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ON GENERALIZED CLASS OF p-VALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. In this paper we introduce and study new class $F_{p,\theta}(\gamma,\beta)$ of p-valent functions with negative coefficients. We obtain coefficients inequalities, distortion theorems, extreme points and radii of close to convexity, starlikeness and convexity for the class $F_{p,\theta}(\gamma,\beta)$. Also modified Hadamard products of several functions belonging to the class $F_{p,\theta}(\gamma,\beta)$ are study here. Finally, we investigate several distortion inequalities involving fractional calculus.

Keywords: Analytic, $p-$valent functions, Hadamard product, fractional calculus operators.

AMS Subject Classification: 30C45.

1. Introduction

Let $T_p(\theta)$ denote the class of functions of the form:

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \; (e^{i\theta} a_{p+k} \geq 0;\ |\theta| < \frac{\pi}{2};\ p \in \mathbb{N} = \{1, 2, ...\}), \tag{1}$$

which are analytic and $p-$valent in the open unit disc $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also, let $F_{p,\theta}(\gamma,\beta)$ denote the class of functions $f(z) \in T_p(\theta)$ which satisfy

$$Re\left\{e^{i\theta}\left((1-\gamma)\frac{f(z)}{z^p} + \gamma\frac{f'(z)}{pz^{p-1}}\right)\right\} > \frac{\beta}{p}, \tag{2}$$

where $0 \leq \dfrac{\beta}{p} < \cos\theta,\ |\theta| < \dfrac{\pi}{2},\ \gamma \geq 0,\ p \in \mathbb{N}$ and $z \in U$.

We note that for suitable choices of γ, θ and p, we obtain the following subclasses:
(1) $F_{p,0}(\gamma,\beta) = F_p(\gamma,\beta)$ $(0 \leq \beta < p,\ \gamma \geq 0,\ p \in \mathbb{N})$ (see Lee et al. [4] and Aouf and Darwish [2]);
(2) $F_{1,0}(\gamma,\beta) = F(\gamma,\beta)$ $(0 \leq \beta < 1,\ \gamma \geq 0)$ (see Bhoosnurmath and Swamy [3]);
(3) $F_{1,\theta}(1,\beta) = A(\theta,\beta)$ $(0 \leq \beta < \cos\theta,\ |\theta| < \frac{\pi}{2})$ (see Sekine [7]).

2. Coefficient estimates

Unless otherwise mentioned, we assume throughout this paper that

$$e^{i\theta} a_{p+k} \geq 0,\ 0 \leq \frac{\beta}{p} < \cos\theta,\ |\theta| < \frac{\pi}{2},\ \gamma \geq 0 \text{ and } p, k \in \mathbb{N}.$$

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Theorem 2.1. *Let the function $f(z)$ be given by (1.1). Then $f(z) \in F_{p,\theta}(\gamma,\beta)$ if and only if*

$$\sum_{k=1}^{\infty} e^{i\theta}\left(1+\frac{\gamma k}{p}\right) a_{p+k} \leq \cos\theta - \frac{\beta}{p}. \tag{3}$$

Assume that the condition (3) holds true, then it is sufficient to show that the value for

$$e^{i\theta}\left((1-\gamma)\frac{f(z)}{z^p}+\gamma\frac{f'(z)}{pz^{p-1}}\right),$$

lie in a circle centered at a point $e^{i\theta}$ whose radius is $\cos\theta - \frac{\beta}{p}$. Indeed, we have

$$\left|e^{i\theta}\left((1-\gamma)\frac{f(z)}{z^p}+\gamma\frac{f'(z)}{pz^{p-1}}\right)-e^{i\theta}\right| =$$

$$\begin{aligned} &= \left|e^{i\theta}\sum_{k=1}^{\infty}\left(1+\frac{\gamma k}{p}\right)a_{p+k}z^k\right| \leq \\ &\leq \sum_{k=1}^{\infty} e^{i\theta}\left(1+\frac{\gamma k}{p}\right)a_{p+k} \leq \\ &\leq \cos\theta-\frac{\beta}{p}. \end{aligned}$$

