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SOME FAMILIES OF MITTAG-LEFFLER TYPE FUNCTIONS AND ASSOCIATED OPERATORS OF FRACTIONAL CALCULUS *(SURVEY)*

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Abstract. Our main objective in this survey-cum-expository article is essentially to present a review of some recent developments involving various classes of the Mittag-Leffler type functions which are associated with several family of generalized Riemann-Liouville and other related fractional derivative operators. Specifically, we consider various compositional properties, which are associated with the Mittag-Leffler type functions and the Hardy-type inequalities for a certain generalized fractional derivative operator. We also present solutions of many different classes of fractional differential equations with constant coefficients and variable coefficients and some general Volterra-type differintegral equations in the space of Lebesgue integrable functions as well as a number of interesting particular cases of these general solutions and certain recently investigated fractional kinetic differintegral equations.

Keywords: Riemann-Liouville, Liouville-Caputo fractional derivative operator, generalized Mittag-Leffler function, Volterra differintegral, Lebesgue integrable functions, Fox-Wright hypergeometric functions, Hurwitz-Lerch and related zeta functions.

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1. Introduction, definitions and preliminaries

During the past three decades or so, the subject of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance, which is due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. It does indeed provide several potentially useful tools for solving differential and integral equations, and various other problems involving special functions of mathematical physics as well as their extensions and generalizations in one and more variables. In a wide variety of applications of fractional calculus, one requires fractional derivatives of different (and, occasionally, *ad hoc*) kinds (see, for example, [12] to [17], [26], [27], [33], [34], [40], [44], [47], [49] and [50]). Differentiation and integration of fractional order are traditionally defined by the right-sided Riemann-Liouville fractional integral operator I^p_{a+} and the left-sided Riemann-Liouville fractional integral operator I^p_{a-}, and the corresponding Riemann-Liouville fractional derivative operators D^p_{a+} and D^p_{a-}, as follows (see, for example, [8, Chapter 13], [22, pp. 69–70] and [32]):

$$\left(I^{\mu}_{a+}f\right)(x)=\frac{1}{\Gamma(\mu)}\int_a^x \frac{f(t)}{(x-t)^{1-\mu}}\,dt \qquad \big(x>a;\ \Re(\mu)>0\big), \tag{1.1}$$

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$$\left(I_{a-}^{\mu}f\right)(x)=\frac{1}{\Gamma(\mu)}\int_x^a\frac{f(t)}{(t-x)^{1-\mu}}\,dt\qquad\left(x<a;\ \Re(\mu)>0\right)\tag{1.2}$$

and

$$\left(D_{a\pm}^{\mu}f\right)(x)=\left(\pm\frac{d}{dx}\right)^n\left(I_{a\pm}^{n-\mu}f\right)(x)\qquad\left(\Re(\mu)\geq 0;\ n=[\Re(\mu)]+1\right),\tag{1.3}$$

where the function f is locally integrable, $\Re(\mu)$ denotes the real part of the complex number $\mu\in\mathbb{C}$ and $[\Re(\mu)]$ means the greatest integer in $\Re(\mu)$.
An interesting family of generalized Riemann-Liouville fractional derivatives of order α $(0<\alpha<1)$ and type β $(0\leq\beta\leq 1)$ were introduced recently as follows (see [12], [13] and [14]; see also [16], [17] and [33]).

Definition 1.1. *The right-sided fractional derivative* $D_{a+}^{\alpha,\beta}$ *and the left-sided fractional derivative* $D_{a-}^{\alpha,\beta}$ *of order* α $(0<\alpha<1)$ *and type* β $(0\leq\beta\leq 1)$ *with respect to* x *are defined by*

$$\left(D_{a\pm}^{\alpha,\beta}f\right)(x)=\left(\pm I_{a\pm}^{\beta(1-\alpha)}\frac{d}{dx}\left(I_{a\pm}^{(1-\beta)(1-\alpha)}f\right)\right)(x),\tag{1.4}$$

where it is tacitly assumed that the second member of (1.4) *exists. Obviously, this generalization* (1.4) *yields the classical Riemann-Liouville fractional derivative operator when* $\beta=0$*. Moreover, for* $\beta=1$*, it leads to the fractional derivative operator introduced by Liouville* [24, p. 10]*, which is often attributed to Caputo now-a-days and which should more appropriately be referred to as the Liouville-Caputo fractional derivative. Many authors (see, for example,* [26] *and* [49]*) called the general operators in* (1.4) *the Hilfer fractional derivative operators. Several applications of the Hilfer fractional derivative operator* $D_{a\pm}^{\alpha,\beta}$ *can indeed be found in* [14].

In this survey-cum-expository article, we aim mainly at presenting a brief review of interesting and potentially useful properties of the aforementioned family of generalized Riemann-Liouville fractional derivative operators $D_{a\pm}^{\alpha,\beta}$ of order α and type β (see Definition 1.1 above). In particular, we consider various compositional properties, which are associated with the Mittag-Leffler type functions and the Hardy-type inequalities for the generalized fractional derivative operator $D_{a\pm}^{\alpha,\beta}$. By applying some techniques based upon the Laplace transform, we present solutions of many different classes of fractional differential equations with constant coefficients and variable coefficients and some general Volterra-type differintegral equations in the space of Lebesgue integrable functions. We also include various special cases of these general solutions and a brief discussion about some recently-investigated fractional kinetic equations.

First of all, by using the formulas (1.1) and (1.2) in conjunction with (1.3) when $n=1$, the fractional derivative operator $D_{a\pm}^{\alpha,\beta}$ can be rewritten in the following form:

$$\left(D_{a\pm}^{\alpha,\beta}f\right)(x)=\left(\pm I_{a\pm}^{\beta(1-\alpha)}\left(D_{a\pm}^{\alpha+\beta-\alpha\beta}f\right)\right)(x).\tag{1.5}$$

The significant difference between fractional derivatives of *different* types would become apparent from a closer look at their Laplace transformations. For example, it is found for $0<\alpha<1$ that (see [12], [13] and [49])

$$\mathcal{L}\left[\left(D_{0+}^{\alpha,\beta}f\right)(x)\right](s)=s^{\alpha}\mathcal{L}[f(x)](s)-s^{\beta(\alpha-1)}\left(I_{0+}^{(1-\beta)(1-\alpha)}f\right)(0+),\tag{1.6}$$

$$(0<\alpha<1),$$

where

$$\left(I_{0+}^{(1-\beta)(1-\alpha)}f\right)(0+)$$

is the Riemann-Liouville fractional integral of order $(1-\beta)(1-\alpha)$ evaluated in the limit as $t \to 0+$, it being understood (as usual) that (see, for details, [7, Chapters 4 and 5])

$$\mathcal{L}[f(x)](s) := \int_0^\infty e^{-sx} f(x)\, dx =: F(s), \tag{1.7}$$

provided that the defining integral in (1.7) exists.

