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GENERALIZED QUATERNIONS AND ROTATION IN 3-SPACE $\mathbf{E}^3_{\alpha\beta}$

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Abstract. The paper explains how a unit generalized quaternion is used to represent a rotation of a vector in 3-dimensional $\mathbf{E}^3_{\alpha\beta}$ space. We review of some algebraic properties of generalized quaternions and operations between them and then show their relation with the rotation matrix.

Keywords: generalized quaternion, quasi-orthogonal matrix, rotation.

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1. Introduction

The quaternions algebra was invented by W.R. Hamilton as an extension to the complex numbers. He was able to find connections between this new algebra and spatial rotations. The unit quaternions form a group that is isomorphic to the group $SU(2)$ and is a double cover of $SO(3)$, the group of 3-dimensional rotations. Under these isomorphisms the quaternion multiplication operation corresponds to the composition operation of rotations [18]. Kula and Yaylı [13] showed that unit split quaternions in H' determined a rotation in semi-Euclidean space E^4_2. In [15], is demonstrated how timelike split quaternions are used to perform rotations in the Minkowski 3-space E^3_1. Rotations in a complex 3-dimensional space are considered in [20] and applied to the treatment of the Lorentz transformation in special relativity.

A brief introduction of the generalized quaternions is provided in [16]. Also, this subject have investigated in algebra [19]. Recently, we studied the generalized quaternions, and gave some of their algebraic properties [7]. It is shown that the set of all unit generalized quaternions with the group operation of quaternion multiplication is a Lie group of 3-dimension. Their Lie algebra and properties of the bracket multiplication are looked for. Also, a matrix corresponding to Hamilton operators that is defined for generalized quaternions is determined a Homothetic motion in $E^4_{\alpha\beta}$[9]. Furthermore, we showed how these operators can be used to describe rotation in $E^4_{\alpha\beta}$[10]. In this paper, after developing some elementary properties of generalized quaternions, it is shown how unit generalized quaternions can be used to described rotation in 3-dimensional space $E^3_{\alpha\beta}$.

2. Preliminaries

Quaternions are hypercomplex numbers used to represent spatial rotations in three dimensions. The basic definition of a real quaternion given in [3, 4] as follows:

Definition 2.1. *A real quaternion is defined as*

$$q = a_0 + a_1 i + a_2 j + a_3 k$$

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where a_0, a_1, a_2 and a_3 are real numbers and $1, i, j, k$ of q may be interpreted as the four basic vectors of Cartesian set of coordinates; and they satisfy the non-commutative multiplication rules

$$\begin{aligned} i^2 &= j^2 = k^2 = -1 \\ ij &= k = -ji, \quad jk = i = -kj \end{aligned}$$

and

$$ki = j = -ik, \ ijk = -1.$$

The quaternion algebra H is the even subalgebra of the Clifford algebra of the 3-dimensional Euclidean space. The Clifford algebra $Cl(E_p^n) = Cl_{n-p,p}$ for the n-dimensional non-degenerate vector space E_p^n having an orthonormal base $\{e_1, e_2, ..., e_n\}$ with the signature $(p, n-p)$ is the associative algebra generated by 1 and $\{e_i\}$ with satisfying the relations $e_ie_j + e_je_i = 0$ for $i \neq j$ and