Conversely, assume that

$$Re\left\{e^{i\theta}\left((1-\gamma)\frac{f(z)}{z^p}+\gamma\frac{f'(z)}{pz^{p-1}}\right)\right\} > \frac{\beta}{p},$$

which is equivalent to

$$Re\left\{\sum_{k=1}^{\infty} e^{i\theta}\left(1+\frac{\gamma k}{p}\right)a_{p+k}z^k\right\} < \cos\theta-\frac{\beta}{p}.$$

Choose values of z on the real axis so that

$$\sum_{k=1}^{\infty} e^{i\theta}\left(1+\frac{\gamma k}{p}\right)a_{p+k}z^k,$$

is real. Letting $z \to 1^-$ along the real axis, we have

$$\sum_{k=1}^{\infty} e^{i\theta}\left(1+\frac{\gamma k}{p}\right)a_{p+k} \leq \cos\theta-\frac{\beta}{p},$$

and hence the proof of Theorem 2.1 is completed.

Corollary 2.1. *Let the function $f(z)$ defined by (1.1) be in the class $F_{p,\theta}(\gamma,\beta)$. Then*

$$|a_{p+k}| \leq \frac{p\cos\theta-\beta}{p+k\gamma}. \tag{4}$$

The result is sharp for the function

$$f(z) = z^p - \frac{p\cos\theta-\beta}{p+k\gamma}e^{-i\theta}z^{p+k}. \tag{5}$$

3. Distortion theorems

Theorem 3.1. *Let the function $f(z)$ defined by (1) be in the class $F_{p,\theta}(\gamma,\beta)$, then for $z \in U$, we have*

$$|z|^p - \frac{p\cos\theta-\beta}{p+\gamma}|z|^{p+1} \le |f(z)| \le |z|^p + \frac{p\cos\theta-\beta}{p+\gamma}|z|^{p+1}. \tag{6}$$

Furthermore

$$p|z|^{p-1} - \frac{(p+1)(p\cos\theta-\beta)}{(p+\gamma)}|z|^p \le \left|f'(z)\right| \le p|z|^{p-1} + \frac{(p+1)(p\cos\theta-\beta)}{(p+\gamma)}|z|^p. \tag{7}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{p\cos\theta-\beta}{p+\gamma}e^{-i\theta}z^{p+1}(z=\pm|z|e^{i\theta}). \tag{8}$$

Proof. It is easy to see from Theorem 2.1 that

$$\frac{p+\gamma}{p\cos\theta-\beta}\sum_{k=1}^{\infty}|a_{p+k}| \le \sum_{k=1}^{\infty}\frac{p+k\gamma}{p\cos\theta-\beta}|a_{p+k}| \le 1.$$

Then

$$\sum_{k=1}^{\infty}|a_{p+k}| \le \frac{p\cos\theta-\beta}{p+\gamma}. \tag{9}$$

Making use of (9), we have

$$|f(z)| \;\ge\; |z|^p - |z|^{p+1}_{k=1}{}^{\infty}|a_{p+k}| \ge |z|^p - \frac{p\cos\theta-\beta}{p+\gamma}|z|^{p+1}, \tag{10}$$

and

$$|f(z)| \;\le\; |z|^p + |z|^{p+1}_{k=1}{}^{\infty}|a_{p+k}| \le |z|^p + \frac{p\cos\theta-\beta}{p+\gamma}|z|^{p+1}, \tag{11}$$

which proves the assertion (6).
From (9) and Theorem 2.1, it follows also that

$$\sum_{k=1}^{\infty}(p+k)|a_{p+k}| \le \frac{(p+1)(p\cos\theta-\beta)}{(p+\gamma)}. \tag{12}$$

Consequently, we have

$$\left|f'(z)\right| \;\ge\; p|z|^{p-1} - |z|^p_{k=1}{}^{\infty}(p+k)|a_{p+k}| \ge p|z|^{p-1} - \frac{(p+1)(p\cos\theta-\beta)}{(p+\gamma)}|z|^p, \tag{13}$$

and

$$\left|f'(z)\right| \;\le\; p|z|^{p-1} + |z|^p_{k=1}{}^{\infty}(p+k)|a_{p+k}| \le p|z|^{p-1} + \frac{(p+1)(p\cos\theta-\beta)}{(p+\gamma)}|z|^p, \tag{14}$$

which proves the assertion (7). Since each of equalities in (6) and (7) is satisfied by the function $f(z)$ given by (8), our proof of Theorem 3.1 is thus completed. □