We now turn to the familiar Mittag-Leffler functions $E_\mu(z)$ and $E_{\mu,\nu}(z)$ which are defined (as usual) by means of the following series:

$$E_\mu(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu n+1)} =: E_{\mu,1}(z) \qquad \big(z \in \mathbb{C};\ \Re(\mu) > 0\big) \tag{1.8}$$

and

$$E_{\mu,\nu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu n+\nu)} \qquad \big(z, \nu \in \mathbb{C};\ \Re(\mu) > 0\big), \tag{1.9}$$

respectively. The Mittag-Leffler functions $E_\mu(z)$ and $E_{\mu,\nu}(z)$ are *natural* extensions of the exponential, hyperbolic and trigonometric functions, since it is easily verified that

$$E_1(z) = e^z, \quad E_2\left(z^2\right) = \cosh z, \quad E_2\left(-z^2\right) = \cos z,$$

$$E_{1,2}(z) = \frac{e^z - 1}{z} \quad \text{and} \quad E_{2,2}(z^2) = \frac{\sinh z}{z}.$$

For a reasonably detailed account of the various properties, generalizations and applications of the Mittag-Leffler functions $E_\mu(z)$ and $E_{\mu,\nu}(z)$, the reader may refer to the recent works by (for example) Gorenflo *et al.* [9], Haubold *et al.* [11] and Kilbas *et al.* ([20], [21] and [22, Chapter 1]). The Mittag-Leffler function $E_\mu(z)$ given by (1.8) and some of its various generalizations have only recently been calculated numerically in the whole complex plane (see, for example, [18] and [36]). By means of the series representation, a generalization of the Mittag-Leffler function $E_{\mu,\nu}(z)$ of (1.9) was introduced by Prabhakar [31] as follows:

$$E^\lambda_{\mu,\nu}(z) = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{\Gamma(\mu n+\nu)} \frac{z^n}{n!} \qquad \big(z, \nu, \lambda \in \mathbb{C};\ \Re(\mu) > 0\big), \tag{1.10}$$

where (*and throughout our presentation*) $(\lambda)_\nu$ denotes the familiar Pochhammer symbol or the *shifted* factorial, since

$$(1)_n = n! \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ \mathbb{N} := \{1, 2, 3, \cdots\}),$$

defined (for $\lambda, \nu \in \mathbb{C}$ and in terms of the familiar Gamma function) by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases}$$

it being assumed *conventionally* that $(0)_0 := 1$. Clearly, we have the following special cases:

$$E^1_{\mu,\nu}(z) = E_{\mu,\nu}(z) \qquad \text{and} \qquad E^1_{\mu,1}(z) = E_\mu(z). \tag{1.11}$$

Indeed, as already observed earlier by Srivastava and Saxena [47, p. 201, Equation (1.6)], the generalized Mittag-Leffler function $E^\lambda_{\mu,\nu}(z)$ itself is actually a very specialized case of a rather extensively investigated function ${}_p\Psi_q$ as indicated below (see also [22, p. 45, Equation (1.9.1)]):

$$E^\lambda_{\mu,\nu}(z) = \frac{1}{\Gamma(\lambda)}\, {}_1\Psi_1 \left[\begin{array}{c} (\lambda, 1); \\ \\ (\nu, \mu); \end{array} z \right], \tag{1.12}$$

where, *and in what follows*, ${}_p\Psi_q\ \ (p,q\in\mathbb{N}_0)$ or ${}_p\Psi_q^*\ \ (p,q\in\mathbb{N}_0)$ denotes the the Fox-Wright) generalization of the relatively more familiar hypergeometric function ${}_pF_q\ \ (p,q\in\mathbb{N}_0)$, with p numerator parameters $a_1,\ \cdots,a_p$ and q denominator parameters $b_1,\ \cdots,b_q$ such that

$$a_j\in\mathbb{C}\quad (j=1,\ \cdots,p)\qquad\text{and}\qquad b_j\in\mathbb{C}\setminus\mathbb{Z}_0^-\quad (j=1,\ \cdots,q),$$

which is defined by (see, for details, [7, p. 183] and [46, p. 21]; see also [22, p. 56], [19, p. 65] and [45, p. 19])

$$\begin{aligned}{}_p\Psi_q^*&\left[\begin{array}{c}(a_1,A_1),\ \cdots,(a_p,A_p);\\ \\ (b_1,B_1),\ \cdots,(b_q,B_q);\end{array}\ z\right]\\ &:=\sum_{n=0}^{\infty}\frac{(a_1)_{A_1n}\ \cdots(a_p)_{A_pn}}{(b_1)_{B_1n}\ \cdots(b_q)_{B_qn}}\ \frac{z^n}{n!}\\ &=\frac{\Gamma(b_1)\ \cdots\Gamma(b_q)}{\Gamma(a_1)\ \cdots\Gamma(a_p)}\ {}_p\Psi_q\left[\begin{array}{c}(a_1,A_1),\ \cdots,(a_p,A_p);\\ \\ (b_1,B_1),\ \cdots,(b_q,B_q);\end{array}\ z\right]\end{aligned}\tag{1.13}$$

$$\left(\Re(A_j)>0\ \ (j=1,\cdots,p);\ \Re(B_j)>0\ \ (j=1,\cdots,q);\ 1+\Re\left(\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j\right)\geqq 0\right),$$

where we have assumed, in general, that

$$a_j,A_j\in\mathbb{C}\quad (j=1,\cdots,p)\qquad\text{and}\qquad b_j,B_j\in\mathbb{C}\quad (j=1,\cdots,q)$$

and that the equality in the convergence condition holds true only for suitably bounded values of $|z|$ given by

$$|z|<\nabla:=\left(\prod_{j=1}^{p}A_j^{-A_j}\right)\cdot\left(\prod_{j=1}^{q}B_j^{B_j}\right).$$

Various special higher transcendental functions of the Mittag-Leffler and the Fox-Wright types type are known to play an important rôle in the theory of fractional and operational calculus and their applications in the basic processes of evolution, relaxation, diffusion, oscillation, and wave propagation. Just as we have remarked above, the Mittag-Leffler type functions have only recently been calculated numerically in the whole complex plane (see, for example, [18] and [36]; see also [1] and [29]). Furthermore, several general families of Mittag-Leffler type functions were investigated and applied recently by Srivastava and Tomovski [49]). Clearly, therefore, we can deduce the following relationships with the Mittag-Leffler type function $E_{\kappa,\nu}^{(a)}(s;z)$ of Barnes [5]:

$$E_\alpha(z)=\lim_{s\to 0}\left\{E_{\alpha,1}^{(a)}(s;z)\right\}\qquad\text{and}\qquad E_{\alpha,\beta}(z)=\lim_{s\to 0}\left\{E_{\alpha,\beta}^{(a)}(s;z)\right\}.\tag{1.14}$$

More interestingly, we have

$$\lim_{\kappa\to 0}\left\{E_{\kappa,1}^{(a)}(s;z)\right\}=\frac{1}{\Gamma(\nu)}\ \Phi(z,s,a)$$

in terms of the classical Hurwitz-Lerch zeta function $\Phi(z,s,a)$ defined by (*cf.*, *e.g.*, [7, p. 27. Eq. 1.11 (1)]; see also [41, p. 121, *et seq.*])

$$\Phi(z,s,a):=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s}\tag{1.15}$$

$$\left(a\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ s\in\mathbb{C}\ \text{when}\ \ |z|<1;\ \Re(s)>1\ \text{when}\ \ |z|=1\right).$$

