TWMS J. Pure Appl. Math., V.12, N.2, 2021, pp.278-288

THE CAUCHY PROBLEM FOR DEGENERATE PARABOLIC CONVOLUTION EQUATION

HUMMET K. MUSAEV[1,2]

Abstract. This paper explores the maximal regularity and separability properties of abstract differential operator equations in weighted spaces. Using Fourier multiplier theorems we will obtain the coercive properties of convolution differential-operator equations (CDOEs) with unbounded operator coefficients in weighted L_p spaces. Finally, these results are applied to establish well-posedness of the Cauchy problem for degenerate parabolic convolution differential-operator equations. In particular we study the $R-$positivity of the corresponding convolution-elliptic operators.

Keywords: convolution differential-operator equation, R -positivity, weighted spaces, operator-valued functions, degenerate convolution equations, weighted multiplier condition, $R-$ boundedness.

AMS Subject Classification: 34G10, 45J05.

1. Introduction

Some properties (like separability and maximal regularity for differential operator equations) have been recently introduced; for example, in [1-6, 9-13, 16, 21-25] and the references therein. Furthermore, convolution differential equations have been investigated, e.g., in [18-20]. Moreover, convolution differential-operator equations (CDOEs) in weighted spaces are studied in [15, 17].

In the years CDOEs are an under-researched topic. In [19, 23], the parabolic and elliptic type CDOEs were established in L_p- spaces. Regularity and separability properties of degenerate CDOEs have been studied, e.g., in [10, 14, 18]. This article aims to obtain the Cauchy problem for the following degenerate parabolic convolution differential operator equation

$$\frac{\partial u}{\partial t}+\sum_{i=0}^{l} a_i * \frac{\partial^{[i]} u}{\partial x^{[i]}}+A*u=f(t,x),\ u(0,x)=0 \tag{1}$$

in $E-$ valued weighted $L_{p,\gamma}$ spaces. Here E is a Banach space, $A=A(x)$ is a linear operator in E, $a_i=a_i(x)$ are complex-valued functions, λ is a complex parameter, $\gamma(x)$ is a measurable positive function in $\mathbb{R}=(-\infty;\infty)$, and $u^{[i]}=\left(\gamma(x)\frac{d}{dx}\right)^i u$.

Using the operator-valued Fourier multiplier theorems and the regularity properties of the corresponding equations, we obtained an element: the well-posedness of the problem (1). Subsequently, we obtained the sharp estimate in weighted mixed $L_{\mathbf{p},\gamma}, \mathbf{p}=(p,p_1)$ spaces.

[1] Department of Differential and Integral Equations, Baku State University, Baku, Azerbaijan
[2] Institute of Applied Mathematics, Baku State University, Baku, Azerbaijan
e-mail: gkm55@mail.ru
Manuscript received April 2021.

By virtue of the $R-$ positivity properties of the corresponding convolution operator and the semigroup theory we may drew the conclusion that the Cauchy problem (1) presents a unique solution that satisfies the coercive estimate.

2. Notation and definitions

The set of natural numbers is indicated by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$ and of complex numbers by $\mathbb{C}$. If we suppose that E_1 and E_2 are Banach spaces. The space of bounded linear operators from E_1 to E_2 is shown by $\mathcal{L}(E_1, E_2)$. For $E_1 = E_2 = E$,we write $\mathcal{L}(E)$ instead of $\mathcal{L}(E, E)$.

Let $\gamma = \gamma(x)$, $x = (x_1, x_2, ..., x_n)$ be a positive measurable real-valued function on a measurable subset $\Omega \subset \mathbb{R}^n$. By the symbol $L_{p,\gamma}(\Omega; E)$ we mean the space of all strongly E -valued functions on a $\Omega \subset \mathbb{R}^n$ with the norm

$$\|f\|_{L_{p,\gamma}} = \|f\|_{L_{p,\gamma}(\Omega;E)} = \left(\int_\Omega \|f(x)\|_E^p \gamma(x) dx \right)^{1/p}, \ 1 \le p < \infty.$$

For $\gamma(x) \equiv 1$, the space $L_{p,y}(\Omega, E)$ will be denoted by $L_p = L_p(\Omega; E)$.

$$\|f\|_{L_{\infty,\gamma}(\Omega;E)} = ess \sup_{x\in\Omega} \left[\gamma(x) \|f(x)\|_E\right].$$

The weight function $\gamma(x)$ is said to satisfy the A_p condition, i.e., $\gamma(x) \in A_p, 1 < p < \infty$ if there is such a positive constant C that

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \gamma(x) dx \right) \left(\frac{1}{|Q|} \int_Q \gamma^{-\frac{1}{p-1}}(x) dx \right)^{p-1} \le C$$

for all compact sets $Q \subset \mathbb{R}^n$. (*Example*: the weighted function $\gamma(x) = |x|^\nu$, $x \in \mathbb{R}$, $-1 < \nu < p-1$ belong to the A_p class.)

Suppose that

$$S_\varphi = \{\lambda; \ \lambda \in \mathbb{C}, \quad |\arg \lambda| \le \varphi\} \cup \{0\}, \quad 0 \le \varphi < \pi.$$

A closed linear operator function $A = A(x)$, $x \in \mathbb{R}$is believed to be uniformly $\varphi-$ positive in Banach space E provided that $D(A(x))$ is dense in E and is not dependent on x and the positive constant M fulfills the condition below

$$\left\|(A(x) + \lambda I)^{-1}\right\|_{\mathcal{L}(E)} \le M(1 + |\lambda|)^{-1}$$

for every $x \in \mathbb{R}$ and $\lambda \in S_\varphi$, $\varphi \in [0, \pi)$, where I is an identity operator in E. For a scalar λ we may also write $A + \lambda$ or A_λ instead of $A + \lambda I$.