4. Closure theorems

Let the functions $f_j(z)$ be defined, for $j = 1, 2, ..., m$, by

$$f_j(z) = z^p - \sum_{k=1}^{\infty} a_{p+k,j} z^{p+k} \ (e^{i\theta} a_{p+k,j} \geq 0; \ |\theta| < \frac{\pi}{2}). \tag{15}$$

Theorem 4.1. *Let the functions $f_j(z)$ $(j = 1, 2, ..., m)$ defined by (15) be in the class $F_{p,\theta}(\gamma, \beta)$. Then the function $h(z)$ defined by*

$$h(z) = z^p - \sum_{k=1}^{\infty} b_{p+k} z^{p+k}, \tag{16}$$

also belongs to the class $F_{p,\theta}(\gamma, \beta)$, where

$$b_{p+k} = \frac{1}{m} \sum_{j=1}^{m} a_{p+k,j}. \tag{17}$$

Proof. Since $f_j(z)$ $(j = 1, 2, ..., m)$ are in the class $F_{p,\theta}(\gamma, \beta)$, it follows from Theorem 2.1, that

$$\sum_{k=1}^{\infty} e^{i\theta} \left(1 + \frac{\gamma k}{p}\right) a_{p+k,j} \leq \cos\theta - \frac{\beta}{p},$$

for every $j = 1, 2, ..., m$. Hence

$$\sum_{k=1}^{\infty} e^{i\theta} \left(1 + \frac{\gamma k}{p}\right) b_{p+k} = \sum_{k=1}^{\infty} e^{i\theta} \left(1 + \frac{\gamma k}{p}\right) \left(\frac{1}{m} \sum_{j=1}^{m} a_{p+k,j}\right) =$$

$$= \frac{1}{m} \sum_{j=1}^{m} \left(\sum_{k=1}^{\infty} e^{i\theta} \left(1 + \frac{\gamma k}{p}\right) a_{p+k,j}\right) \leq \frac{1}{m} \sum_{j=1}^{m} \left(\cos\theta - \frac{\beta}{p}\right) \leq \cos\theta - \frac{\beta}{p}.$$

By Theorem 2.1, it follows that $h(z) \in F_{p,\theta}(\gamma, \beta)$. This completes the proof of Theorem 4.1. □

Theorem 4.2. *Let the functions $f_j(z)$ $(j = 1, 2, ..., m)$ defined by (15) be in the class $F_{p,\theta}(\gamma, \beta_j)$. Then the function $h(z)$ defined by*

$$h(z) = z^p - \sum_{k=1}^{\infty} \left(\frac{1}{m} \sum_{j=1}^{m} a_{p+k,j}\right) z^{p+k}, \tag{18}$$

is in the class $F_{p,\theta}(\gamma, \beta)$, where

$$\beta = \min_{1 \leq j \leq m} \{\beta_j\}. \tag{19}$$

Proof. Since $f_j(z)$ $(j = 1, 2, ..., m)$ are in the class $F_{p,\theta}(\gamma, \beta_j)$, it follows from Theorem 2.1, that

$$\sum_{k=1}^{\infty} e^{i\theta} \left(1 + \frac{\gamma k}{p}\right) a_{p+k,j} \leq \cos\theta - \frac{\beta_j}{p},$$

for every $j = 1, 2, ..., m$. Hence

$$\sum_{k=1}^{\infty} e^{i\theta} \left(1 + \frac{\gamma k}{p}\right) \left(\frac{1}{m} \sum_{j=1}^{m} a_{p+k,j}\right) = \frac{1}{m} \sum_{j=1}^{m} \left(\sum_{k=1}^{\infty} e^{i\theta} \left(1 + \frac{\gamma k}{p}\right) a_{p+k,j}\right) \leq$$

$$\leq \frac{1}{m} \sum_{j=1}^{m} \left(\cos\theta - \frac{\beta_j}{p}\right) \leq \frac{1}{m} \sum_{j=1}^{m} \left(\cos\theta - \frac{\beta}{p}\right) \leq \cos\theta - \frac{\beta}{p}.$$