$$e_i^2 = \{ {}^{-1, \ if \ i=1,2,...,p}_{1, \ if \ i=p+1,...,n}$$

The Clifford algebra $Cl_{n-p,p}$ has the basis$\{e_{i_1}, e_{i_2}, ..., e_{i_k} : 1 \leq i_1 < i_2 < ... < i_k < n\}$ that is the division algebra of quaternions H is isomorphic with the even subalgebra $Cl_{3,0}^+$ of the Clifford algebra $Cl_{3,0}$ such that $Cl_{3,0}^+$ has the basis $\{1, e_1e_2 \to i, e_2e_3 \to j, e_1e_3 \to k\}$. The conjugate of the quaternion $q = S_q + v_q$ is denoted by $\overline{q}$, and defined as $\overline{q} = S_q - v_q$. The norm of a quaternion $q = (a_0, a_1, a_2, a_3)$ is defined by $q\overline{q} = \overline{q}q = a_0^2 + a_1^2 + a_2^2 + a_3^2$ and is denoted by N_q and say that $q_0 = \frac{q}{N_q}$ is unit quaternion where $q \neq 0$. Unit quaternions provide a convenient mathematical notation for representing orientations and rotations of objects in three dimensions[20]. One can represent a quaternion $q = a_0 + a_1i + a_2j + a_3k$ by a 2×2 complex matrix (with i' being the usual complex imaginary);

$$A = \begin{bmatrix} a_0 + i'a_1 & -i'a_1 + a_2 \\ -i'a_1 - a_2 & a_0 - i'a_3 \end{bmatrix}$$

or by a 4×4 real matrix

$$A = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}.$$

The Euler's and De-Moivre's formulae for the matrix A are studied in [8]. It is shown that as the De Moivre's formula implies, there are uncountably many matrices of unit quaternion $A^n = I_4$ satisfying for $n > 2$.

In geometry and linear algebra, a rotation is a transformation in a plane or in space that describes the motion of a rigid body around a fixed point. There are at least eight methods used commonly to represent rotation, including: i) orthogonal matrices, ii) axis and angle, iii) Euler angles, iv) Gibbs vector, v) Pauli spin matrices, vi) Cayley-Klein parameters, vii) Euler or Rodrigues parameters, and viii) Hamilton's quaternions [6]. But to use the unit quaternions is a more useful, natural, and elegant way to perceive rotations compared to other methods.

Theorem 2.1. *All the rotation about lines through the origin in ordinary space form a group, homomorphic to the group of all unit quaternions* [2].

If a simple rotation is only in the three space dimensions, i.e. about a plane that is entirely in space, then this rotation is the same as a spatial rotation in three dimensions. But a simple rotation about a plane spanned by a space dimension and a time dimension is a "boost", a transformation between two different reference frames, which together with other properties of spacetime determines the relativistic relationship between the frames. The set of these rotations forms the Lorentz group [1].

With Cartesian point coordinates in 3-space, a rotation in 3-space about the origin can be represented by the orthogonal matrix

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix},$$

where $RR^T = I_3$ and $\det R = 1$. It is known that unit quaternions can represent rotations about the origin. Wittenburg [21] gives the following conversion formulae. For any unit quaternion q the entries of the rotation matrix are

$$\begin{aligned} r_{11} &= 2(a_0^2 + a_1^2) - 1,\ r_{21} = 2(a_1a_2 + a_0a_3),\ r_{31} = 2(a_1a_3 - a_0a_2) \\ r_{12} &= 2(a_1a_2 - a_0a_3),\ r_{22} = 2(a_0^2 + a_2^2) - 1,\ r_{32} = 2(a_2a_3 - a_0a_2) \\ r_{13} &= 2(a_1a_3 - a_0a_2),\ r_{23} = 2(a_2a_3 - a_0a_2),\ r_{33} = 2(a_0^2 + a_2^2) - 1 \ [5]. \end{aligned}$$

Example 2.1. *For the unit real quaternion* $q = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(1, -1, 0)$, *the rotation matrix is*

$$R_q = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix},$$

the axis of this rotation is spanned by the vector $(1, -1, 0)$ *and the angle of rotation is* $\phi = \frac{\pi}{2}$.