$S = S(\mathbb{R}^n; E)$ indicates the Schwartz space of rapidly decreasing smooth E -valued functions on $\mathbb{R}^n$. In $E = \mathbb{C}$ this space is denoted by the symbol $S = S(\mathbb{R}^n; \mathbb{C})$. $S'(\mathbb{R}^n; E)$ represents the space of linear continuous mappings from S to E dubbed the Schwarts space of E-valued distributions. $S(\mathbb{R}^n; E)$ is a norm dense in $L_{p,\gamma}(\mathbb{R}^n; E)$ when $1 \le p < \infty$, $\gamma \in A_p$.

Let Ω be a domain in $\mathbb{R}^n$. $C(\Omega, E)$ and $C^{(m)}(\Omega; E)$ represent the spaces of E -valued bounded, uniformly strongly continuous and m -times continuously differentiable functions on Ω, respectively. For $E = \mathbb{C}$ the space $C^{(m)}(\Omega, E)$ will be shown by $C^{(m)}(\Omega)$.

An E-valued generalized function $D^{\alpha}f$ is called a generalized derivative in the sense of Schwartz distributions of the function $f \in S'(\mathbb{R}^n, E)$, if the equality

$$\langle D^{\alpha} f, \varphi \rangle = (-1)^{|\alpha|} \langle f, D^{\alpha}\varphi \rangle$$

holds for all $\varphi \in S$, where $\alpha = (\alpha_1,\alpha_2,...,\alpha_n)$, $|\alpha| = \sum\limits_{k=1}^{n}\alpha_k$, $D^{\alpha} = D_1^{\alpha_1}D_2^{\alpha_2},...,D_n^{\alpha_n}$, α_i are integers.

Suppose F represents the Fourier transform of the function f which will be denoted here by $\widehat{f}$, $Ff = \widehat{f}$ and $F^{-1}f = \check{f}$. It is known that

$$F(D_x^{\alpha}f) = (i\xi_1)^{\alpha_1}...(i\xi_n)^{\alpha_n}\widehat{f}, \quad D_{\xi}^{\alpha}(F(f)) = F\left[(-ix_1)^{\alpha_1}...(-ix_n)^{\alpha_n}f\right]$$

for all $f \in S^{'}(\mathbb{R}^n; E)$.

A function $\Psi \in L_{\infty}(\mathbb{R}^n; \mathcal{L}(E_1, E_2))$ is dubbed a multiplier from $L_{p,\gamma}(\mathbb{R}^n; E_1)$ to $L_{p,\gamma}(\mathbb{R}^n; E_2)$ for $p \in (1,\infty)$ if the map $u \to Bu = F^{-1}\Psi(\xi)Fu$, $u \in S(\mathbb{R}^n; E_1)$ is well defined and extends to a bounded linear operator

$$B : L_{p,\gamma}(\mathbb{R}^n; E_1) \to L_{p,\gamma}(\mathbb{R}^n; E_2).$$

The collection of all Fourier multipliers from $L_{p,\gamma}(\mathbb{R}^n; E_1)$ to $L_{p,\gamma}(\mathbb{R}^n; E_2)$ will be indicated by $M_{p,\gamma}^{p,\gamma}(E_1, E_2)$. For $E_1 = E_2 = E$ it is simply denoted by $M_{p,\gamma}^{p,\gamma}(E)$. Let $M(h)$ denote a set of some parameters.

Consider the family $B_h = \{\Psi_h;\ \Psi_h \in M_{p,\gamma}^{p,\gamma}(E_1, E_2),\ h \in M(h)\}$ of multipliers from the collection $M_{p,\gamma}^{p,\gamma}(E_1, E_2)$. The multipliers Ψ_h are said to be uniformly bounded (UBM) with respect to h if there is a positive constant M which acts independently from $h \in M(h)$ that

$$\left\|F^{-1}\Psi_h Fu\right\|_{L_{p,\gamma}(R^n;E_2)} \leq M \left\|u\right\|_{L_{p,\gamma}(R^n;E_1)}$$

for all $h \in M(h)$ and $u \in S(\mathbb{R}^n; E_1)$.

The Banach space E is designated as UMD -space ([8, 11]) if the Hilbert operator of a function $f \in S(\mathbb{R}; E)$ is defined by $Hf = \frac{1}{\pi}PV\left(\frac{1}{t}\right) * f$, that is to say,

$$(Hf)(t) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int\limits_{|\tau|>\varepsilon}\frac{f(t-\tau)}{\tau}d\tau$$

is bounded in $L_p(\mathbb{R}; E)$, for $p \in (1,\infty)$ (see e.g. [7, 12]). UMD spaces include e.g. L_p, l_p spaces, Hilbert spaces, Sobolev spaces and Lorentz spaces L_{pq}, $p, q \in (1,\infty)$.