By Theorem 2.1, it follows that $h(z) \in F_{p,\theta}(\gamma,\beta)$. This completes the proof of Theorem 4.2. $\square$

Theorem 4.3. *Let the functions $f_j(z)$ $(j = 1, 2, ..., m)$ defined by (15) be in the class $F_{p,\theta}(\gamma,\beta)$. Then the function $h(z)$ defined by*

$$h(z) = \sum_{j=1}^{m} c_j f_j(z), \tag{20}$$

is also in the class $F_{p,\theta}(\gamma,\beta)$, where

$$\sum_{j=1}^{m} c_j = 1. \tag{21}$$

Proof. Assume that

$$h(z) \;=\; \sum_{j=1}^{m} c_j f_j(z) = z^p - \sum_{k=1}^{\infty} \left(\sum_{j=1}^{m} c_j a_{p+k,j} \right) z^{p+k}. \tag{22}$$

Then it follows that

$$\sum_{k=1}^{\infty} \left(1 + \frac{\gamma k}{p}\right) e^{i\theta} \left(\sum_{j=1}^{m} c_j a_{p+k,j} \right) = \sum_{j=1}^{m} c_j \left(\sum_{k=1}^{\infty} e^{i\theta} \left[1 + \frac{\gamma k}{p}\right] a_{p+k,j} \right) \le$$

$$\le \;\; \left(\cos\theta - \tfrac{\beta}{p}\right) \textstyle\sum_{j=1}^{m} c_j \le \cos\theta - \tfrac{\beta}{p}.$$

By Theorem 2.1, it follows that $h(z) \in F_{p,\theta}(\gamma,\beta)$. This completes the proof of Theorem 4.3. $\square$

Theorem 4.4. *Let $f_p(z) = z^p$ and*

$$f_{p+k}(z) = z^p - \frac{p\cos\theta - \beta}{p + k\gamma} e^{-i\theta} z^{p+k}. \tag{23}$$

Then $f(z)$ is in the class $F_{p,\theta}(\gamma,\beta)$ if and only if can be expressed in the form

$$f(z) =_{k=0}^{\infty} \mu_{p+k} f_{p+k}(z), \tag{24}$$

where $\mu_{p+k} \ge 0$ and ${}_{k=0}^{\infty}\mu_{p+k} = 1$.

Proof. Assume that

$$f(z) \;=\; {}_{k=0}^{\infty}\mu_{p+k} f_{p+k}(z) = z^p - \sum_{k=1}^{\infty} \frac{p\cos\theta - \beta}{p + k\gamma} e^{-i\theta} \mu_{p+k} z^{p+k}.4.11 \tag{25}$$

Then it follows that

$$\sum_{k=1}^{\infty} e^{i\theta} \left(1 + \frac{\gamma k}{p}\right) \left(\frac{p\cos\theta - \beta}{p + k\gamma}\right) e^{-i\theta} \mu_{p+k} = \left(\cos\theta - \frac{\beta}{p}\right) \sum_{k=1}^{\infty} \mu_{p+k} =$$

$$= \left(\cos\theta - \frac{\beta}{p}\right) (1 - \mu_p) \le \cos\theta - \frac{\beta}{p},$$

which implies that $f(z) \in F_{p,\theta}(\gamma,\beta)$.
Conversely, assume that the function $f(z)$ defined by (1) be in the class $F_{p,\theta}(\gamma,\beta)$. Then

$$a_{p+k} \le \frac{(p\cos\theta - \beta)}{(p + k\gamma)} e^{-i\theta}.$$

Setting

$$\mu_{p+k}=\frac{(p+k\gamma)}{(p\cos\theta-\beta)}e^{i\theta}a_{p+k},$$

where

$$\mu_p=1-\sum_{k=1}^{\infty}\mu_{p+k}\ ,$$

we can see that $f(z)$ can be expressed in the form (24). This completes the proof of Theorem 4.4. □

Corollary 4.1. *The extreme points of the class $F_{p,\theta}(\gamma,\beta)$ are the functions $f_p(z)=z^p$ and*

$$f_{p+k}(z)=z^p-\frac{p\cos\theta-\beta}{p+k\gamma}e^{-i\theta}z^{p+k}. \tag{26}$$

5. Radii of close-to-convexity, starlikeness and convexity

Theorem 5.1. *Let the function $f(z)$ defined by* (1) *be in the class $F_{p,\theta}(\gamma,\beta)$. Then $f(z)$ is p-valent close-to-convex of order δ $(0\le\delta<p)$ in $|z|\le r_1$, where*

$$r_1=\inf_{k\ge1}\left\{\frac{(p+k\gamma)(p-\delta)}{(p\cos\theta-\beta)(k+p)}\right\}^{\frac{1}{k}}. \tag{27}$$

The result is sharp and the extremal function is given by (5).