The Hurwitz-Lerch zeta function $\Phi(z,s,a)$ defined by (1.15) contains, as its *special* cases, not only the Riemann zeta function $\zeta(s)$ and the Hurwitz (or generalized) zeta function $\zeta(s,a)$:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \Phi(1,s,1) \qquad \text{and} \qquad \zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} = \Phi(1,s,a) \tag{1.16}$$

and the Lerch zeta function $\ell_s(\xi)$ defined by (see, for details, [7, Chapter I] and [41, Chapter 2])

$$\ell_s(\xi) := \sum_{n=1}^{\infty} \frac{e^{2n\pi i\xi}}{n^s} = e^{2\pi i\xi}\ \Phi\left(e^{2\pi i\xi}, s, 1\right) \tag{1.17}$$

$$(\xi \in \mathbb{R};\ \Re(s) > 1),$$

but also such other important functions of *Analytic Number Theory* as the Polylogarithmic function (or *de Jonquière's function*) $\mathrm{Li}_s(z)$:

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\ \Phi(z,s,1) \tag{1.18}$$

$$\left(s \in \mathbb{C} \quad \text{when} \quad |z| < 1;\ \Re(s) > 1 \quad \text{when} \quad |z| = 1\right)$$

and the Lipschitz-Lerch zeta function (*cf.* [41, p. 122, Eq. 2.5 (11)]):

$$\phi(\xi,a,s) := \sum_{n=0}^{\infty} \frac{e^{2n\pi i\xi}}{(n+a)^s} = \Phi\left(e^{2\pi i\xi}, s, a\right) =: L(\xi,s,a) \tag{1.19}$$

$$\left(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \Re(s) > 0 \quad \text{when} \quad \xi \in \mathbb{R} \setminus \mathbb{Z};\ \Re(s) > 1 \quad \text{when} \quad \xi \in \mathbb{Z}\right),$$

which was first studied by Rudolf Lipschitz (1832-1903) and Matyáš Lerch (1860-1922) in connection with Dirichlet's famous theorem on primes in arithmetic progressions.

Asymptotic expansions of the Mittag-Leffler type function $E_{\kappa,\nu}^{(a)}(s;z)$ and the classical Mittag-Leffler function $E_\alpha(z)$ defined by (1.8) are discussed by Barnes [5]. Moreover, as already pointed out categorically by Srivastava *et al.* [48, p. 503, Eq. (6.3)], the following generalized M-series was introduced recently by Sharma and Jain [37] by

$$\begin{aligned} {}_{p}\overset{\alpha,\beta}{M}_{q}(a_1,\cdots,a_p;b_1,\cdots,b_q;z) &= \sum_{k=0}^{\infty} \frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\ \frac{z^k}{\Gamma(\alpha k+\beta)} \\ &= \frac{1}{\Gamma(\beta)}\ {}_{p+1}\Psi_{q+1}^{*}\left[\begin{array}{l} (a_1,1),\ \cdots,(a_p,1),(1,1); \\ \\ (b_1,1),\ \cdots,(b_q,1),(\beta,\alpha); \end{array} z\right], \end{aligned} \tag{1.20}$$

in which the last relationship exhibits the fact that the so-called generalized M-series is, in fact, an *obvious* (rather trivial) variant of the Fox-Wright function ${}_p\Psi_q^*$ defined by (1.13).

A natural unification and generalization of the Fox-Wright function ${}_p\Psi_q^*$ defined by (1.13) as well as the Hurwitz-Lerch zeta function $\Phi(z,s,a)$ defined by (1.15) was indeed accomplished by introducing essentially arbitrary numbers of numerator and denominator parameters in the definition (1.15). For this purpose, in addition to the symbol ∇^* defined by

$$\nabla^* := \left(\prod_{j=1}^{p} \rho_j^{-\rho_j}\right) \cdot \left(\prod_{j=1}^{q} \sigma_j^{\sigma_j}\right), \tag{1.21}$$

the following notations will be employed:

$$\Delta := \sum_{j=1}^{q} \sigma_j - \sum_{j=1}^{p} \rho_j \qquad \text{and} \qquad \Xi := s + \sum_{j=1}^{q} \mu_j - \sum_{j=1}^{p} \lambda_j + \frac{p-q}{2}. \tag{1.22}$$

Then the extended Hurwitz-Lerch zeta function

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a)$$

is defined by [48, p. 503, Equation (6.2)] (see also [38] and [42])

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a) := \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\lambda_j)_{n\rho_j}}{n! \cdot \prod_{j=1}^{q} (\mu_j)_{n\sigma_j}} \frac{z^n}{(n+a)^s} \tag{1.23}$$

$$\Big(p,q \in \mathbb{N}_0;\ \lambda_j \in \mathbb{C}\ (j=1,\cdots,p);\ a,\mu_j \in \mathbb{C} \setminus Z_0^-\ (j=1,\cdots,q);$$

$$\rho_j, \sigma_k \in \mathbb{R}^+\ (j=1,\cdots,p;\ k=1,\cdots,q); \Delta > -1;$$

when $s,z \in \mathbb{C};\ \Delta = -1$ and $s \in \mathbb{C}$ when $|z| < \nabla^*;\ \Delta = -1$ and $\Re(\Xi) > \frac{1}{2}$; when $|z| = \nabla^* \Big)$.

For an interesting and potentially useful family of λ-generalized Hurwitz-Lerch zeta functions, which *further* extend the multi-parameter Hurwitz-Lerch zeta function

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a)$$

defined by (1.23), was introduced and investigated systematically in a recent paper by Srivastava [39], who also discussed their potential application in Number Theory by appropriately constructing a presumably new continuous analogue of Lippert's Hurwitz measure and also considered some other statistical applications of these families of the λ-generalized Hurwitz-Lerch zeta functions in probability distribution theory (see also the references to several related earlier works cited by Srivastava [39]).

Remark 1.1. *If we set*

$$s = 0, \quad p \mapsto p+1 \qquad (\rho_1 = \cdots = \rho_p = 1;\ \lambda_{p+1} = \rho_{p+1} = 1)$$

and

$$q \mapsto q+1 \qquad (\sigma_1 = \cdots = \sigma_q = 1;\ \mu_{q+1} = \beta;\ \sigma_{q+1} = \alpha),$$

then (1.23) *reduces at once to the M-series in* (1.20).