A family of operators $T \subset \mathcal{L}(E_1, E_2)$ is called R-bounded (see [7, 11, 24]) if there is such a constant $C > 0$ that for all $T_1, T_2, ..., T_n \in T$ and $u_1, u_2, ..., u_n \in E_1$, $n \in \mathbb{N}$

$$\int\limits_0^1\left\|\sum_{j=1}^{n} r_j(y)T_j u_j\right\|_{E_2} dy \leq C\int\limits_0^1\left\|\sum_{j=1}^{n} r_j(y)u_j\right\|_{E_1} dy,$$

where $\{r_j\}$ is a sequence of independent symmetric $\{-1; 1\}$-valued random variables on $[0, 1]$. The smallest C in which the above-mentioned holds is named the R-bound of the collection T represented by $R(T)$.

The definition of $R-$boundedness has it that every $\mathbb{R}-$bounded family of operators is (uniformly) bounded (it is enough to take $n = 1$).

A Banach space E is believed to be a space satisfying a weighted multiplier condition with respect to $p \in (1, \infty)$ and weighted function γ if for any $\Psi \in L_\infty(\mathbb{R}, \mathcal{L}(E))$ the R-boundedness of the set

$$\left\{ |\xi|^k D^k \Psi(\xi) : \xi \in \mathbb{R} \setminus \{0\}, \quad k = 0, 1 \right\}$$

implies that Ψ is a Fourier multiplier in $L_{p,\gamma}(\mathbb{R}; E)$, i.e. $\Psi \in M_{p,\gamma}^{p,\gamma}(E)$ for any $p \in (1, \infty)$. If $E = \mathbb{C}$ and $\gamma \in A_p, p \in (1, \infty)$, then $\Psi \in M_{p,\gamma}^{p,\gamma}(\mathbb{C})$.

If E is UMD space and $\gamma(x) \equiv 1$, the space E satisfies the multiplier condition in view of [7], [12] and [24]. The UMD spaces satisfy the uniformly multiplier condition.

As is shown in [7] and [11], any Hilbert space satisfies the multiplier condition. That said, we may have Banach spaces that are not Hilbert spaces but may satisfy the multiplier condition.

A positive operator A is said to be R-positive in the Banach space E if there exists such $\varphi \in [0, \pi)$ that the set

$$\left\{ \xi \left(A + \xi I \right)^{-1}; \quad \xi \in S_\varphi \right\}$$

is R-bounded.

Any bounded set is a Hilbert space is supposed to be R-bounded, so the notion of R-boundedness in such space equals the boundedness of a family of operators. Moreover, in Hilbert spaces all positive operators are necessarily R-positive (see [7] and [11]).

If $A = A(x), x \in \mathbb{R}$ is a closed linear operator in E with domain definition $D(A)$ independent of x and $u \in L_p(\mathbb{R}; E(A))$. Then define

$$(A * u)(x) = \int\limits_{\mathbb{R}} A(x - y) u(y) dy.$$

We consider the $E-$valued weighted space

$$W_{p,\gamma}^l \left(\mathbb{R}^n; E_0, E \right) = \left\{ u; u \in L_{p,\gamma} \left(\mathbb{R}^n; E_0 \right), \ D_k^{l_k} u \in L_{p,\gamma} \left(\mathbb{R}^n; E \right) \right\},$$

where $l = (l_1, l_2, ..., l_n)$, l_k are positive integers, $D_k^{l_k} = \dfrac{\partial^{l_k}}{\partial x_k^{l_k}}$, $k = 1, 2, ..., n$, E_0 and E are Banach spaces, E_0 is continuously and densely embedded into E,

$$\|u\|_{W_{p,\gamma}^l(\mathbb{R}^n; E_0, E)} = \|u\|_{L_{p,\gamma}(\mathbb{R}^n; E_0)} + \sum_{k=1}^{n} \left\| D_k^{l_k} u \right\|_{L_{p,\gamma}(\mathbb{R}^n; E)} < \infty, \quad 1 \leq p < \infty.$$

For a positive measurable function $\gamma(x)$ on $\mathbb{R}$, let

$$D_k^{[i]} = \left(\gamma(x) \frac{\partial}{\partial x_k} \right)^i,$$

and

$$W_{p,\gamma}^{[l]}(\mathbb{R}^n; E_0, E) = \left\{ u; u \in L_p \left(\mathbb{R}^n; E_0 \right), \ D_k^{[l_k]} u \in L_p \left(\mathbb{R}^n; E \right), \right.$$

$$\left. \|u\|_{W_{p,\gamma}^{[l]}(\mathbb{R}^n; E_0, E)} = \|u\|_{L_p(\mathbb{R}^n; E_0)} + \sum_{k=1}^{n} \left\| D_k^{[l_k]} u \right\|_{L_p(\mathbb{R}^n; E)} < \infty \right\}.$$

3. The Cauchy problem for degenerate parabolic convolution differential-operator equations

In this section we shall consider the maximal regularity and separability properties of the degenerate parabolic CDOEs.

The degenerate parabolic CDOEs (1) are said to be uniformly separable in $L_p(\mathbb{R};E)$ if for $f \in L_p(\mathbb{R};E)$ equation (1) has a unique solution $u \in W^{1,[l]}_{\mathbf{p},\gamma}(\mathbb{R};E_0,E)$ and the following coercive estimate holds:

$$\left\|\frac{\partial u}{\partial t}\right\|_{L_p(\mathbb{R};E)} + \sum_{i=0}^{l}\left\|a_k * \frac{\partial^{[i]} u}{\partial x^{[i]}}\right\|_{L_p(\mathbb{R};E)} + \|A * u\|_{L_p(\mathbb{R};E)} \le C\|f\|_{L_p(\mathbb{R};E)},$$

where the constant C is independent of f.