Proof. We must show that

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|\le p-\delta\ for\ \ |z|\le r_1, \tag{28}$$

where r_1 is given by (27). Indeed we find from (1) that

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|\le\sum_{k=1}^{\infty}(p+k)\left|a_{p+k}\right|\left|z\right|^k.$$

Thus

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|\le p-\delta,$$

if

$$\sum_{k=1}^{\infty}\left(\frac{k+p}{p-\delta}\right)\left|a_{p+k}\right|\left|z\right|^k\le1. \tag{29}$$

But by using Theorem 2.1, (29) will be true if

$$\left(\frac{k+p}{p-\delta}\right)|z|^k\le\left(\frac{p+k\gamma}{p\cos\theta-\beta}\right).$$

Then

$$|z|\le\left\{\frac{(p+k\gamma)(p-\delta)}{(p\cos\theta-\beta)(k+p)}\right\}^{\frac{1}{k}}. \tag{30}$$

The result follows easily from (30). □

Theorem 5.2. *Let the function $f(z)$ defined by* (1) *be in the class $F_{p,\theta}(\gamma,\beta)$. Then $f(z)$ is p-valent starlike of order δ $(0\le\delta<p)$ in $|z|\le r_2$, where*

$$r_2=\inf_{k\ge1}\left\{\frac{(p+k\gamma)(p-\delta)}{(p\cos\theta-\beta)(k+p-\delta)}\right\}^{\frac{1}{k}}. \tag{31}$$

The result is sharp and the extremal function is given by (5).

Proof. We must show that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq p - \delta \ for \ |z| \leq r_2, \tag{32}$$

where r_2 is given by (31). Indeed we find from (1) that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq \frac{\sum_{k=1}^{\infty} k\,|a_{p+k}|\,|z|^k}{1 - \sum_{k=1}^{\infty} |a_{p+k}|\,|z|^k}.$$

Thus

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq p - \delta,$$

if

$$\sum_{k=1}^{\infty} \left(\frac{k+p-\delta}{p-\delta}\right) |a_{p+k}|\,|z|^k \leq 1. \tag{33}$$

But by using Theorem 2.1, (33) will be true if

$$\left(\frac{k+p-\delta}{p-\delta}\right) |z|^k \leq \left(\frac{p+k\gamma}{p\cos\theta - \beta}\right).$$

Then

$$|z| \leq \left\{\frac{(p+k\gamma)(p-\delta)}{(p\cos\theta-\beta)(k+p-\delta)}\right\}^{\frac{1}{k}}. \tag{34}$$

The result follows easily from (34). □

Corollary 5.1. *Let the function $f(z)$ defined by* (1) *be in the class $F_{p,\theta}(\gamma,\beta)$. Then $f(z)$ is in p-valent convex of order δ $(0 \leq \delta < p)$ in $|z| \leq r_3$, where*

$$r_3 = \inf_{k\geq 1} \left\{\frac{p(p+k\gamma)(p-\delta)}{(k+p)(p\cos\theta-\beta)(k+p-\delta)}\right\}^{\frac{1}{k}}. \tag{35}$$

The result is sharp and the extremal function is given by (5).