Definition 2.2. *Let* $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in \mathbb{R}^3$. *If* $\alpha, \beta \in \mathbb{R}^+$, *the generalized inner product of* u *and* v *is defined by*

$$\langle u, v \rangle = \alpha u_1 v_1 + \beta u_2 v_2 + \alpha\beta u_3 v_3.$$

It could be written

$$\langle u, v \rangle = u^T \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha\beta \end{bmatrix} v.$$

Also, if $\alpha > 0, \beta < 0$, $\langle u, v \rangle$ *is called the generalized semi-Euclidean inner product. We put* $E^3_{\alpha\beta} = \left(\mathbb{R}^3, \langle , \rangle\right)$. *The vector product in* $E^3_{\alpha\beta}$ *is defined by*

$$\begin{aligned} u \times v &= \begin{vmatrix} \beta i & \alpha j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= \beta(u_2v_3 - u_3v_2)i + \alpha(u_3v_1 - u_1v_3)j + (u_1v_2 - u_2v_1)k, \end{aligned}$$

where $i \times j = k$, $j \times k = \beta i$, $k \times i = \alpha j$.

Special cases:

1. If $\alpha = \beta = 1$, then $E^3_{\alpha\beta}$ is an Euclidean 3-space E^3.
2. If $\alpha = 1, \beta = -1$, then $E^3_{\alpha\beta}$ is a semi-Euclidean 3-space E^3_2.

Definition 2.3. *A generalized quaternion is an expression of form*

$$q = a_0 + a_1 i + a_2 j + a_3 k$$

where a_0, a_1, a_2 *and* a_3 *are real numbers and* i, j, k *are quaternionic units which satisfy the equalities*

$$\begin{aligned} i^2 &= -\alpha, \quad j^2 = -\beta, \quad k^2 = -\alpha\beta \\ ij &= k = -ji \ , \quad jk = \beta i = -kj \end{aligned}$$

and

$$ki = \alpha j = -ik, \quad \alpha, \beta \in \mathbb{R}.$$

The set of all generalized quaternions are denoted by $H_{\alpha\beta}$. *A generalized quaternion* q *is a sum of a scalar and a vector, called scalar part,* $S_q = a_0$, *and vector part* $V_q = a_1 i + +a_2 j + a_3 k \in \mathbb{R}^3_{\alpha\beta}$. *Therefore,* $H_{\alpha\beta}$ *forms a* 4*-dimensional real space which contains the real axis* $\mathbb{R}$ *and a 3-dimensional real linear space* $E^3_{\alpha\beta}$, *so that,* $H_{\alpha\beta} = \mathbb{R} \oplus E^3_{\alpha\beta}$.

Special cases:
1) If $\alpha = \beta = 1$ is considered, then $H_{\alpha\beta}$ is the algebra of real quaternions [4].
2) If $\alpha = 1, \beta = -1$ is considered, then $H_{\alpha\beta}$ is the algebra of split quaternions [13].
3) If $\alpha = 1, \beta = 0$ is considered, then $H_{\alpha\beta}$ is the algebra of semi-quaternions [14].
4) If $\alpha = -1, \beta = 0$ is considered, then $H_{\alpha\beta}$ is the algebra of split semi-quaternions [12].
5) If $\alpha = 0, \beta = 0$ is considered, then $H_{\alpha\beta}$ is the algebra of $\frac{1}{4}-$quaternions [11,17].

The multiplication rule for generalized quaternions is defined as

$$qp = S_p S_q - \langle V_q, V_p \rangle + S_q V_p + S_p V_q + V_p \times V_q,$$

where

$$\begin{aligned} S_p &= a_0, \ S_q = b_0, \ \langle V_q, V_p \rangle = \alpha a_1 b_1 + \beta a_2 b_2 + \alpha\beta a_3 b_3, \\ V_p \times V_q &= \beta(a_2 b_3 - a_3 b_2) i + \alpha(a_3 b_1 - a_1 b_3) j + (a_1 b_2 - a_2 b_1) k. \end{aligned}$$

It could be expressed as

$$qp = \begin{bmatrix} a_0 & -\alpha a_1 & -\beta a_2 & -\alpha\beta a_3 \\ a_1 & a_0 & -\beta a_3 & \beta a_2 \\ a_2 & \alpha a_3 & a_0 & -\alpha a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Obviously, quaternion multiplication is an associative and distributive with respect to addition and subtraction, but the commutative law does not hold in general.

