\alpha}, Br^{\alpha-\beta}) u_0' + ABr^{\alpha-\beta+1} E_{\alpha,\alpha-\beta,\alpha-\beta+2}(Ar^{\alpha}, Br^{\alpha-\beta}) u_0'.
\end{aligned}$$

Similarly, we obtain:

$$\begin{aligned}
B\left({}^C D_{0+}^{\beta} u\right)(r) &= AB\, {}^C D_{0+}^{\beta} \left(r^{\alpha} E_{\alpha,\alpha-\beta,\alpha+1}(Ar^{\alpha}, Br^{\alpha-\beta})\right) u_0 \\
&+ AB\, {}^C D_{0+}^{\beta} \left(r^{\alpha+1} E_{\alpha,\alpha-\beta,\alpha+2}(Ar^{\alpha}, Br^{\alpha-\beta})\right) u_0' \\
&= ABr^{\alpha-\beta} E_{\alpha,\alpha-\beta,\alpha-\beta+1}(Ar^{\alpha}, Br^{\alpha-\beta}) u_0 \\
&+ ABr^{\alpha-\beta+1} E_{\alpha,\alpha-\beta,\alpha-\beta+2}(Ar^{\alpha}, Br^{\alpha-\beta}) u_0'.
\end{aligned}$$

and

$$Au(r) = Au_0 + A^2 r^\alpha E_{\alpha,\alpha-\beta,\alpha+1}(Ar^\alpha, Br^{\alpha-\beta})u_0 + Aru_0'$$
$$+ Ar^{\alpha+1} E_{\alpha,\alpha-\beta,\alpha+2}(Ar^\alpha, Br^{\alpha-\beta})u_0'.$$

Taking a linear combination of above results, we attain the desired result. □

Secondly, we consider the Cauchy problem for the inhomogeneous linear generalized multidimensional Bagley-Torvik equation in the form of:

$$\begin{cases} \left({}^C D_{0+}^\alpha u\right)(r) - B\left({}^C D_{0+}^\beta u\right)(r) - Au(r) = g(r), \quad r \in (0,T], \\ u(0) = u_0, \quad u'(0) = u_0', \end{cases} \tag{25}$$

where $\alpha \in (1,2]$ and $\beta \in (1,2]$.

Theorem 4.2. *The inhomogeneous generalized multidimensional Bagley-Torvik equation* (16) *with zero initial conditions has the following solution:*

$$u(r) = \int_0^r (r-s)^{\alpha-1} E_{\alpha,\alpha-\beta,\alpha}(A(r-s)^\alpha, B(r-s)^{\alpha-\beta}) g(s) \mathrm{d}s, \quad r > 0.$$

Proof. In a similiar way, we can prove this theorem according to the proof of Theorem 3.2. □

Therefore, we find the explicit formula of solutions to linear inhomogeneous generalized multidimensional Bagley-Torvik equations by applying the superposition principle to find solution of (25).

Theorem 4.3. *A unique solution $u \in C^2([0,T], \mathbb{R}^n)$ of the inhomogeneous generalized multidimensional Bagley-Torvik equation* (25) *has the formula:*

$$u(r) = \Big(1 + r^\alpha A E_{\alpha,\alpha-\beta,\alpha+1}(Ar^\alpha, Br^{\alpha-\beta})\Big)u_0 + \Big(r + r^{\alpha+1} A E_{\alpha,\alpha-\beta,\alpha+2}(Ar^\alpha, Br^{\alpha-\beta})\Big)u_0'$$
$$+ \int_0^r (r-s)^{\alpha-1} E_{\alpha,\alpha-\beta,\alpha}(A(r-s)^\alpha, B(r-s)^{\alpha-\beta}) g(s) \mathrm{d}s, \quad r > 0. \tag{26}$$

Proof: The proofs of Theorem 4.3 is straightforward approaches by following the general case above, so we omit it here. □

4.1. **Special case: Bagley-Torvik equation of $\frac{3}{2}$-order derivative.** If we choose $\alpha = 2$, $\beta = \frac{3}{2}$ and $A = -\frac{2s\sqrt{\mu\rho}}{m}$, $B = -\frac{k}{m}$, we get classical scalar Bagley-Torvik equation of $\frac{3}{2}$-order derivative.