6. Modified Hadamard products

For the functions $f_j(z)$ $(j = 1, 2)$ defined by (15) and belonging to the class $T_p(\theta)$, the modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^p - \sum_{k=1}^{\infty} a_{p+k,1}a_{p+k,2}z^{p+k}. \tag{36}$$

Theorem 6.1. *Let the functions $f_j(z)$ $(j = 1, 2)$ defined by* (15) *be in the class $F_{p,\theta}(\gamma,\beta)$. Then $(f_1 * f_2)(z) \in F_{p,2\theta}(\gamma,\alpha)$ where*

$$\alpha = p\cos 2\theta - \frac{(p\cos\theta-\beta)^2}{(p+\gamma)}. \tag{37}$$

The result is sharp for the functions $f_j(z)$ given by

$$f_j(z) = z^p - \left(\frac{p\cos\theta-\beta}{p+\gamma}\right) e^{-i\theta} z^{p+1} \ (j = 1, 2). \tag{38}$$

Proof. Employing the technique used ealier by Schild and Silverman [6]. We need only to find the largest α such that

$$\sum_{k=1}^{\infty} e^{2i\theta}\left(\frac{p+k\gamma}{p\cos 2\theta-\alpha}\right) a_{p+k,1}a_{p+k,2} \le 1. \tag{39}$$

Since $f_j(z)$ $(j=1,2)$ are in the class $F_{p,\theta}(\gamma,\beta)$, it follows from Theorem 2.1, that

$$\sum_{k=1}^{\infty} e^{i\theta}\left(\frac{p+k\gamma}{p\cos\theta-\beta}\right) a_{p+k,j} \le 1, \tag{40}$$

for every $j=1,2$. By the Cauchy Schwarz inequality we have

$$\sum_{k=1}^{\infty} e^{i\theta}\left(\frac{p+k\gamma}{p\cos\theta-\beta}\right) \sqrt{a_{p+k,1}a_{p+k,2}} \le 1. \tag{41}$$

Therefore, (39) will be satisfied if

$$e^{2i\theta}\left(\frac{p+k\gamma}{p\cos 2\theta-\alpha}\right) a_{p+k,1}a_{p+k,2} \le e^{i\theta}\left(\frac{p+k\gamma}{p\cos\theta-\beta}\right)\sqrt{a_{p+k,1}a_{p+k,2}}.$$

Then

$$\sqrt{a_{p+k,1}a_{p+k,2}} \le \left(\frac{p\cos 2\theta-\alpha}{p\cos\theta-\beta}\right) e^{-i\theta}. \tag{42}$$

Since (41) implies

$$\sqrt{a_{p+k,1}a_{p+k,2}} \le \left(\frac{p\cos\theta-\beta}{p+k\gamma}\right) e^{-i\theta}. \tag{43}$$

From ((42) and (43) we have

$$\alpha \le p\cos 2\theta - \frac{(p\cos\theta-\beta)^2}{(p+k\gamma)}. \tag{44}$$

Now defining the function $G(k)$ by

$$G(k) = p\cos 2\theta - \frac{(p\cos\theta-\beta)^2}{(p+k\gamma)}, \tag{45}$$

we see that $G(k)$ is an increasing function of k $(k\in\mathbb{N})$. Therefore, we conclude that

$$\alpha \le G(1) = p\cos 2\theta - \frac{(p\cos\theta-\beta)^2}{(p+\gamma)}, \tag{46}$$

which evidently completes the proof of Theorem 6.1. □

Using arguments similiar to those in the proof of Theorem 6.1, we obtain the following theorem.

Theorem 6.2. *Let the function $f_1(z)$ defined by (15) be in the class $F_{p,\theta}(\gamma,\beta)$. Suppose also that the function $f_2(z)$ defined by (15) be in the class $F_{p,\theta}(\gamma,\phi)$. Then $(f_1 * f_2)(z)\in F_{p,2\theta}(\gamma,\zeta)$, where*

$$\zeta = p\cos 2\theta - \frac{(p\cos\theta-\beta)(p\cos\theta-\phi)}{(p+\gamma)}. \tag{47}$$

The result is sharp for the functions $f_j(z)$ $(j=1,2)$ given by

$$f_1(z) = z^p - \left(\frac{p\cos\theta-\beta}{p+\gamma}\right) e^{-i\theta} z^{p+1}, \tag{48}$$

and

$$f_2(z) = z^p - \left(\frac{p\cos\theta-\phi}{p+\gamma}\right) e^{-i\theta} z^{p+1}. \tag{49}$$

Theorem 6.3. . *Let the functions $f_j(z)$ $(j = 1, 2)$ defined by* (15) *be in the class $F_{p,\theta}(\gamma, \beta)$. Then the function*

$$h(z) = z^p - \sum_{k=1}^{\infty}(a_{p+k,1}^2 + a_{p+k,2}^2)z^{p+k}, \tag{50}$$

also belongs to the class $F_{p,2\theta}(\gamma, \eta)$, where

$$\eta(p; \beta, \gamma; \theta) = p\cos 2\theta - \frac{2(p\cos\theta - \beta)^2}{(p+\gamma)}. \tag{51}$$

The result is sharp for the functions given by (38).