Corollary 2.1. $H_{\alpha\beta}$ *with addition and multiplication has all the properties of a number field expect commutativity of the multiplication. It is therefore called the skew field of quaternions.*

The conjugate of the quaternion $q = S_q + V_q$ is denoted by $\overline{q}$ and defined as $\overline{q} = S_q - V_q$. The norm of a quaternion $q = (a_0, a_1, a_2, a_3)$ is defined by $N_q = |q\overline{q}| = |\overline{q}q| = \left|a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2\right|$ and say that $q_0 = \frac{q}{N_q}$ is a unit generalized quaternion where $q \neq 0$. The set of unit generalized quaternions, G, with the group operation of quaternion multiplication is a Lie group of 3-dimension. The inverse of q is defined as $q^{-1} = \frac{\overline{q}}{N_q}$, $N_q \neq 0$.

The scalar product of two generalized quaternions $q = S_q + V_q$ and $p = S_p + V_p$ is defined as

$$\langle q, p \rangle_s = S_p S_q + \langle V_q, V_p \rangle = S_{p\overline{q}}.$$

Also, using the scalar product we can defined an angle λ between two quaternions q, p to be such

$$\cos\lambda = \frac{S_{p\bar{q}}}{\sqrt{N_p}\sqrt{N_q}}.$$

Definition 2.4. *A matrix $A_{3\times3}$ is called a quasi-orthogonal matrix if $A^T\varepsilon A = \varepsilon$ and $\det A = 1$ where*

$$\varepsilon = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha\beta \end{bmatrix},$$

and $\alpha, \beta \in \mathbb{R} - \{0\}$. The set of all quasi-orthogonal matrices with the operation of matrix multiplication is called rotation group in 3-space $E^3_{\alpha\beta}$.

Definition 2.5. *A matrix*

$$S_{3\times3} = \begin{bmatrix} 0 & -\beta s_3 & \beta s_2 \\ \alpha s_3 & 0 & -\alpha s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}$$

is called a generalized skew-symmetric matrix if $S^T\varepsilon = -\varepsilon S$ where

$$\varepsilon = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha\beta \end{bmatrix},$$

and $\alpha, \beta \in \mathbb{R} - \{0\}$.

3. Relationship of generalized quaternion to rotation

In this section, we show that a unit generalized quaternion represents a rotation in 3-space $E^3_{\alpha\beta}$. Let q be a unit generalized quaternion. The map φ acting on a pure quaternion ω:

$$\varphi \quad : \quad E^3_{\alpha\beta} \to E^3_{\alpha\beta} \qquad \varphi(\omega) = q\omega q^{-1}$$

is a 3D vector, a length-preserving function of 3D vectors, a linear transformation and does not have a reflection component. Since $E^3_{\alpha\beta} = span\{i, j, k\}$ and if $q = a_0 + a_1 i + a_2 j + a_3 k \in G$ then

$$\begin{aligned} \varphi(i) &= (a_0^2 + \alpha a_1^2 - \beta a_2^2 - \alpha\beta a_3^2)i + 2\alpha(a_1a_2 + a_0\alpha_3)\ j + 2(\alpha a_1\alpha_3 - a_0a_2)k, \\ \varphi(j) &= 2\beta(a_1a_2 - a_0a_3)i + (a_0^2 - \alpha a_1^2 + \beta a_2^2 - \alpha\beta a_3^2)\ j + 2(\beta a_2\alpha_3 + a_0a_1)k, \\ \varphi(k) &= 2\beta(a_0a_2 + \alpha a_1a_3)i + 2\alpha(\beta a_2\alpha_3 - a_0a_1)\ j + (a_0^2 - \alpha a_1^2 - \beta a_2^2 + \alpha\beta a_3^2)k. \end{aligned}$$