We consider the Cauchy problem for homogeneous linear scalar Bagley-Torvik equation in the form of:

$$\begin{cases} u''(r) + \frac{2s\sqrt{\mu\rho}}{m}({}^C D_{0+}^{\frac{3}{2}} u)(r) + \frac{k}{m}u(r) = 0, \quad r \in (0,T], \\ u(0) = u_0, \quad u'(0) = u_0', \end{cases} \tag{27}$$

has the following solution:

Theorem 4.4. *A unique solution $u \in C^2([0,T], \mathbb{R})$ of the homogeneous Bagley-Torvik equation* (27) *with the inhomogeneous initial conditions has the formula:*

$$u(r) = \Big(1 - \frac{k}{m} r^2 E_{2,\frac{1}{2},3}(-\frac{k}{m}r^2, -\frac{2s\sqrt{\mu\rho}}{m} r^{\frac{1}{2}})\Big)u_0$$
$$+ \Big(r - \frac{k}{m} r^3 E_{2,\frac{1}{2},4}(-\frac{k}{m}r^2, -\frac{2s\sqrt{\mu\rho}}{m} r^{\frac{1}{2}})\Big)u_0', r > 0. \tag{28}$$

The solution of the Cauchy problem for inhomogeneous linear sacalar Bagley-Torvik equation in the form of:

$$\begin{cases} u''(r) + \frac{2s\sqrt{\mu\rho}}{m}({}^C D^{\frac{3}{2}}_{0+}u)(r) + \frac{k}{m}u(r) = \frac{1}{m}g(r), \quad r \in (0, T], \\ u(0) = u_0, \quad u'(0) = u'_0, \end{cases} \tag{29}$$

has the the following integral representation:

Theorem 4.5. *A unique solution* $u \in C^2([0,T], \mathbb{R})$ *of the inhomogeneous Bagley-Torvik equation* (29) *with the inhomogeneous initial conditions has the formula:*

$$u(r) = \Big(1 - \frac{k}{m}r^2 E_{2,\frac{1}{2},3}(-\frac{k}{m}r^2, -\frac{2s\sqrt{\mu\rho}}{m}r^{\frac{1}{2}})\Big)u_0 + \Big(r - \frac{k}{m}r^3 E_{2,\frac{1}{2},4}(-\frac{k}{m}r^2, -\frac{2s\sqrt{\mu\rho}}{m}r^{\frac{1}{2}})\Big)u'_0$$
$$+ \frac{1}{m}\int_0^r (r-s)E_{2,\frac{1}{2},2}(-\frac{k}{m}(r-s)^2, -\frac{2s\sqrt{\mu\rho}}{m}(r-s)^{\frac{1}{2}})g(s)\mathrm{d}s, \quad r > 0. \tag{30}$$

The exact solution of the Cauchy problem for inhomogeneous linear Bagley-Torvik equation in the form of :

$$\begin{cases} u''(r) + \frac{2s\sqrt{\mu\rho}}{m}({}^C D^{\frac{1}{2}}_{0+}u)(r) + \frac{k}{m}u(r) = \frac{1}{m}g(r), \quad r \in (0, T], \\ u(0) = 0, \quad u'(0) = 0, \end{cases} \tag{31}$$

has the following solution:

Theorem 4.6. *A unique solution* $u \in C^2([0,T], \mathbb{R})$ *of the inhomogeneous Bagley-Torvik equation with the homogeneous initial conditions* (31) *has the formula:*

$$u(r) = \frac{1}{m}\int_0^r G(r-s)g(s)\mathrm{d}s, \quad r > 0, \tag{32}$$

where

$$G(r) = rE_{2,\frac{1}{2},2}(-\frac{k}{m}r^2, -\frac{2s\sqrt{\mu\rho}}{m}r^{\frac{1}{2}}).$$

Remark 4.1. *The Cauchy problem* (25) *has also a solution in terms of Fox-Wright matrix* [14, 45] *functions below:*