Finally, we recall that a Laplace transform formula for the generalized Mittag-Leffler function $E^{\lambda}_{\mu,\nu}(z)$ was given by Prabhakar [31] as follows:

$$\mathcal{L}\left[x^{\nu-1} E^{\lambda}_{\mu,\nu}(\omega x^{\mu})\right](s) = \frac{s^{\lambda\mu-\nu}}{(s^{\mu}-\omega)^{\lambda}} \tag{1.24}$$

$$\left(\lambda,\mu,\omega \in \mathbb{C};\ \Re(\nu) > 0;\ \Re(s) > 0;\ \left|\frac{\omega}{s^{\mu}}\right| < 1\right).$$

Prabhaker [31] also introduced the following fractional integral operator:

$$\left(\mathbf{E}^{\lambda}_{\mu,\nu,\omega;a+}\varphi\right)(x) = \int_a^x (x-t)^{\nu-1} E^{\lambda}_{\mu,\nu}\big(\omega(x-t)^{\mu}\big)\varphi(t)\ dt \qquad (x > a) \tag{1.25}$$

in the space $L(\mathfrak{a},\mathfrak{b})$ of Lebesgue integrable functions on a finite closed interval $[\mathfrak{a},\mathfrak{b}]$ $(\mathfrak{b} > \mathfrak{a})$ of the real line $\mathbb{R}$ given by

$$L(\mathfrak{a},\mathfrak{b}) = \left\{f:\ \|f\|_1 = \int_{\mathfrak{a}}^{\mathfrak{b}} |f(x)|\,dx < \infty\right\}, \tag{1.26}$$

it being *tacitly* assumed (*throughout the present investigation*) that, in situations such as those occurring in (1.25) and in conjunction with the usages of the definitions in (1.3), (1.4) and (1.5),

$\mathfrak{a}$ in all such function spaces as (for example) the function space $L(\mathfrak{a},\mathfrak{b})$ coincides precisely with the *lower* terminal a in the integrals involved in the definitions (1.3), (1.4) and (1.5).

The fractional integral operator (1.25) was investigated earlier by Kilbas *et al.* [20] and its generalization involving a family of more general Mittag-Leffler type functions than $E^{\lambda}_{\mu,\nu}(z)$ was studied recently by Srivastava and Tomovski [49].

2. The Mittag-Leffler type functions: basic properties and relationships

Here, in this section, we present several continuity properties of the generalized fractional derivative operator $D^{\alpha,\beta}_{a+}$. Each of the following results (Lemma 2.1 as well as Theorems 2.1 and 2.2) are easily derivable by suitably specializing the corresponding general results proven recently by Srivastava and Tomovski [49].

Lemma 2.1. (see [49]). *The following fractional derivative formula holds true*:

$$\left(D^{\alpha,\beta}_{a+}\left[(t-a)^{\nu-1}\right]\right)(x)=\frac{\Gamma(\nu)}{\Gamma(\nu-\alpha)}(x-a)^{\nu-\alpha-1} \tag{2.1}$$

$$\big(x>a;\ 0<\alpha<1;\ 0\leq\beta\leq1;\ \Re(\nu)>0\big).$$

Theorem 2.1. (see [49]).*The following relationship holds true*:

$$\left(D^{\alpha,\beta}_{a+}\left[(t-a)^{\nu-1}E^{\lambda}_{\mu,\nu}\left[\omega(t-a)^{\mu}\right]\right]\right)(x)=(x-a)^{\nu-\alpha-1}E^{\lambda}_{\mu,\nu-\alpha}\left[\omega(x-a)^{\mu}\right] \tag{2.2}$$

$$\big(x>a;\ 0<\alpha<1;\ 0\leq\beta\leq1;\ \lambda,\omega\in\mathbb{C};\ \Re(\mu)>0;\ \Re(\nu)>0\big).$$

Theorem 2.2. (see [49]). *The following relationship holds true for any Lebesgue integrable function* $\varphi\in L(\mathfrak{a},\mathfrak{b})$:

$$D^{\alpha,\beta}_{a+}\left(\mathbf{E}^{\lambda}_{\mu,\nu,\omega;a+}\varphi\right)=\mathbf{E}^{\lambda}_{\mu,\nu-\alpha,\omega;a+}\varphi \tag{2.3}$$

$$\big(x>a\ (a=\mathfrak{a});\ 0<\alpha<1;\ 0\leq\beta\leq1;\ \lambda,\omega\in\mathbb{C};\ \Re(\mu)>0;\ \Re(\nu)>0\big).$$

In addition to the space $L(\mathfrak{a},\mathfrak{b})$ given by (1.26), we shall need the weighted L^p-space

$$X^p_{\mathfrak{c}}(\mathfrak{a},\mathfrak{b})\qquad(\mathfrak{c}\in\mathbb{R};\ 1\leq p\leq\infty),$$

which consists of those complex-valued Lebesgue integrable functions f on $(\mathfrak{a},\mathfrak{b})$ for which

$$\|f\|_{X^p_{\mathfrak{c}}}<\infty$$

with

$$\|f\|_{X^p_{\mathfrak{c}}}=\left(\int_{\mathfrak{a}}^{\mathfrak{b}}|t^{\mathfrak{c}}f(t)|^p\,\frac{dt}{t}\right)^{1/p}\qquad(1\leq p<\infty).$$

In particular, when

$$\mathfrak{c}=\frac{1}{p},$$

the space $X^p_{\mathfrak{c}}(\mathfrak{a},\mathfrak{b})$ coincides with the $L^p(\mathfrak{a},\mathfrak{b})$-space, that is,

$$X^p_{1/p}(\mathfrak{a},\mathfrak{b})=L^p(\mathfrak{a},\mathfrak{b}).$$

We also introduce here a suitable *fractional* Sobolev space $W^{\alpha,p}_{a+}(\mathfrak{a},\mathfrak{b})$ defined, for a closed interval $[\mathfrak{a},\mathfrak{b}]$ $(\mathfrak{b}>\mathfrak{a})$ in $\mathbb{R}$, by

$$W^{\alpha,p}_{a+}(\mathfrak{a},\mathfrak{b})=\left\{f:f\in L^p(\mathfrak{a},\mathfrak{b})\quad\text{and}\quad D^{\alpha}_{a+}f\in L^p(\mathfrak{a},\mathfrak{b})\quad(0<\alpha\leq1)\right\},$$

where $D_{a+}^{\alpha}f$ denotes the *fractional* derivative of f of order α $(0 < \alpha \leq 1)$ Alternatively, in Theorems 2.3 and 2.4 below, we can make use of a suitable p-variant of the space $\mathbf{L}_{a+}^{\alpha}(\mathfrak{a},\mathfrak{b})$ which was defined, for $\Re(\alpha) > 0$, by Kilbas *et al.* [22, p. 144, Equation (3.2.1)]:

$$\mathbf{L}_{a+}^{\alpha}(\mathfrak{a},\mathfrak{b}) = \left\{ f : f \in L(\mathfrak{a},\mathfrak{b}) \quad \text{and} \quad D_{a+}^{\alpha}f \in L(\mathfrak{a},\mathfrak{b}) \quad \big(\Re(\alpha) > 0\big) \right\}.$$

See also the notational convention mentioned in connection with (1.26).