So that the matrix representation of the map φ is

$$M = \begin{bmatrix} a_0^2 + \alpha a_1^2 - \beta a_2^2 - \alpha\beta a_3^2 & 2\beta(a_1a_2 - a_0a_3) & 2\beta(a_0a_2 + \alpha a_1a_3) \\ 2\alpha(a_1a_2 + a_0\alpha_3) & a_0^2 - \alpha a_1^2 + \beta a_2^2 - \alpha\beta a_3^2 & 2\alpha(\beta a_2\alpha_3 - a_0a_1) \\ 2\alpha a_1\alpha_3 - 2a_0a_2 & 2\beta a_2\alpha_3 + 2a_0a_1 & a_0^2 - \alpha a_1^2 - \beta a_2^2 + \alpha\beta a_3^2 \end{bmatrix}.$$

We investigate matrix M in two different cases.
Case 1: Let α, β are positive numbers.

Theorem 3.1. *Let q be a unit generalized quaternion, then the matrix M can be written as*

$$M_{(\phi,S)} = I_3 + \sin\phi\ S + (1 - \cos\phi)S^2, \tag{1}$$

where φ is a elliptic, such that $cos\frac{\varphi}{2} = a_0$, $\sin\frac{\varphi}{2} = \sqrt{\alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2}$, and S is a generalized skew-symmetric matrix. The formula (1.1) known as Rodrigues rotation formula.

Proof. Every unit generalized quaternion $q = a_0 + a_1 i + a_2 j + a_3 k$ can be written in polar form

$$q = \cos\frac{\phi}{2} + S\sin\frac{\phi}{2},$$

then the matrix M can be written as

$$M = \begin{bmatrix} \cos^2\frac{\phi}{2} + (\alpha s_1^2 - \beta s_2^2 - \alpha\beta s_3^2)\sin^2\frac{\phi}{2} & 2\beta(s_1 s_2 \sin^2\frac{\phi}{2} - s_3\cos\frac{\phi}{2}\sin\frac{\phi}{2}) & 2\beta(\alpha s_1 s_2 \sin^2\frac{\phi}{2} - s_2\cos\frac{\phi}{2}\sin\frac{\phi}{2}) \\ 2\alpha(s_1 s_2\sin^2\frac{\phi}{2} + s_3\cos\frac{\phi}{2}\sin\frac{\phi}{2}) & \cos^2\frac{\phi}{2} + (-\alpha s_1^2 + \beta s_2^2 - \alpha\beta s_3^2)\sin^2\frac{\phi}{2} & 2\alpha(\beta s_2 s_3\sin^2\frac{\phi}{2} - s_1\cos\frac{\phi}{2}\sin\frac{\phi}{2}) \\ 2(\alpha s_1 s_3\sin^2\frac{\phi}{2} - s_2\cos\frac{\phi}{2}\sin\frac{\phi}{2}) & 2(\beta s_2 s_3 \sin^2\frac{\phi}{2} + s_1\cos\frac{\phi}{2}\sin\frac{\phi}{2}) & \cos^2\frac{\phi}{2} + (-\alpha s_1^2 - \beta s_2^2 + \alpha\beta s_3^2)\sin^2\frac{\phi}{2} \end{bmatrix}$$