$$u(r) = \Bigg\{\sum_{l=0}^{\infty}\frac{A^l r^{l\alpha}}{l!} 1\Psi_1\left[\begin{array}{c}(l+1,1)\\(l\alpha+1,\alpha-\beta)\end{array}\Big| Br^{\alpha-\beta}\right]$$
$$- B\sum_{l=0}^{\infty}\frac{A^l r^{l\alpha+\alpha-\beta}}{l!} 1\Psi_1\left[\begin{array}{c}(l+1,1)\\(l\alpha+\alpha-\beta+1,\alpha-\beta)\end{array}\Big| Br^{\alpha-\beta}\right]\Bigg\}u_0$$
$$+ \Bigg\{\sum_{l=0}^{\infty}\frac{A^l r^{l\alpha+1}}{l!} 1\Psi_1\left[\begin{array}{c}(l+1,1)\\(l\alpha+2,\alpha-\beta)\end{array}\Big| Br^{\alpha-\beta}\right]$$
$$- B\sum_{l=0}^{\infty}\frac{A^l r^{l\alpha+\alpha-\beta+1}}{l!} 1\Psi_1\left[\begin{array}{c}(l+1,1)\\(l\alpha+\alpha-\beta+2,\alpha-\beta)\end{array}\Big| Br^{\alpha-\beta}\right]\Bigg\}u'_0$$
$$+ \int_0^r (r-s)^{\alpha-1}G_{\alpha,\beta;\lambda,\mu}(r-s)g(s)\mathrm{d}s, \quad r > 0,$$

where

$$G_{\alpha,\beta;\lambda,\mu}(r)\sum_{l=0}^{\infty}\frac{A^l r^{l\alpha}}{l!} 1\Psi_1\left[\begin{array}{c}(l+1,1)\\(l\alpha+\alpha,\alpha-\beta)\end{array}\Big| Br^{\alpha-\beta}\right].$$

Proof. Using the definition of Fox-Wright function [14, 45], we arrive at

$$
\begin{aligned}
u(r) = &\left\{\sum_{l=0}^{\infty}\frac{A^l r^{l\alpha}}{l!}1\Psi_1\left[\begin{array}{c}(l+1,1)\\(l\alpha+1,\alpha-\beta)\end{array}\Big|Br^{\alpha-\beta}\right]\right.\\
&\left.-B\sum_{l=0}^{\infty}\frac{A^l r^{l\alpha+\alpha-\beta}}{l!}1\Psi_1\left[\begin{array}{c}(l+1,1)\\(l\alpha+\alpha-\beta+1,\alpha-\beta)\end{array}\Big|Br^{\alpha-\beta}\right]\right\}u_0\\
&+\left\{\sum_{l=0}^{\infty}\frac{A^l r^{l\alpha+1}}{l!}1\Psi_1\left[\begin{array}{c}(l+1,1)\\(l\alpha+2,\alpha-\beta)\end{array}\Big|Br^{\alpha-\beta}\right]\right.\\
&\left.-B\sum_{l=0}^{\infty}\frac{A^l r^{l\alpha+\alpha-\beta+1}}{l!}1\Psi_1\left[\begin{array}{c}(l+1,1)\\(l\alpha+\alpha-\beta+2,\alpha-\beta)\end{array}\Big|Br^{\alpha-\beta}\right]\right\}u_0'\\
&+\int_0^r\sum_{l=0}^{\infty}\frac{A^l (r-s)^{l\alpha+\alpha-1}}{l!}1\Psi_1\left[\begin{array}{c}(l+1,1)\\(l\alpha+\alpha,\alpha-\beta)\end{array}\Big|B(r-s)^{\alpha-\beta}\right]g(s)\mathrm{d}s\\
&=\left\{\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{A^l B^p r^{l\alpha+p(\alpha-\beta)}}{\Gamma(l\alpha+p(\alpha-\beta)+1)}\right.\\
&\left.-\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{A^l B^{p+1} r^{l\alpha+(p+1)(\alpha-\beta)}}{\Gamma(l\alpha+(p+1)(\alpha-\beta)+1)}\right\}u_0\\
&+\left\{\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{A^l B^p r^{l\alpha+p(\alpha-\beta)+1}}{\Gamma(l\alpha+p(\alpha-\beta)+2)}\right.\\
&\left.-\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{A^l B^{p+1} r^{l\alpha+(p+1)(\alpha-\beta)+1}}{\Gamma(l\alpha+(p+1)(\alpha-\beta)+2)}\right\}u_0'\\
&+\int_0^r\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{A^l B^p r^{l\alpha+p(\alpha-\beta)+\alpha-1}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha)}g(s)\mathrm{d}s\\
&=\left\{1+\sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\binom{l+p-1}{p}\frac{A^l B^p r^{l\alpha+p(\alpha-\beta)}}{\Gamma(l\alpha+p(\alpha-\beta)+1)}\right.\\
&+\sum_{l=0}^{\infty}\sum_{p=1}^{\infty}\binom{l+p-1}{p-1}\frac{A^l B^p r^{l\alpha+p(\alpha-\beta)}}{\Gamma(l\alpha+p(\alpha-\beta)+1)}\\
&\left.-\sum_{l=0}^{\infty}\sum_{p=1}^{\infty}\binom{l+p-1}{p-1}\frac{A^l B^p r^{l\alpha+p(\alpha-\beta)}}{\Gamma(l\alpha+p(\alpha-\beta)+1)}\right\}u_0\\
&+\left\{r+\sum_{l=1}^{\infty}\sum_{p=0}^{\infty}\binom{l+p-1}{p}\frac{A^l B^p r^{l\alpha+p(\alpha-\beta)+1}}{\Gamma(l\alpha+p(\alpha-\beta)+2)}\right.\\
&+\sum_{l=0}^{\infty}\sum_{p=1}^{\infty}\binom{l+p-1}{p-1}\frac{A^l B^p r^{l\alpha+p(\alpha-\beta)+1}}{\Gamma(l\alpha+p(\alpha-\beta)+2)}\\
&\left.-\sum_{l=0}^{\infty}\sum_{p=1}^{\infty}\binom{l+p-1}{p-1}\frac{A^l B^p r^{l\alpha+p(\alpha-\beta)}}{\Gamma(l\alpha+p(\alpha-\beta)+1)}\right\}u_0'
\end{aligned}
$$