Theorem 2.3. (see [50]). *For $0 < \alpha < 1$ and $0 < \beta < 1$, the operator $D_{a+}^{\alpha,\beta}$ is bounded in the space $W_{a+}^{\alpha+\beta-\alpha\beta,1}(\mathfrak{a},\mathfrak{b})$ and*

$$\left\| D_{a+}^{\alpha,\beta} \right\|_1 \leq A \left\| D_{a+}^{\alpha+\beta-\alpha\beta} \right\|_1 \qquad \left(A := \frac{(b-a)^{\beta(1-\alpha)}}{\beta(1-\alpha)\,\Gamma[\beta(1-\alpha)]} \right). \tag{2.4}$$

Proof. By applying a known result [32, Equation (2.72)], we find that

$$\begin{aligned} \left\| D_{a+}^{\alpha,\beta}\varphi \right\|_1 &= \left\| I_{a+}^{\beta(1-\alpha)} \left(D_{a+}^{\alpha+\beta-\alpha\beta}\varphi \right) \right\|_1 \\ &\leq \frac{(b-a)^{\beta(1-\alpha)}}{\beta(1-\alpha)\,\Gamma[\beta(1-\alpha)]} \left\| D_{a+}^{\alpha+\beta-\alpha\beta}\varphi \right\|_1 . \end{aligned}$$

□

The weighted Hardy-type inequality for the integral operator I_{a+}^{α} is stated as the following lemma.

Lemma 2.2. (see [22, p. 82]). *If $1 < p < \infty$ and $\alpha > 0$, then the integral operator I_{0+}^{α} is bounded from $L^p(0,\infty)$ into $X_{1/p-\alpha}^{p}(0,\infty)$ as follows:*

$$\begin{aligned} &\left(\int_0^{\infty} x^{-\alpha p} \left| \left(I_{0+}^{\alpha} f \right)(x) \right|^p dx \right)^{1/p} \\ &\leq \frac{\Gamma(1/p')}{\Gamma(\alpha+1/p)} \left(\int_0^{\infty} |f(x)|^p dx \right)^{1/p} \qquad \left(\frac{1}{p} + \frac{1}{p'} = 1 \right). \end{aligned} \tag{2.5}$$

Applying the last two inequalities to the fractional derivative operator $D_{a+}^{\alpha,\beta}$, we get

$$\begin{aligned} &\left(\int_0^{\infty} x^{-\alpha p} \left| \left(D_{0+}^{\alpha,\beta} f \right)(x) \right|^p dx \right)^{1/p} \\ &\leq \frac{\Gamma(1/p')}{\Gamma\big(\beta(1-\alpha)+1/p\big)} \left(\int_0^{\infty} \left| \left(D_{0+}^{\alpha+\beta-\alpha\beta} f \right)(x) \right|^p dx \right)^{1/p} \qquad \left(\tfrac{1}{p} + \tfrac{1}{p'} = 1 \right). \end{aligned} \tag{2.6}$$

We thus arrive at the following result.

Theorem 2.4. (see [50]). *If $1 < p < \infty$, $0 < \alpha < 1$ and $0 \leq \beta \leq 1$, then the fractional derivative operator $D_{0+}^{\alpha,\beta}$ is bounded from $W_{0+}^{\alpha+\beta-\alpha\beta,p}(0,\infty)$ into $X_{1/p-\alpha}^{p}(0,\infty)$.*

3. Families of fractional differintegral equations with constant coefficients and variable coefficients

The eigenfunctions of the Riemann-Liouville fractional derivatives are defined as the solutions of the following fractional differential equation:

$$\left(D_{0+}^{\alpha} f \right)(x) = \lambda f(x), \tag{3.1}$$

where λ is the eigenvalue. The solution of (3.1) is given by

$$f(x) = x^{1-\alpha} E_{\alpha,\alpha}(\lambda x^{\alpha}). \tag{3.2}$$

More generally, the eigenvalue equation for the fractional derivative $D_{0+}^{\alpha,\beta}$ of order α and type β reads as follows:

$$\left(D_{0+}^{\alpha,\beta} f\right)(x) = \lambda f(x) \tag{3.3}$$

and its solution is given by (see [13, Equation (124)])

$$f(x) = x^{(1-\beta)(1-\alpha)} E_{\alpha,\alpha+\beta(1-\alpha)}\left(\lambda x^{\alpha}\right), \tag{3.4}$$

which, in the special case when $\beta = 0$, corresponds to (3.2). A second important special case of (3.3) occurs when $\beta = 1$:

$$\left(D_{0+}^{\alpha,1} f\right)(x) = \lambda f(x). \tag{3.5}$$

In this case, the eigenfunction is given by

$$f(x) = E_{\alpha}\left(\lambda x^{\alpha}\right). \tag{3.6}$$

We now divide this section in the following four subsections (see, for details, [50]).

3.1. In this subsection, we assume that

$$0 < \alpha_1 \leqq \alpha_2 < 1,\ 0 \leqq \beta_1 \leqq 1, 0 \leqq \beta_2 \leqq 1 \quad \text{and} \quad a, b, c \in \mathbb{R}$$

and consider the following fractional differential equation:

$$a\left(D_{0+}^{\alpha_1,\beta_1} y\right)(x) + b\left(D_{0+}^{\alpha_2,\beta_2} y\right)(x) + cy(x) = f(x) \tag{3.7}$$

in the space of Lebesgue integrable functions $y \in L(0,\infty)$ with the initial conditions:

$$\left(I_{0+}^{(1-\beta_i)(1-\alpha_i)} y\right)(0+) = c_i \qquad (i = 1, 2), \tag{3.8}$$

where, without any loss of generality, we assume that

$$(1-\beta_1)(1-\alpha_1) \leqq (1-\beta_2)(1-\alpha_2).$$

If $c_1 < \infty$, then

$$c_2 = 0 \quad \text{unless} \quad (1-\beta_1)(1-\alpha_1) = (1-\beta_2)(1-\alpha_2).$$

An equation of the form (3.7) was introduced in [14] for dielectric relaxation in glasses. While the Laplace transformed relaxation function and the corresponding dielectric susceptibility were found, its general solution was not given in [14]. We now proceed to find its general solution.

Theorem 3.1. (see [50]). *The fractional differential equation* (3.7) *with the initial conditions* (3.8) *has its solution in the space* $L(0,\infty)$ *given by*

$$\begin{aligned} y(x) = \frac{1}{b}\sum_{m=0}^{\infty}\left(-\frac{a}{b}\right)^m \Bigg[& ac_1 x^{(\alpha_2-\alpha_1)m+\alpha_2+\beta_1(1-\alpha_1)-1} E^{m+1}_{\alpha_2,(\alpha_2-\alpha_1)m+\alpha_2+\beta_1(1-\alpha_1)}\left(-\frac{c}{b}x^{\alpha_2}\right) \\ & + bc_2 x^{(\alpha_2-\alpha_1)m+\alpha_2+\beta_2(1-\alpha_2)-1} E^{m+1}_{\alpha_2,(\alpha_2-\alpha_1)m+\alpha_2+\beta_2(1-\alpha_2)}\left(-\frac{c}{b}x^{\alpha_2}\right) \\ & + \left(\mathbf{E}^{m+1}_{\alpha_2,(\alpha_2-\alpha_1)m+\alpha_2,-\frac{c}{b};0+} f\right)\left(-\frac{c}{b}x^{\alpha_2}\right)\Bigg]. \end{aligned} \tag{3.9}$$