This section aims to obtain separability property of the following degenerate parabolic CDOE

$$\frac{\partial u}{\partial t} + \sum_{i=0}^{l} a_i * \frac{\partial^{[i]} u}{\partial x^{[i]}} + A * u = f(t,x), \tag{2}$$

where $\dfrac{\partial^{[i]} u}{\partial x^{[i]}} = \left(\gamma(x)\dfrac{\partial}{\partial x}\right)^{i} u(x)$, $\gamma(x)$ is a positive measurable function in $\mathbb{R}$, $A = A(x)$ is a linear operator in Banach space E, $a_i = a_i(x)$ are complex-valued functions.

First, the Cauchy problem for the convolution parabolic equation is examined here

$$\begin{cases} \dfrac{\partial u}{\partial t} + \displaystyle\sum_{i=0}^{l} a_i * \dfrac{\partial^{i} u}{\partial x^{i}} + A * u = f(t,x) \\ u(0,x) = 0,\ t \in \mathbb{R}_+, x \in \mathbb{R}. \end{cases} \tag{3}$$

For this purpose we will indicate the space of all $\mathbf{p} = (p,p_1)-$summable $E-$valued functions with mixed norm through $L_{\mathbf{p},\gamma}\left(\mathbb{R}^2_+;E\right)$, where $\mathbb{R}^2_+ = \mathbb{R} \times \mathbb{R}_+$. Therefore, $L_{\mathbf{p},\gamma}\left(\mathbb{R}^2_+;E\right)$ denotes the space of all measurable $E-$valued functions defined on $\mathbb{R}^2_+$ with the norm

$$\|f\|_{L_{\mathbf{p},\gamma}\left(\mathbb{R}^2_+;E\right)} = \left(\int\limits_{\mathbb{R}}\left(\int\limits_{\mathbb{R}_+}\|f(t,x)\|^p_E\,\gamma(x)dx\right)^{\frac{p_1}{p}} dt\right)^{\frac{1}{p_1}} < \infty.$$

Respectively, we define $W^{1,l}_{\mathbf{p},\gamma}\left(\mathbb{R}^2_+;E_0,E\right)$. Let E_0 and E be two Banach spaces, where E_0 is continuously and densely embedded into E_0. Consequently $W^{1,l}_{\mathbf{p},\gamma}\left(\mathbb{R}^2_+;E_0,E\right)$ denotes the space of all functions $u \in L_{\mathbf{p},\gamma}\left(\mathbb{R}^2_+;E\right)$ with the norm

$$\|u\|_{W^{1,l}_{\mathbf{p},\gamma}\left(\mathbb{R}^2_+;E_0,E\right)} = \|u\|_{L_{\mathbf{p},\gamma}\left(\mathbb{R}^2_+;E\right)} + \|D_t u\|_{L_{\mathbf{p},\gamma}\left(\mathbb{R}^2_+;E\right)} + \sum_{i=1}^{n}\left\|D^i_x u\right\|_{L_{\mathbf{p},\gamma}\left(\mathbb{R}^2_+;E\right)}.$$

First we show that the operator L is defined by $D(L) = W^{l}_{p,\widetilde{\gamma}}(\mathbb{R};E(A),E)$, $Lu = \sum\limits_{i=0}^{l} a_i * \dfrac{\partial^i u}{\partial x^i} + A * u$, is $R-$positive in $L_{p,\gamma}(\mathbb{R};E)$.

Condition 3.1. Suppose the followings are satisfied:

1) $L(\xi) = \sum\limits_{i=0}^{l} \widehat{a}_i(\xi)(i\xi)^i \in S_{\varphi_1},\ \ \varphi_1 + \varphi < \pi, \varphi \in [0,\pi),$

$|L(\xi)| \ge C|\xi|^l \sum\limits_{i=0}^{l} |\widehat{a}_i(\xi)|,\ \xi \in \mathbb{R}\setminus\{0\};$

2) $\widehat{A}(\xi)$ is uniformly $R-$positive in E;
3) $a_i \in C^{(1)}(\mathbb{R})$, $\widehat{A}\prime(\xi)A^{-1}(\xi_0) \in C\left(\mathbb{R}; \mathcal{L}(E)\right)$, and
$|\xi| \left\|\widehat{A}\prime(\xi)A^{-1}(\xi_0)\right\| < C_1$, $|\xi| \left|\widehat{a}_i^{\prime}(\xi)\right| < C_2, \xi_0 \in \mathbb{R}\backslash \{0\}$;
4) $\gamma \in A_p$, $p \in (1, \infty)$.

By using Condition 3.1 and the assumption on $\hat{A}(\xi)$, we get the following:

Lemma 3.1. Suppose Condition 3.1.holds for $0 \leq \varphi < \pi$, E is a Banach space and satisfies the uniform weighted multiplier condition. Then the operator $A_L(\xi) = \left[\hat{A}(\xi) + L(\xi)\right]$ is uniformly $R-$positive in $L_{p,\gamma}(\mathbb{R}; E)$.