$$+\int_0^r \sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{A^l B^p r^{l\alpha+p(\alpha-\beta)+\alpha-1}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha)}g(s)\mathrm{d}s$$

$$=\left\{1+A\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{A^l B^p r^{l\alpha+p(\alpha-\beta)+\alpha}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha+1)}\right\}u_0$$

$$+\left\{r+A\sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{A^l B^p r^{l\alpha+p(\alpha-\beta)+\alpha+1}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha+2)}\right\}u_0'$$

$$+\int_0^r \sum_{l=0}^{\infty}\sum_{p=0}^{\infty}\binom{l+p}{p}\frac{A^l B^p r^{l\alpha+p(\alpha-\beta)+\alpha-1}}{\Gamma(l\alpha+p(\alpha-\beta)+\alpha)}g(s)\mathrm{d}s$$

$$=\Big(1+r^{\alpha}AE_{\alpha,\alpha-\beta,\alpha+1}(Ar^{\alpha},Br^{\alpha-\beta})\Big)u_0$$

$$+\Big(r+r^{\alpha+1}AE_{\alpha,\alpha-\beta,\alpha+2}(Ar^{\alpha},Br^{\alpha-\beta})\Big)u_0'$$

$$+\int_0^r (r-s)^{\alpha-1}E_{\alpha,\alpha-\beta,\alpha}(Ar^{\alpha},Br^{\alpha-\beta})g(s)\mathrm{d}s, \quad r>0.$$

Therefore, our solution in terms of M-L type matrix functions coincide with the solution by means of Fox-Wright matrix functions shown in [20]. □

5. Illustrative examples

To end this paper, we give various examples to illustrate the above theoretical results for scalar Bagley-Torvik equations.

Example 5.1. Let $\alpha = 2, \quad \beta = \frac{1}{2}$. Consider the following Cauchy problem for Liouville-Caputo type Bagley-Torvik equation of $\frac{1}{2}$-order derivative:

$$\begin{cases} u''(r)+8({}^C D_{0+}^{0.5}u)(r)+3u(r)=0.5sin(r), & r\in(0,1],\\ u(0)=2, \quad u'(0)=4, \end{cases} \tag{33}$$

where $m=2, s=4, \mu=1, \rho=4, k=6$ and $g(r)=sin(r)$.

Using by Formula 21, the explicit analytical representation of solution to the IVP (33), can be represented in the integral form:

$$u(r)=\Big(2-6r^2E_{2,\frac{3}{2},3}(-3r^2,-8r^{\frac{3}{2}})\Big)+4rE_{2,\frac{3}{2},2}(-3r^2,-8r^{\frac{3}{2}})$$
$$+\frac{1}{2}\int_0^r (r-s)E_{2,\frac{3}{2},2}(-3(r-s)^2,-8(r-s)^{\frac{3}{2}})sin(s)\mathrm{d}s, \quad r>0. \tag{34}$$

Now we are going to illustrate Example 1 for $u(r)$:

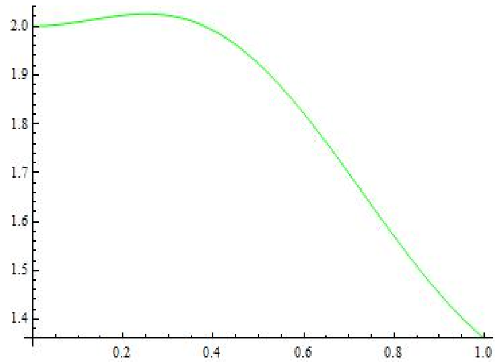

Figure 1. The graph of the solution $u(r)$ to the Bagley-Torvik equation of $\frac{1}{2}$-order derivative .