Proof. Our demonstration of Theorem 5 is based upon the Laplace transform method. Indeed, if we make use of the notational convention depicted in (1.7) and the Laplace transform formula (1.6), by applying the operator $\mathcal{L}$ to both sides of (3.7), it is easily seen that

$$Y(s) = ac_1\frac{s^{\beta_1(\alpha_1-1)}}{as^{\alpha_1}+bs^{\alpha_2}+c} + bc_2\frac{s^{\beta_2(\alpha_2-1)}}{as^{\alpha_1}+bs^{\alpha_2}+c} + \frac{F(s)}{as^{\alpha_1}+bs^{\alpha_2}+c}.$$

Furthermore, since

$$\frac{s^{\beta_i(\alpha_i-1)}}{as^{\alpha_1}+bs^{\alpha_2}+c}=\frac{1}{b}\left(\frac{s^{\beta_i(\alpha_i-1)}}{s^{\alpha_2}+\frac{c}{b}}\right)\left(\frac{1}{1+\frac{a}{b}\left(\frac{s^{\alpha_1}}{s^{\alpha_2}+\frac{c}{b}}\right)}\right)=\frac{1}{b}\sum_{m=0}^{\infty}\left(-\frac{a}{b}\right)^m\frac{s^{\alpha_1 m+\beta_i\alpha_i-\beta_i}}{\left(s^{\alpha_2}+\frac{c}{b}\right)^{m+1}}$$

$$=\mathcal{L}\left[\frac{1}{b}\sum_{m=0}^{\infty}(-1)^m\left(\frac{a}{b}\right)^m x^{(\alpha_2-\alpha_1)m+\alpha_2+\beta_i(1-\alpha_i)-1}\right.$$

$$\left.\cdot E^{m+1}_{\alpha_2,(\alpha_2-\alpha_1)m+\alpha_2+\beta_i(1-\alpha_i)}\left(-\frac{c}{b}x^{\alpha_2}\right)\right]\qquad(i=1,2)$$

and

$$\frac{F(s)}{as^{\alpha_1}+bs^{\alpha_2}+c}=\frac{1}{b}\sum_{m=0}^{\infty}\left(-\frac{a}{b}\right)^m\left(\frac{s^{\alpha_1 m}}{\left(s^{\alpha_2}+\frac{c}{b}\right)^{m+1}}F(p)\right)=\mathcal{L}\left[\frac{1}{b}\sum_{m=0}^{\infty}\left(-\frac{a}{b}\right)^m\right.$$

$$\left.\cdot\left(x^{(\alpha_2-\alpha_1)m+\alpha_2-1}E^{m+1}_{\alpha_2,(\alpha_2-\alpha_1)m+\alpha_2}\left(-\frac{c}{b}x^{\alpha_2}\right)*f(x)\right)\right]$$

$$=\mathcal{L}\left[\frac{1}{b}\sum_{m=0}^{\infty}(-1)^m\left(\frac{a}{b}\right)^m\left(\mathbf{E}^{m+1}_{\alpha_2,(\alpha_2-\alpha_1)m+\alpha_2,-\frac{c}{b};0+}f\right)\left(-\frac{c}{b}x^{\alpha_2}\right)\right]$$

in terms of the Laplace convolution, by applying the inverse Laplace transform, we get the solution (3.9) asserted by Theorem 3.1. □

3.2. The next problem is to solve the fractional differential equation (3.7) in the space of Lebesgue integrable functions $y\in L(\mathfrak{a},\mathfrak{b})$ when

$$\alpha_1=\alpha_2=\alpha\qquad\text{and}\qquad\beta_1\neq\beta_2,$$

that is, the following fractional differential equation:

$$a\left(D^{\alpha,\beta_1}_{0+}y\right)(x)+b\left(D^{\alpha,\beta_2}_{0+}y\right)(x)+cy(x)=f(x)\tag{3.10}$$

under the following initial conditions:

$$\left(I^{(1-\beta_i)(1-\alpha)}_{0+}y\right)(0+)=c_i\qquad(i=1,2).\tag{3.11}$$

We now assume, without loss of generality, that $\beta_2\leq\beta_1$. If $c_1<\infty$, then

$$c_2=0\quad\text{unless}\quad\beta_1=\beta_2.$$

Corollary 3.1. (see [50]). *The fractional differential equation* (3.10) *under the initial conditions* (3.11) *has its solution in the space* $L(0,\infty)$ *given by*

$$y(x)=\left(\frac{ac_1}{a+b}\right)x^{\beta_1+\alpha(1-\beta_1)-1}E_{\alpha,\beta_1+\alpha(1-\beta_1)}\left(-\frac{c}{a+b}x^{\alpha}\right)$$

$$+\left(\frac{bc_2}{a+b}\right)x^{\beta_2+\alpha(1-\beta_2)-1}E_{\alpha,\beta_2+\alpha(1-\beta_2)}\left(-\frac{c}{a+b}x^{\alpha}\right)$$

$$+\left(\frac{1}{a+b}\right)\left(\mathbf{E}^{1}_{\alpha,1,-\frac{c}{a+b};0+}f\right)(x).\tag{3.12}$$

Proof. Our proof of Corollary 3.1 is much akin to that of Theorem 3.1. We choose to omit the details involved (see [50]). □

3.3. Let

$$0 < \alpha_1 \leq \alpha_2 \leq \alpha_3 < 1 \quad \text{and} \quad 0 \leq \beta_i \leq 1 \qquad (i = 1, 2, 3;\ a, b, c, e \in \mathbb{R}).$$

Consider the following fractional differential equation:

$$a\left(D_{0+}^{\alpha_1,\beta_1}y\right)(x) + b\left(D_{0+}^{\alpha_2,\beta_2}y\right)(x) + c\left(D_{0+}^{\alpha_3,\beta_3}y\right)(x) + ey(x) = f(x) \tag{3.13}$$

in the space of Lebesgue integrable functions $y \in L(\mathfrak{a}, \mathfrak{b})$ with the initial conditions given by

$$\left(I_{0+}^{(1-\beta_i)(1-\alpha_i)}y\right)(0+) = c_i \qquad (i = 1, 2, 3). \tag{3.14}$$

Without loss of generality, we assume that

$$(1-\beta_1)(1-\alpha_1) \leq (1-\beta_2)(1-\alpha_2) \leq (1-\beta_3)(1-\alpha_3).$$

If $c_1 < \infty$, then

$$c_2 = 0 \quad \text{unless} \quad (1-\beta_1)(1-\alpha_1) = (1-\beta_2)(1-\alpha_2)$$

and

$$c_3 = 0 \quad \text{unless} \quad (1-\beta_1)(1-\alpha_1) = (1-\beta_2)(1-\alpha_2) = (1-\beta_3)(1-\alpha_3).$$

Hilfer [14] observed that a particular case of the fractional differential equation (3.13) when

$$\alpha_1 = 1,\ \beta_i = 1 \quad (i = 1, 2, 3),\ e = 1 \quad \text{and} \quad f(x) = 0$$

describes the process of dielectric relaxation in glycerol over 12 decades in frequency.