Theorem 3.1. Suppose Condition 3.1 holds for $0 \leq \varphi < \pi$, E is a Banach space and satisfies the uniform weighted multiplier condition. Then for $f \in L_{p,\gamma}(\mathbb{R}; E)$ and $\lambda \in S_\varphi$ the equation $(L + \lambda) u = f$ has a unique solution $u \in W_{p,\gamma}^l(\mathbb{R}; E(A), E)$ and the following coercive uniform estimate holds

$$\sum_{i=0}^{l} |\lambda|^{1-\frac{i}{l}} \left\|a_i * \frac{d^i u}{dx^i}\right\|_{L_{p,\gamma}(\mathbb{R};E)} + \|A * u\|_{L_{p,\gamma}(\mathbb{R};E)} \leq C \|f\|_{L_{p,\gamma}(\mathbb{R};E)}. \tag{4}$$

Proof. If the Fourier transform is applied to the equation

$$(L + \lambda) u = f, \tag{5}$$

we get $\hat{u}(\xi) = [A_L(\xi) + \lambda]^{-1} \hat{f}(\xi)$, where $\hat{a}_i(\xi)$, $\hat{A}(\xi)$, $\hat{u}(\xi)$ and $\hat{f}(\xi)$ show the Fourier transforms of $a_i(x)$, $A(x)$, $u(x)$ and $f(x)$, respectively.

It is easy to see that $\left(\hat{A}(\xi) + L(\xi) + \lambda\right)^{-1} \in \mathcal{L}(E)$, and

$$u(x) = F^{-1}\left[\hat{A}(\xi) + L(\xi) + \lambda\right]^{-1} \hat{f}.$$

In a similar manner as in [15] and in view of Lemma 3.1, one can easly show that the operator-valued functions $\hat{a}_i(\xi)(i\xi)^k [A_L(\xi) + \lambda]^{-1}$, $\hat{A}(\xi)[A_L(\xi) + \lambda]^{-1}$ produced in the solution of equation (5) are Fourier multipliers from $L_{p,\gamma}(\mathbb{R}; E)$ to $L_{p,\gamma}(\mathbb{R}; E)$. By using a similar technique as in [14] and [15], we may conclude that for $f \in L_{p,\gamma}(\mathbb{R}; E)$ and $\lambda \in S_\varphi$ equation (5) has a unique solution $u \in W_{p,\gamma}^l(\mathbb{R}; E(A), E)$ and the following coercive uniform estimate holds

$$\sum_{i=0}^{l} |\lambda|^{1-\frac{i}{l}} \left\|a_i * \frac{d^i u}{dx^i}\right\|_{L_{p,\gamma}(\mathbb{R};E)} + \|A * u\|_{L_{p,\gamma}(\mathbb{R};E)} \leq C \|f\|_{L_{p,\gamma}(\mathbb{R};E)}. \tag{6}$$

It implies that for all $\lambda \in S_\varphi$ the resolvent of the operator L exists and the following sharp estimate holds

$$\sum_{i=0}^{l} |\lambda|^{1-\frac{i}{l}} \left\|a_i * \frac{d^i}{dx^i}(L + \lambda)^{-1}\right\|_{\mathcal{L}(L_{p,\gamma}(\mathbb{R};E))} + \left\|A * (L + \lambda)^{-1}\right\|_{\mathcal{L}(L_{p,\gamma}(\mathbb{R};E))} +$$
$$\left\|\lambda (L + \lambda)^{-1}\right\|_{\mathcal{L}(L_{p,\gamma}(\mathbb{R};E))} \leq C. \tag{7}$$

The estimate (7) particularly, implies that the operator L is positive in $L_{p,\gamma}(\mathbb{R}; E)$. To prove the $R-$positivity, we need to prove the $R-$boundedness of the set $\left\{\lambda (L + \lambda)^{-1}; \lambda \in S_\varphi\right\}$. From the representation of the solution of the equation $(L + \lambda) u = f$, it is clear that

$$\lambda (L + \lambda)^{-1} = F^{-1}\left[\lambda \left(\widehat{A}(\xi) + \lambda + L(\xi)\right)^{-1}\right] \widehat{f}.$$

Like in [14, Lemma 2.1] one can easily show that $\lambda\left(\widehat{A}(\xi)+\lambda+L(\xi)\right)^{-1}$ is a uniformly bounded multiplier (UBM) in $L_{p,\gamma}(\mathbb{R};E)$. Then, by definition of $R-$boundedness we obtain

$$\int\limits_0^1 \left\|\sum_{i=1}^k r_i(y)F^{-1}\left[\lambda_i\left(\widehat{A}(\xi)+\lambda_i+L(\xi)\right)^{-1}\right]\widehat{f}_i\right\|_{L_{p,\gamma}(\mathbb{R};E)} dy =$$

$$\int\limits_0^1 \left\|F^{-1}\sum_{i=1}^k r_i(y)\lambda_i\left(\widehat{A}(\xi)+\lambda_i+L(\xi)\right)^{-1}\widehat{f}_i\right\|_{L_{p,\gamma}(\mathbb{R};E)} dy \leq$$

$$\leq C\int\limits_0^1 \left\|\sum_{i=1}^k r_i(y)f_i\right\|_{L_{p,\gamma}(\mathbb{R};E)} dy$$

for all $\xi \in \mathbb{R}$, $\lambda_i \in S_\varphi$, $f_i \in L_{p,\gamma}(\mathbb{R};E)$, $i=\overline{1,k}$, where $\{r_i\}$ is a sequence of independent symmetric $\{-1;1\}-$valued random variables on [0,1]. So, from this we get that the set $\left\{\lambda(L+\lambda)^{-1};\ \lambda\in S_\varphi\right\}$ is $R-$bounded. We presume that the operator L is a generator of the analytic semigroup in $L_{p,\gamma}(\mathbb{R};E)$, for $\varphi\in\left(\frac{\pi}{2},\pi\right)$.