$$= I_3 + \begin{bmatrix} (\alpha s_1^2 - \beta s_2^2 - \alpha\beta s_3^2 - 1)\sin^2\frac{\phi}{2} & 2\beta s_1 s_2\sin^2\frac{\phi}{2} - \beta s_3\sin\phi & 2\beta\alpha s_1 s_2\sin^2\frac{\phi}{2} - \beta s_2\cos\frac{\phi}{2}\sin\phi \\ 2\alpha s_1 s_2\sin^2\frac{\phi}{2} + \alpha s_3\sin\phi & (-\alpha s_1^2 + \beta s_2^2 - \alpha\beta s_3^2 - 1)\sin^2\frac{\phi}{2} & 2\alpha\beta s_2 s_3\sin^2\frac{\phi}{2} + \alpha s_1\sin\phi \\ 2\alpha s_1 s_3\sin^2\frac{\phi}{2} - s_2\sin\phi & 2\beta s_2 s_3\sin^2\frac{\phi}{2} + s_1\sin\phi & (-\alpha s_1^2 - \beta s_2^2 + \alpha\beta s_3^2 + 1)\sin^2\frac{\phi}{2} \end{bmatrix},$$

with used of $2\sin^2\frac{\phi}{2} = 1 - \cos\phi$ and $\alpha s_1^2 + \beta s_2^2 + \alpha\beta s_3^2 = 1$, we have

$$M = I_3 + \sin\phi \begin{bmatrix} 0 & -\beta s_3 & \beta s_2 \\ \alpha s_3 & 0 & -\alpha s_1 \\ -s_2 & s_1 & 0 \end{bmatrix} + (1-\cos\phi)\begin{bmatrix} -\alpha\beta s_3^2 - \beta s_2^2 & \beta s_1 s_2 & \alpha\beta s_1 s_3 \\ \alpha s_1 s_2 & -\alpha s_1^2 - \alpha\beta s_3^2 & \alpha\beta s_2 s_3 \\ \alpha s_1 s_3 & \beta s_2 s_3 & -\alpha s_1^2 - \beta s_2^2 \end{bmatrix},$$

so, proof is complete. □

Special case:

For the case $\alpha = \beta = 1$, we have $M = (M_H)$ for real quaternion H. M_H is a orthogonal matrix, then the map φ corresponds to a rotation in E^3. If we take the rotation axis to be $S = (s_1, s_2, s_3)$, then the Rodrigues rotation formula is

$$M_H = I_3 + S\sin\theta + (1 - \cos\theta)S^2$$

where S is a skew-symmetric matrix,

$$S = \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}.$$

Theorem 3.2. *Let q be a unit generalized quaternion. Then M is a quasi-orthogonal matrix, i.e. $M^T\varepsilon M = \varepsilon$ and $\det M = 1$ where*

$$\varepsilon = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha\beta \end{bmatrix}.$$

Corollary 3.1. *Let $q = \cos\frac{\phi}{2} + S\sin\frac{\phi}{2}$ be a unit generalized quaternion. The linear map $\varphi(\omega) = q\omega q^{-1}$ represent a rotation of the original vector ω by an angle ϕ around the axis S in 3-space $E^3_{\alpha\beta}$.*

Example 3.1. *Let $q = \frac{1}{\sqrt{2}} + \frac{1}{2}(\frac{1}{\sqrt{\alpha}}, \frac{1}{\sqrt{\beta}}, 0)$ be a unit generalized quaternion. The rotation matrix is*

$$M = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{\beta}}{2\sqrt{\alpha}} & \frac{\sqrt{\beta}}{\sqrt{2}} \\ \frac{\sqrt{\alpha}}{2\sqrt{\beta}} & \frac{1}{2} & -\frac{\sqrt{\alpha}}{\sqrt{2}} \\ -\frac{1}{\sqrt{2\beta}} & \frac{1}{\sqrt{2\alpha}} & 0 \end{bmatrix}.$$

Case 2. Let α be a positive number and β be a negative number.

Theorem 3.3. *Let q be a unit generalized quaternion and the norm of its vector part is negative. i.e. $\alpha s_1^2+\beta s_2^2+\alpha\beta s_3^2<0$, then the matrix M can be written as*

$$M_{(\phi,C)}=I_3+\sinh\phi C+(-1+\cosh\phi)C^2,$$

where is a hyperbolic angle, such that $\cosh\phi=a_0$, $\sinh\phi=\sqrt{-\alpha s_1^2-\beta s_2^2-\alpha\beta s_3^2}$ and C is a generalized skew-symmetric matrix.