Proof. By using Theorem 2.1, we have

$$\sum_{k=1}^{\infty}\left[e^{i\theta}\frac{p+k\gamma}{p\cos\theta-\beta}\right]^2 a_{p+k,1}^2 \le \left[\sum_{k=1}^{\infty} e^{i\theta}\frac{p+k\gamma}{p\cos\theta-\beta}a_{p+k,1}\right]^2 \le 1, \tag{52}$$

and

$$\sum_{k=1}^{\infty}\left[e^{i\theta}\frac{p+k\gamma}{p\cos\theta-\beta}\right]^2 a_{p+k,2}^2 \le \left[\sum_{k=1}^{\infty} e^{i\theta}\frac{p+k\gamma}{p\cos\theta-\beta}a_{p+k,2}\right]^2 \le 1. \tag{53}$$

It follow from (52) and (53) that

$$\sum_{k=1}^{\infty}\frac{1}{2}\left[e^{i\theta}\frac{p+k\gamma}{p\cos\theta-\beta}\right]^2 (a_{p+k,1}^2 + a_{p+k,2}^2) \le 1. \tag{54}$$

Therefore, we need to find the largest η such that

$$e^{2i\theta}\frac{p+k\gamma}{p\cos 2\theta-\eta} \le \frac{1}{2}\left[e^{i\theta}\frac{p+k\gamma}{p\cos\theta-\beta}\right]^2, \tag{55}$$

that is

$$\eta \le p\cos 2\theta - \frac{2(p\cos\theta-\beta)^2}{(p+k\gamma)}.$$

Since

$$D(k) = p\cos 2\theta - \frac{2(p\cos\theta-\beta)^2}{(p+k\gamma)},$$

is an increasing function of $k (k \in \mathbb{N})$, we obtain

$$\eta \le D(1) = p\cos 2\theta - \frac{2(p\cos\theta-\beta)^2}{(p+\gamma)},$$

and hence the proof of Theorem 6.3 is completed. □

Remark 6.1. *(1) Putting $\theta = 0$ in our results, we obtain the results obtained by Lee et al.* [4]*; (2) Putting $\gamma = p = 1$ in our results, we obtain the results obtained by Sekine* [7].

7. Definitions and applications of fractional calculus

Many essentially equivalent definitions of fractional calculus (that is, fractional derivatives and fractional integrals) have been given in the literature (cf., e.g. [1], [9] and [10]. We find it to be convenient to recall here the following definitions which were used recently by Owa [5] and by Srivastava and Owa [8]).

Definition 7.1. *The fractional integral of order μ is defined, for a function $f(z)$, by*

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(t)}{(z-t)^{1-\mu}} dt \ (\mu > 0), \tag{56}$$

where $f(z)$ is an analytic function in a simply-connected region of the complex $z-$plane containing the origin and the multiplicity of $(z-t)^{\mu-1}$ is removed by requiring $log(z-t)$ to be real when $z-t>0$.

Definition 7.2. *The fractional derivative of order μ is defined, for a function $f(z)$, by*

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\mu}} dt \ (0 \le \mu < 1), \tag{57}$$

where $f(z)$ is an analytic function in a simply-connected region of the complex $z-$plane containing the origin and the multiplicity of $(z-t)^{-\mu}$ is removed by requiring $log(z-t)$ to be real when $z-t>0$.