Now we prove one of the main propositions of this section.

Theorem 3.2. Assume the conditions of Theorem 3.1 hold for $\varphi\in\left(\frac{\pi}{2},\pi\right)$. Then for all $f\in L_{\mathbf{p},\gamma}\left(\mathbb{R}_+^2;E\right)$ the problem (3) has a unique solution $u\in W_{\mathbf{p},\gamma}^l\left(\mathbb{R}_+^2;E(A),E\right)$ satisfying the estimate

$$\left\|\frac{\partial u}{\partial t}\right\|_{L_{\mathbf{p},\gamma}(\mathbb{R}_+^2;E)}+\sum_{i=0}^l\left\|a_i*\frac{\partial^i u}{\partial x^i}\right\|_{L_{\mathbf{p},\gamma}(\mathbb{R}_+^2;E)}+\|A*u\|_{L_{\mathbf{p},\gamma}(\mathbb{R}_+^2;E)}\leq C\|f\|_{L_{\mathbf{p},\gamma}(\mathbb{R}_+^2;E)}. \tag{8}$$

Proof. By Fubini‘s theorem we have

$$\|f\|_{L_{\mathbf{p},\gamma}(\mathbb{R}_+^2;E)}=\left(\int\limits_{\mathbb{R}}\left(\int\limits_{\mathbb{R}_+}\|f(t,x)\|_E^p\gamma(x)dx\right)^{\frac{p_1}{p}}dt\right)^{\frac{1}{p_1}}=$$

$$\left(\int\limits_{\mathbb{R}_+}\|f(x,t)\|_{L_{p,\gamma}(\mathbb{R};E)}^{p_1}dt\right)^{\frac{1}{p_1}}=\|f\|_{L_{p_1}(\mathbb{R}_+;L_{p,\gamma}(\mathbb{R};E))}.$$

In a similar way we have

$$\|u\|_{W_{p,\gamma}^{1,l}\left(\mathbb{R}_+^2;E(A),E\right)}=\|u\|_{W_{p_1}^1(\mathbb{R}_+;D(L),L_{p,\gamma}(\mathbb{R};E))}.$$

Moreover, by definition of spaces $W_{p,\gamma}^l(\mathbb{R};E(A),E)$, $Z_0=W_{\mathbf{p},\gamma}^{1,l}\left(\mathbb{R}_+^2;E(A),E\right)$ for $E_0=E(A)$ and $Z=L_{p_1}(\mathbb{R}_+;L_{p,\gamma}(\mathbb{R};E))$ we obtain

$$\|u\|_{Z_0}=\|u\|_{Z(A)}+\left\|\frac{du}{dt}\right\|_Z+\|Lu\|_Z\simeq$$

$$\left\|\frac{\partial u}{\partial t}\right\|_{L_{\mathbf{p},\gamma}\left(\mathbb{R}_+^2:E\right)}+\|Lu\|_{L_{\mathbf{p},\gamma}\left(\mathbb{R}_+^2;E\right)}\simeq$$

$$\|Au\|_{L_{\mathbf{p},\gamma}\left(\mathbb{R}_+^2;E\right)} + \left\|\frac{\partial u}{\partial t}\right\|_{L_{\mathbf{p},\gamma}\left(\mathbb{R}_+^2;E\right)} + \sum_{k=1}^{n} \left\|D_k^l u\right\|_{L_{\mathbf{p},\gamma}\left(\mathbb{R}_+^2;E\right)} \simeq \|u\|_{Z_0},$$

where

$$Z(A) = L_{p_1}(\mathbb{R}_+; X(A)), \ X(A) = L_{p,\gamma}(\mathbb{R}; E(A)).$$

From this we get that the problem (3) can be expressed as

$$\frac{du}{dt} + Lu(t) = f(t), \ u(0) = 0, \ t \in \mathbb{R}_+. \tag{9}$$

Given of [3, Theorem 4.5.2] and [11], X is a Banach space satisfying the multiplier condition with respect to $p \in (1,\infty)$. Then in view of R-positivity of operator L with $\varphi \in \left(\frac{\pi}{2};\pi\right)$, by virtue of [24, Theorem 4.2] we obtain that for $f \in L_{p_1}(\mathbb{R}_+; L_{p,\gamma}(\mathbb{R};E))$ (see [7], [19]) the equation (9) has a unique solution $u \in W_{p_1}^1(\mathbb{R}_+; D(L), L_{p,\gamma}(\mathbb{R};E))$ satisfying

$$\left\|\frac{du}{dt}\right\|_{L_{p_1}(\mathbb{R}_+;L_{p,\gamma}(\mathbb{R};E))} + \|Lu\|_{L_{p_1}(\mathbb{R}_+;L_{p,\gamma}(\mathbb{R};E))} \leq C\|f\|_{L_{p_1}(\mathbb{R}_+;L_{p,\gamma}(\mathbb{R};E))}. \tag{10}$$

In view of estimate (6) and from the estimate (10) we get (8).