Example 5.2. Let $\alpha = 2, \quad \beta = \frac{3}{2}$. Consider the following initial value problem for Liouville-Caputo type Bagley-Torvik equation of $\frac{3}{2}$-order derivative:

$$\begin{cases} u''(r) + 20({}^C D^{1.5}_{0^+} u)(r) + 4u(r) = cos(r), \quad r \in (0, 1], \\ u(0) = 1, \quad u'(0) = 3, \end{cases} \tag{35}$$

where $m = 1, s = 5, \mu = 2, \rho = 2, k = 4$ and $g(r) = cos(r)$.

Using by Formula 30, the exact analytical representation of solution to the Cauchy problem (35), can be expressed in the following form:

$$\begin{aligned} u(r) = &\Big(1 - 4r^2 E_{2,\frac{1}{2},3}(-4r^2, -20r^{\frac{1}{2}})\Big) + \Big(3r - 12r^3 E_{2,\frac{1}{2},2}(-4r^2, -20r^{\frac{1}{2}}) \\ &+ \int_0^r (r-s) E_{2,\frac{1}{2},2}(-4(r-s)^2, -20(r-s)^{\frac{1}{2}}) cos(s) \mathrm{d}s, \quad r > 0. \end{aligned} \tag{36}$$

Now, we are going to illustrate Example 2 for $u(r)$:

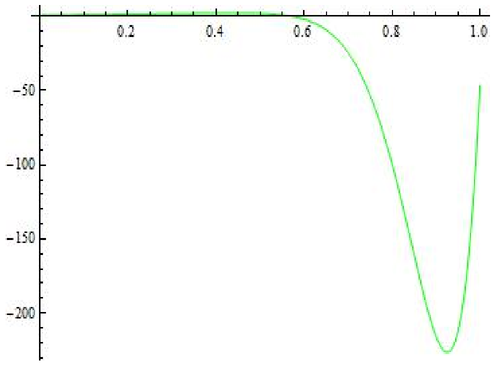

Figure 2. The graph of the solution $u(r)$ to the Bagley-Torvik equation of $\frac{3}{2}$-order derivative.

6. Conclusions and future work

The main contributions of our research work are as follows:

- we introduce the exact analytical representation of homogeneous and inhomogeneous linear generalized multidimensional Bagley-Torvik equations with two fractional orders satisfying $\alpha \in (1, 2]$, $\beta \in (0, 1]$ by means of a M-L type matrix function via double infinite series;

- we propose the exact analytical representation of homogeneous and inhomogeneous linear generalized Bagley-Torvik matrix equations with two fractional orders satisfying $\alpha \in (1,2]$, $\beta \in (1,2]$ in terms of a M-L type matrix function via double infinite series;
- we obtain the explicit analytical representation of solutions to homogeneous and inhomogeneous linear scalar Bagley-Torvik equations of $\frac{1}{2}$-order derivative and $\frac{3}{2}$-order derivative;
- we verify that our solutions with regard to M-L type matrix functions are identical with the results by means of generalized Wright matrix functions for multi-term FDEs;
- we propose a new representation of solutions to the Bagley-Torvik equations with matrix or constant coefficients.

There are a number of potential directions in which the results acquired here can be extended. Our future work will proceed to study the asymptotic stability of the trivial solution with the help of the Lyapunov methods [21] and relative controllability results of solutions with the aid of Gramian matrix and rank criterion to the Bagley-Torvik equations with a constant delay [22].

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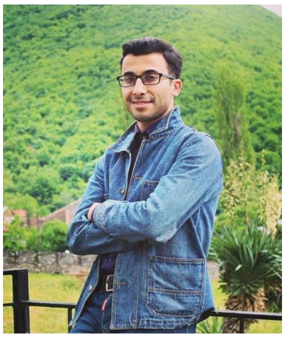

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