Theorem 3.2. (see [50]). *The fractional differential equation* (3.13) *with the initial conditions* (3.14) *has its solution in the space* $L(0, \infty)$ *given by*

$$\begin{aligned} y(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{c^{m+1}} \sum_{k=0}^{m} \binom{m}{k} a^k b^{m-k} x^{(\alpha_3-\alpha_2)m+(\alpha_2-\alpha_1)k+\alpha_3-1} \\ \cdot \Big[ac_1 x^{\beta_1(1-\alpha_1)} E_{\alpha_3,(\alpha_3-\alpha_2)m+(\alpha_2-\alpha_1)k+\alpha_3+\beta_1(1-\alpha_1)}\left(-\frac{e}{c}x^{\alpha_3}\right) \\ + bc_2 x^{\beta_2(1-\alpha_2)} E_{\alpha_3,(\alpha_3-\alpha_2)m+(\alpha_2-\alpha_1)k+\alpha_3+\beta_2(1-\alpha_2)}\left(-\frac{e}{c}x^{\alpha_3}\right) \\ + cc_3 x^{\beta_3(1-\alpha_3)} E_{\alpha_3,(\alpha_3-\alpha_2)m+(\alpha_2-\alpha_1)k+\alpha_3+\beta_3(1-\alpha_3)}\left(-\frac{e}{c}x^{\alpha_3}\right)\Big] \\ + \sum_{m=0}^{\infty} \frac{(-1)^m}{c^{m+1}} \sum_{k=0}^{m} \binom{m}{k} a^k b^{m-k} \left(\mathbf{E}_{\alpha_3,(\alpha_3-\alpha_2)m+(\alpha_2-\alpha_1)k+\alpha_3}^{m+1} f\right)\left(-\frac{e}{c}x^{\alpha_3}\right). \end{aligned} \tag{3.15}$$

Proof. Making use of above-demonstrated technique based upon the Laplace and the inverse Laplace transformations once again, it is not difficult to deduce the solution (3.15) just as we did in our proof of Theorem 3.1. □

3.4. Let

$$0 < \alpha < 1 \qquad \text{and} \qquad 0 \leq \beta_i \leq 1\ \ (i = 1, 2, 3).$$

In the space of Lebesgue integrable functions $y \in L(0, \infty)$, we consider special case of the fractional differential equation (3.13) when

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha.$$

$$a\left(D_{0+}^{\alpha,\beta_1}y\right)(x) + b\left(D_{0+}^{\alpha,\beta_2}y\right)(x) + c\left(D_{0+}^{\alpha,\beta_3}y\right)(x) + ey(x) = f(x) \tag{3.16}$$

under the initial conditions:

$$\left(I_{0+}^{(1-\beta_i)(1-\alpha)}y\right)(0+) = c_i \qquad (i = 1, 2, 3). \tag{3.17}$$

We assume, without loss of generality, that $\beta_3 \leq \beta_2 \leq \beta_1$. If $c_1 < \infty$, then

$$c_2 = 0 \qquad \text{unless} \qquad \beta_1 = \beta_2$$

and

$$c_3 = 0 \quad \text{unless} \quad \beta_1 = \beta_2 = \beta_3.$$

We are thus led fairly easily to the following consequence of Theorem 3.2.

Corollary 3.2. (see [50]). *The fractional differential equation* (3.16) *with the initial conditions* (3.17) *has its solution in the space* $L(0,\infty)$ *given by*

$$\begin{aligned} y(x) = {} & \left(\frac{ac_1}{a+b+c}\right) x^{\beta_1+\alpha(1-\beta_1)-1} E_{\alpha,\beta_1+\alpha(1-\beta_1)}\left(-\frac{e}{a+b+c}x^{\alpha}\right) \\ & + \left(\frac{bc_2}{a+b+c}\right) x^{\beta_2+\alpha(1-\beta_2)-1} E_{\alpha,\beta_2+\alpha(1-\beta_2)}\left(-\frac{e}{a+b+c}x^{\alpha}\right) \\ & + \left(\mathbf{E}^1_{\alpha,1,-\frac{e}{a+b+c};0+}f\right)(x). \end{aligned} \tag{3.18}$$

Remark 3.1. *Podlubny* [30] *used the Laplace transform method in order to give an explicit solution for an arbitrary fractional linear ordinary differential equation with constant coefficients involving Riemann-Liouville fractional derivatives in series of multinomial Mittag-Leffler functions.*

3.5. Kilbas *et al.* [22] used the Laplace transform method to derive an explicit solution for the following fractional differential equation with variable coefficients:

$$x\left(D^{\alpha}_{0+}y\right)(x) = \lambda y(x) \qquad \left(x > 0,\ \lambda \in \mathbb{R};\ \alpha > 0;\ l-1 < \alpha \leq l;\ l \in \mathbb{N} \setminus \{1\}\right). \tag{3.19}$$

They proved that the differential equation (3.19) with $0 < \alpha < 1$ is solvable and that its solution is given by (see, for details, [22])

$$y(x) = cx^{\alpha-1}\ \phi\left(\alpha-1, \alpha; -\frac{\lambda}{1-\alpha}x^{\alpha-1}\right),$$

where (see *also* Section 1)

$$\phi = {}_0\Psi_1$$

is the Wright function defined by the following series [22, p. 54]:

$$\phi(\alpha,\beta;z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+\beta)} \frac{z^k}{k!} \qquad (\alpha,\beta,z \in \mathbb{C})$$

and c is an arbitrary real constant.

In the space of Lebesgue integrable functions $y \in L(0,\infty)$, we consider the following more general fractional differential equation than (3.19):

$$x\left(D^{\alpha,\beta}_{0+}y\right)(x) = \lambda y(x) \qquad (x > 0;\ \lambda \in \mathbb{R};\ 0 < \alpha < 1;\ 0 \leq \beta \leq 1)) \tag{3.20}$$

under the initial condition:

$$\left(I^{(1-\beta)(1-\alpha)}_{0+}y\right)(0+) = c_1. \tag{3.21}$$

Theorem 3.3. (see [50]). *The fractional differential equation* (3.20) *with the initial condition* (3.21) *has its solution in the space* $L(0,\infty)$ *given by*

$$y(x) = c_1\beta x^{(\alpha-1)(1-\beta)} \sum_{n=0}^{\infty} \frac{\left(\frac{\lambda}{1-\alpha}\, x^{\alpha-1}\right)^n}{n!\,(\beta-n)}\, \phi\left(\alpha-1, (\beta-n)(1-\alpha)+\alpha, \frac{\lambda}{\alpha-1}\, x^{\alpha-1}\right)$$
$$+ c_2 x^{\alpha-1}\phi\left(\alpha-1, \alpha, \frac{\lambda}{\alpha-1}\, x^{\alpha-1}\right), \tag{3.22}$$

where c_1 *and* c_2 *are arbitrary constants.*

Proof. We first apply the Laplace transform operator $\mathcal{L}$ to each member of the fractional differential equation (3.22) and use the special case $n=1$ of the following formula [7, p. 129, Entry 4.1 (6)]:

$$\frac{\partial^n}{\partial s^n}\big(\mathcal{L}[f(x)](s)\big) = (-1)^n\ \mathcal{L}[x^n f(x)](s) \qquad (n \in \mathbb{N}). \tag{3.23}$$