Now, we consider the Cauchy problem for equation (2), i.e.

$$\begin{cases} \dfrac{\partial u}{\partial t} + \sum\limits_{i=0}^{l} a_i * \dfrac{\partial^{[i]} u}{\partial x^{[i]}} + A * u = f(t,x), \\ u(0,x) = 0, \ t \in \mathbb{R}_+, x \in \mathbb{R}. \end{cases} \tag{11}$$

We use the following substitution

$$y = \int_0^x \gamma^{-1}(z)dz. \tag{12}$$

It is clear to see that, under the substitution (*) the weighted derivatives $\dfrac{\partial^{[i]} u}{\partial x^{[i]}}$ transform to the function $\dfrac{\partial^i u}{\partial x^i}$. Furthermore the spaces $L_p(\mathbb{R};E)$ and $W_{p,\gamma}^{[l]}(\mathbb{R};E(A),E)$ are mapped isomorphically onto the weighted spaces $L_{p,\widetilde{\gamma}}(\mathbb{R};E)$ and $W_{p,\widetilde{\gamma}}^{l}(\mathbb{R};E(A),E)$ respectively, where $\widetilde{\gamma}(y) = \gamma(x(y))$. Therefore, under substitution (12) the degenerate problem is transformed into the nondegenerate problem which is regarded in the weighted space $L_{p,\widetilde{\gamma}}(\mathbb{R};E)$. Similarly, under the substitution (12) the degenerate Cauchy problem (11) examined in $L_p(\mathbb{R};E)$ is transformed into nondegenerate Cauchy problem (3) considered in the weighted space $L_{p,\widetilde{\gamma}}(\mathbb{R};E)$.

Example 3.1. Let we put $\gamma(z) = z^\nu$, $\nu < 1$. Then by using the substitution

$$y = \int_0^x \gamma^{-1}(z)dz,$$

we get

$$y = \int_0^x z^{-\nu}dz = \frac{x^{-\nu+1}}{-\nu+1}, \ x^{1-\nu} = (1-\nu)\,y, \ x = [(1-\nu)\,y]^{\frac{1}{1-\nu}}.$$

Hence,

$$\gamma(x) = x^\nu = [(1-\nu)\,y]^{\frac{\nu}{1-\nu}} = \tilde{\gamma}(y).$$

We note that the nondegenrate Cauchy problem for parabolic CDOE is considered in [19].

Theorem 3.3. Suppose that Condition 3.1 holds for $\varphi \in \left(\frac{\pi}{2}, \pi\right)$, E is a Banach space and satisfies the weighted multiplier condition. Then for all $f \in L_{\mathbf{p}}(\mathbb{R}_+^2; E)$ problem (11) has a unique solution $u(t, x)$ and the following coercive estimate holds:

$$\left\| \frac{\partial u}{\partial t} \right\|_{L_{\mathbf{p}}(\mathbb{R}_+^2;E)} + \sum_{i=0}^{l} \left\| a_i * \frac{\partial^{[i]} u}{\partial x^{[i]}} \right\|_{L_{\mathbf{p}}(\mathbb{R}_+^2;E)} + \|A * u\|_{L_{\mathbf{p}}(\mathbb{R}_+^2;E)} \leq C \|f\|_{L_{\mathbf{p}}(\mathbb{R}_+^2;E)} .$$

Proof. Let H be the operator in $L_{p,\gamma}(\mathbb{R}; E)$ generated by the problem

$$\sum_{i=0}^{l} a_i * \frac{d^{[i]} u}{dx^{[i]}} + A * u = f, \tag{13}$$

i.e., $D(H) = W_{p,\widetilde{\gamma}}^{l}(\mathbb{R}; E(A), E)\,, Hu = \sum_{i=0}^{l} a_i * \frac{d^{[i]} u}{dx^{[i]}} + A * u.$

Under the conditions of Theorem 3.1 and substitution (12) the equation (13) has a unique solution $u(x)$, belongs to the space $W_{p,\gamma}^{[l]}(\mathbb{R}; E(A), E)$ and the coercive uniform estimate

$$\sum_{i=0}^{l} |\lambda|^{1-\frac{i}{l}} \left\| a_i * \frac{d^{[i]} u}{dx^{[i]}} \right\|_{L_p(\mathbb{R};E)} + \|A * u\|_{L_p(\mathbb{R};E)} \leq C \|f\|_{L_p(\mathbb{R};E)} \tag{14}$$

holds. It is clear to see that, under substitution (12) the degenerate Cauchy problem (11) considered in $L_p(\mathbb{R}; E)$ is transformed into the following nondegenerate Cauchy problem considered in the weighted space $L_{p,\widetilde{\gamma}}(\mathbb{R}; E)$,

$$\begin{cases} \dfrac{\partial \widetilde{u}}{\partial t} + \sum\limits_{i=0}^{l} \tilde{a}_i * \dfrac{\partial^i \tilde{u}}{\partial y^i} + \tilde{A} * \tilde{u} = \tilde{f}(t, y) \\ \tilde{u}(0, y) = 0,\ t \in \mathbb{R}_+, y \in \mathbb{R}. \end{cases}$$

In view of Theorem 3.2 from the estimate (14) we get the assertion.

Finally, from estimate (14) and Theorem 3.1, after some transformations we have the following result.

Result 3.1. For all $\lambda \in S_\varphi$ there exists the resolvent of the operator H and has the estimate

$$\sum_{i=0}^{l} |\lambda|^{1-\frac{i}{l}} \left\| a_i * \frac{d^{[i]}}{dx^{[i]}} (H + \lambda)^{-1} \right\|_{\mathcal{L}(L_p(\mathbb{R};E))} + \left\| A * (H + \lambda)^{-1} \right\|_{\mathcal{L}(L_p(\mathbb{R};E))} +$$

$$\left\| \lambda (H + \lambda)^{-1} \right\|_{\mathcal{L}(L_p(\mathbb{R};E))} \leq C.$$

Result 3.2. So, from Theorems 3.1-3.3 we obtained the similar results for power weighted $\gamma(x) = x^\nu$.