Definition 7.3. *Under the hypotheses of definition 2, the fractional derivative of order $n+\mu$ is defined by*

$$D_z^{n+\mu} f(z) = \frac{d^n}{dz^n} D_z^{\mu} f(z) \ (0 \le \mu < 1; \ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{58}$$

Theorem 7.1. *Let the function $f(z)$ defined by (1) be in the class $F_{p,\theta}(\gamma,\beta)$. Then we have*

$$\left|D_z^{-\mu} f(z)\right| \ge \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} |z|^{p+\mu} \left\{1 - \frac{(p+1)(p\cos\theta - \beta)}{(p+\gamma)(p+\mu+1)} |z|\right\}, \tag{59}$$

and

$$\left|D_z^{-\mu} f(z)\right| \le \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} |z|^{p+\mu} \left\{1 + \frac{(p+1)(p\cos\theta - \beta)}{(p+\gamma)(p+\mu+1)} |z|\right\}, \tag{60}$$

for $\mu > 0$ and $z \in U$. The result is sharp.

Proof. Let

$$\begin{aligned} F(z) &= \frac{\Gamma(p+\mu+1)}{\Gamma(p+1)} z^{-\mu} D_z^{-\mu} f(z) \\ &= z^p - \sum_{k=1}^{\infty} \frac{\Gamma(p+k+1)\Gamma(p+\mu+1)}{\Gamma(p+1)\Gamma(p+k+\mu+1)} a_{p+k} z^{p+k}. \end{aligned}$$

Then

$$F(z) = z^p - \sum_{k=1}^{\infty} \Psi(k) a_{p+k} z^{p+k}, \tag{61}$$

where

$$\Psi(k) = \frac{\Gamma(p+k+1)\Gamma(p+\mu+1)}{\Gamma(p+1)\Gamma(p+k+\mu+1)} \ (\mu > 0).$$

Since $\Psi(k)$ is an decreasing function of k ($k \in \mathbb{N}$), then

$$0 < \Psi(k) \le \Psi(1) = \frac{(p+1)}{(p+\mu+1)}. \tag{62}$$

From (61) and (62), we have

$$|F(z)| \ge |z|^p - \Psi(1) |z|^{p+1} \sum_{k=1}^{\infty} |a_{p+k}|. \tag{63}$$

In view of (9) and (63), we have

$$|F(z)| = \left|\frac{\Gamma(p+\mu+1)}{\Gamma(p+1)} z^{-\mu} D_z^{-\mu} f(z)\right| \geq |z|^p - \frac{(p+1)(p\cos\theta-\beta)}{(p+\gamma)(p+\mu+1)} |z|^{p+1},$$

and

$$|F(z)| = \left|\frac{\Gamma(p+\mu+1)}{\Gamma(p+1)} z^{-\mu} D_z^{-\mu} f(z)\right| \leq |z|^p + \frac{(p+1)(p\cos\theta-\beta)}{(p+\gamma)(p+\mu+1)} |z|^{p+1}.$$

which proves the inequalities of Theorem 7.1. Further equalities are attained for the function

$$D_z^{-\mu} f(z) = \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} z^{p+\mu} \left\{1 - \frac{(p+1)(p\cos\theta-\beta)}{(p+\gamma)(p+\mu+1)} z\right\}, \tag{64}$$

or

$$f(z) = z^p - \frac{p\cos\theta-\beta}{p+\gamma} e^{-i\theta} z^{p+1} (z = \pm |z| e^{i\theta}). \tag{65}$$

□

Using arguments similiar to those in the proof of Theorem 7.1, we obtain the following theorem.

Theorem 7.2. *Let the function $f(z)$ defined by* (1) *be in the class $F_{p,\theta}(\gamma,\beta)$. Then we have*

$$|D_z^{\mu} f(z)| \geq \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} |z|^{p-\mu} \left\{1 - \frac{(p+1)(p\cos\theta-\beta)}{(p+\gamma)(p-\mu+1)} |z|\right\}, \tag{66}$$

and

$$|D_z^{\mu} f(z)| \leq \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} |z|^{p-\mu} \left\{1 + \frac{(p+1)(p\cos\theta-\beta)}{(p+\gamma)(p-\mu+1)} |z|\right\}, \tag{67}$$

for $0 \leq \mu < 1$ and $z \in U$. The result is sharp for the function $f(z)$ given by (65).

Remark 7.1. *(1) Putting $\theta = 0$ in our results, we obtain the results obtained by Aouf and Darwish* [2]*;*
(2)Putting $\theta = 0$ and $p = 1$ in our results, we obtain the results obtained by Bhoosnurmath and Swamy [3].

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