Proof. Every unit generalized quaternion $q=a_0+a_1i+a_2j+a_3k$ can be written in polar form

$$q=\cosh\frac{\phi}{2}+C\sinh\frac{\phi}{2},$$

then the matrix M can be written as

$$M=$$

$$\begin{bmatrix} \cosh^2\frac{\phi}{2}+(\alpha c_1^2-\beta c_2^2-\alpha\beta c_3^2)\sinh^2\frac{\phi}{2} & 2\beta(c_1c_2\sinh^2\frac{\phi}{2}-c_3\cosh\frac{\phi}{2}\sinh\frac{\phi}{2}) & 2\beta(\alpha c_1c_3\sinh^2\frac{\phi}{2}+c_2\cosh\frac{\phi}{2}\sinh\frac{\phi}{2}) \\ 2\alpha(c_1c_2\sinh^2\frac{\phi}{2}+c_3\cosh\frac{\phi}{2}\sinh\frac{\phi}{2}) & \cosh^2\frac{\phi}{2}+(-\alpha c_1^2+\beta c_2^2-\alpha\beta c_3^2)\sinh^2\frac{\phi}{2} & 2\alpha(\beta c_2c_3\sinh^2\frac{\phi}{2}+c_1\cosh\frac{\phi}{2}\sinh\frac{\phi}{2}) \\ 2(\alpha c_1c_3\sinh^2\frac{\phi}{2}-c_2\cosh\frac{\phi}{2}\sinh\frac{\phi}{2}) & 2(\beta c_2c_3\sinh^2\frac{\phi}{2}+c_1\cosh\frac{\phi}{2}\sinh\frac{\phi}{2}) & \cosh^2\frac{\phi}{2}+(-\alpha c_1^2-\beta c_2^2+\alpha\beta c_3^2)\sinh^2\frac{\phi}{2} \end{bmatrix}$$

$$=I_3+\begin{bmatrix} (\alpha c_1^2-\beta c_2^2-\alpha\beta c_3^2)\sinh^2\frac{\phi}{2} & 2\beta c_1c_2\sinh^2\frac{\phi}{2}-\beta c_3\sinh\phi & 2\alpha\beta c_1c_3\sinh^2\frac{\phi}{2}+\beta c_2\sinh\phi \\ 2\alpha c_1c_2\sinh^2\frac{\phi}{2}+\alpha c_3\sinh\phi & (-\alpha c_1^2+\beta c_2^2-\alpha\beta c_3^2+1)\sinh^2\frac{\phi}{2} & 2\alpha\beta c_2c_3\sinh^2\frac{\phi}{2}+\alpha c_1\sinh\phi \\ 2\alpha c_1c_3\sinh^2\frac{\phi}{2}-c_2\sinh\phi & 2\beta c_2c_3\sinh^2\frac{\phi}{2}+c_1\sinh\phi & (-\alpha c_1^2-\beta c_2^2+\alpha\beta c_3^2+1)\sinh^2\frac{\phi}{2} \end{bmatrix}$$

with used of $2\sinh^2\frac{\phi}{2}=-1+\cosh\phi$ and $\alpha c_1^2+\beta c_2^2+\alpha\beta c_3^2=-1$, we have

$$M=I_3+\sinh\phi\begin{bmatrix} 0 & -\beta c_3 & \beta c_2 \\ \alpha c_3 & 0 & -\alpha c_1 \\ -c_2 & c_1 & 0 \end{bmatrix}+$$
$$+(-1+\cosh\phi)\begin{bmatrix} -\alpha\beta c_3^2-\beta c_2^2 & \beta c_1c_2 & \alpha\beta c_1c_3 \\ \alpha c_1c_2 & -\alpha c_1^2-\alpha\beta c_3^2 & \alpha\beta c_2c_3 \\ \alpha c_1c_3 & \beta c_2c_3 & -\alpha c_1^2-\beta c_2^2 \end{bmatrix},$$

so, proof is complete. □

Corollary 3.2. *Let $q=\cosh\frac{\phi}{2}+C\sinh\frac{\phi}{2}$ be a unit generalized quaternion. The linear map $\varphi(\omega)$ represents a rotation of the original vector ω by an angle ϕ around the axis C in 3-space $E^3_{\alpha\beta}$.*