4. Conclusion

The Cauchy problem (1) for the degenerate parabolic convolution differential operator equation presents a unique solution that satisfies the coercive estimate. Moreover, it is shown that for all $\lambda \in S_\varphi$ there exists the resolvent of the operator H with a corresponding estimate.

5. Acknowledgements

The author thanks the referee for carefully reading the paper and some detailed comments and suggestions.

References

[1] Agarwal, R. P., Bohner, R., Shakhmurov, V.B., (2005), Maximal regular boundary value problems in Banach-valued weighted spaces, Bound. Value Probl., 1, pp.9-42.

[2] Agarwal, R.P., Ø'regan D., Shakhmurov, V.B., (2008), Separable anisotropic differential operators in weighted abstract spaces and applications, J. Math. Anal. Appl., 338, pp.970-983.

[3] Amann, H., (1995), Linear and Quasilinear Problems, 1, Birkhauser, Basel, 335p.

[4] Ashyralyev, A., Akturk, S., (2015), Positivity of a one-dimensional difference operator in the half-line and its applications, Appl. Comput. Math., 14(2), pp.204-220.

[5] Besov, O.V., Ilin, V.P., (1975), NikoIskii S.M., Integral Representations Of Functions and Embedding Theorems, Nauka, Moscow (in Rusiian).

[6] Bourgain, J., (1983), Some remarks on Banach spaces in which martingale differences are unconditional, Arkiv Math., 21, pp.163-168.

[7] Denk, R., Hieber, M., Prüss, J., (2003), R -Boundedness, Fourier Multipliers And Problems of Elliptic and Parabolic Type, Mem. Amer. Math. Soc., 166, p.788.

[8] Dore, G., Yakubov, S., (2000), Semigroup estimates and non coercive boundary value problems, Semi-group Form, 60, pp.93-121.

[9] Fattorini, H.O., 1983, The Cauchy Problems, Addison-Wesley, Reading, Mass, 636p.

[10] Favini, A., Goldstein, G.R., Goldstein, J.A., Romanelli, S., (2002), Degenerate second order differential operators generating analytic semigroups in L_p and $W^{1,p}$, Math. Nachr., 238, pp.78-102.

[11] Girardi, M., Weis, L., (2003), Operator-valued multiplier theorems on $L_p(X)$ and geometry of Banach spaces, J.Funct. Analysis, 204(2), pp.320-354.

[12] Haller, R., Heck, H., Noll, A., (2002), Mikhlin's theorem for operator-valued Fourier multipliers in n variables, Math. Nachr., 244, pp.110-130.

[13] Mirzoev, S.S., (2017), On a double-point boundary value problem for a second order operator-differential equation and its application, Appl. Comput. Math., 16(3), pp.313-322.

[14] Musaev H.K., Shakhmurov V.B., (2016), Regularity properties of degenerate convolution-elliptic equations, Bound. Value Probl., 50, pp.1-19.

[15] Musaev, H.K., Shakhmurov, V.B., (2017), $B-$coercive convolution equations in weighted function spaces and applications., J.Ukr. Math., 69(10), pp.1385-1405.

[16] Prüss, J., (1993), Evolutionary Integral Equations and Applications, Birkhauser, Basel, 366p.

[17] Shakhmurov, V. B., Musaev, H.K., (2017), Maximal regular convolution differential equations in weighted Besov spaces, Appl. Comput. Math., 16(2), pp.190-200.

[18] Shakhmurov, V.B., (2004), Coercive boundary value problems for regular degenerate differential-operator equations, J. Math. Anal. Appl., 292(2), pp.605-620.

[19] Shakhmurov, V.B., Shahmurov R., (2010), Sectorial operators with convolution term, Math. Inequal. Appl., 13(2), pp.387-404.

[20] Shakhmurov, V.B., Shahmurov R., (2009), Maximal B-Reqular Integro-differential equaions, Chinese Annals of Mathematics, 30(1), pp.39-50.

[21] Shakhmurov, V.B., (2008), Embedding and separable differential operators in Sobolev-Lions type spaces, Mathematical Notes, 84(6), pp.906-926.

[22] Triebel, H., (1978), Interpolation Theory. Function Spaces. Differential Operators, North-Holland, Amsterdam,

[23] Triebel, H., (1977), Spaces of distributions with weights. Multiplier in L_p spaces with weights, Math. Nachr., 78 pp.339-355.

[24] Weis, L., (2001), Operator-valued Fourier multiplier theorems and maximal L_p regularity, Math. Ann., 319, pp.735-758.

[25] Yakubov, S., Yakubov, Ya., (2000), Differential-Operator Equations. Ordinary and Partial Differential Equations, Chapmen and Hall /CRC, Boca Raton, 568p.

Musaev Hummet - was born in1960 in Western Azerbaijan. He graduated from Mechanics-Mathematics Faculty of Baku State University in 1982. He received Ph.D. degree in 1996. From 2002 he is an associate Professor in Differential and Integral Equations Department, Mechanics-Mathematics Faculty of Baku State University. His researched interests are boundary-value problems for the operator partial differential equations, theorems of investments in the abstract spaces.