Example 3.2. *Let $q=\sqrt{2}+\frac{1}{2}(0,1,1)$ be a unit generalized quaternion and $\alpha=1$, $\beta=-2$. The rotation matrix is*

$$M=\begin{bmatrix} 3 & 2\sqrt{2} & -2\sqrt{2} \\ \sqrt{2} & 2 & -1 \\ -\sqrt{2} & -1 & 2 \end{bmatrix}.$$

M is a quasi-orthogonal matrix, i.e., $M^T\varepsilon M=\varepsilon$ and $\det M=1$, where $\varepsilon=\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ and therefore it represents a rotation in 3-space $E^3_{1,-2}$. The axis of this rotation is spanned by the vector $(0,\frac{1}{2},\frac{1}{2})$ and the hyperbolic angle of rotation is 2ϕ, such taht $\cosh\phi=\sqrt{2}$ and $\sinh\phi=1$.

Theorem 3.4. *Let q be a unit generalized quaternion and the norm of its vector part is positive, i.e. $\alpha a_1^2+\beta a_2^2+\alpha\beta a_3^2>0$, then the matrix M can be written as*

$$M_{(\phi,S)} = I_3+\sin\phi S+(1-\cos\ \phi)S^2,$$

where ϕ is a hyperbolic angle, such taht $\cos\frac{\phi}{2} = a_0$, $s_i \sin\frac{\phi}{2} = \sqrt{\alpha a_1^2 + \beta a_2^2 - \alpha\beta a_3^2}$, *and S is a skew-symmetric matrix.*

Proof. The proof is similar to the proof of Theorem 3.1. □

Special case: For the case $\alpha = 1, \beta = -1$, we have $M(= M_{H'})$ for split quaternion H'. $M_{H'}$ is a semi-orthogonal matrix, then the map φ corresponds to a rotation in E_1^3.

The Rodrigues rotation formula is as follows: i. If the rotation axis $C = (c, c, c)$ is timelike,

$$M_{H'} = I_3 + \sin\theta C + (1 - \cos\theta)C^2,$$

ii. If the rotation axis $C = (c, c, c)$ is spacelike,

$$M_{H'} = I_3 + (\sinh\gamma)C + (-1 + \cosh\gamma)C^2,$$

where C is a skew-symmetric matrix, $i.e.$ $C^T = -\chi C\chi$ and

$$C = \begin{bmatrix} 0 & c_3 & -c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix}, \quad \chi = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

That is, kind of rotation matrix in the Minkowski space depends on the rotation axis.

Example 3.3. *Let* $q = \frac{\sqrt{2}}{2} + (1, 0, \frac{1}{2})$ *be a unit generalized quaternion and* $\alpha = 1, \beta = -2$. *The rotation matrix is*

$$M = \begin{bmatrix} 2 & \sqrt{2} & -2\chi \\ \frac{\sqrt{2}}{2} & 0 & -\sqrt{2} \\ 1 & \sqrt{2} & -1 \end{bmatrix}.$$

M is a quasi-orthogonal matrix, i.e., $M^T \varepsilon M = \varepsilon$ *and* $\det M = 1$ *where*

$$\varepsilon = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

and therefore it represents a rotation in 3-space $E^3_{1,-2}$. *The unit quaternion q represents rotation through an angle* 90 *about the axis* $S = \left(\frac{2}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$.

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