We thus find from (3.20) and (3.23) that

$$\frac{\partial}{\partial s}\left(s^\alpha Y(s) - c_1 s^{\beta(\alpha-1)}\right) = -\lambda Y(s),$$

which leads us to the following ordinary linear differential equation of the first order:

$$Y'(s) + \left(\frac{\alpha}{s} + \frac{\lambda}{s^\alpha}\right) Y(s) - c_1\beta(\alpha-1)\, s^{\beta(\alpha-1)-\alpha-1} = 0.$$

Its solution is given by

$$Y(s) = \frac{1}{s^\alpha}\, e^{\left(\frac{\lambda}{\alpha-1}\right)s^{1-\alpha}} \left(c_2 + c_1\beta(\alpha-1)\int_0^s x^{\beta(\alpha-1)-1} e^{-\frac{\lambda}{\alpha-1}x^{1-\alpha}}\, dx\right), \tag{3.24}$$

where c_1 and c_2 are arbitrary constants.

Upon expanding the exponential function in the integrand of (3.24) in a series, if we use term-by-term integration in conjunction with the above Laplace transform method, we eventually arrive at the solution (3.22) asserted by Theorem 3.3. □

4. Solution of Volterra type fractional differintegral equations

4.1. Recently, Al-Saqabi and Tuan [3] made use of an operational method to solve a general Volterra-type differintegral equation of the form:

$$\left(D_{0+}^{\alpha} f\right)(x) + \frac{a}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1} f(t)\ dt = g(x) \qquad \big(\Re(\alpha) > 0;\ \Re(\nu) > 0\big), \tag{4.1}$$

where $a \in \mathbb{C}$ and $g \in L(0,\mathfrak{b})$ $(\mathfrak{b} > 0)$. Here, in this subsection, we consider the following general class of differintegral equations of the Volterra type involving the generalized fractional derivative operators:

$$\left(D_{0+}^{\alpha,\mu} f\right)(x) + \frac{a}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1} f(t)\ dt = g(x) \tag{4.2}$$
$$\big(0 < \alpha < 1;\ 0 \leq \mu \leq 1;\ \Re(\nu) > 0\big)$$

in the space of Lebesgue integrable functions $f \in L(0,\infty)$ with the initial condition:

$$\left(I_{0+}^{(1-\mu)(1-\alpha)} f\right)(0+) = c. \tag{4.3}$$

Theorem 4.1. (see [50]). *The fractional differintegral equation* (4.2) *with the initial condition* (4.3) *has its solution in the space* $L(0,\infty)$ *given by*

$$f(x) = cx^{\alpha-\mu(\alpha-1)-1}E_{\alpha+\nu,\alpha-\mu(\alpha-1)}\left(-ax^{\alpha+\nu}\right) + \left(\mathbf{E}^{1}_{\alpha+\nu,\alpha,-a;0+}g\right)(x), \tag{4.4}$$

where c *is an arbitrary constant.*

Proof. By applying the Laplace transform operator $\mathcal{L}$ to both sides of (4.2) and using the formula (1.6), we readily get

$$F(s) = c\,\frac{s^{\mu(\alpha-1)+\nu}}{s^{\alpha+\nu}+a} + \frac{s^{\nu}}{s^{\alpha+\nu}+a}\,G(s),$$

which, in view of the Laplace transform formula (1.24) and Laplace convolution theorem, yields

$$F(s) = c\mathcal{L}\left[x^{\alpha-\mu(\alpha-1)-1}E_{\alpha+\nu,\alpha-\mu(\alpha-1)}\left(-ax^{\alpha+\nu}\right)\right](s)$$
$$+\,\mathcal{L}\left[\left(x^{\alpha-1}E_{\alpha+\nu,\alpha}\left(-ax^{\alpha+\nu}\right)\right)*g(x)\right](s).$$

The solution (4.4) asserted by Theorem 4.1 would now follow by appealing to the inverse Laplace transform to each member of this last equation. □

We next consider some interesting illustrative examples of the solution given by (4.4).

Example 1. If we put

$$g(x) = x^{\mu-1}$$

in Theorem 4.1 and apply the special case of the following integral formula when $\gamma = 1$ (see [15]):

$$\int_0^x (x-t)^{\alpha-1}\,E^{\gamma}_{\rho,\alpha}\big(\omega(x-t)^{\rho}\big)t^{\mu-1}\,dt = \Gamma(\mu)\,x^{\alpha+\mu-1}E^{\gamma}_{\rho,\alpha+\mu}(\omega x^{\rho}), \tag{4.5}$$

we can deduce a particular case of the solution (4.4) given by Corollary 4.1.

Corollary 4.1. (see [50]). *The following fractional differintegral equation*:

$$\left(D^{\alpha,\mu}_{0+}f\right)(x) + \frac{a}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}f(t)\,dt = x^{\mu-1} \tag{4.6}$$

$$\big(0<\alpha<1;\ 0\leq\mu\leq1;\ \Re(\nu)>0\big)$$

with the initial condition (4.3) *has its solution in the space* $L(0,\infty)$ *given by*

$$f(x) = x^{\alpha-\mu(\alpha-1)-1}\left[cE_{\alpha+\nu,\alpha-\mu(\alpha-1)}\left(-ax^{\alpha+\nu}\right) + \Gamma(\mu)\,x^{\alpha\mu}E_{\alpha+\nu,\alpha+\mu}\left(-ax^{\alpha+\nu}\right)\right]. \tag{4.7}$$

Example 2. If, in Theorem 8, we put

$$g(x) = x^{\mu-1}E_{\alpha+\nu,\mu}\left(-ax^{\alpha+\nu}\right)$$

and apply the special case of the following integral formula when $\gamma = \sigma = 1$ (see [49]):

$$\int_0^x (x-t)^{\mu-1}\,E^{\gamma}_{\rho,\mu}\big(\omega(x-t)^{\rho}\big)t^{\nu-1}E^{\sigma}_{\rho,\nu}(\omega t^{\rho})\,dt = x^{\mu+\nu-1}E^{\gamma+\sigma}_{\rho,\mu+\nu}(\omega x^{\rho}), \tag{4.8}$$

we get another particular case of the solution (4.4) given by Corollary 4.2 below.

Corollary 4.2. (see [50]). *The following fractional differintegral equation*:

$$\left(D^{\alpha,\mu}_{0+}f\right)(x) + \frac{a}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}f(t)\,dt = x^{\mu-1}E_{\alpha+\nu,\mu}\left(-ax^{\alpha+\nu}\right) \tag{4.9}$$

$$\big(0<\alpha<1;\ 0\leq\mu\leq1;\ \Re(\nu)>0\big)$$

with the initial condition (4.3) *has its solution in the space* $L(0,\infty)$ *given by*

$$f(x) = x^{\alpha-\mu(\alpha-1)-1}\left[cE_{\alpha+\nu,\alpha-\mu(\alpha-1)}\left(-ax^{\alpha+\nu}\right) + x^{\alpha\mu}E^{2}_{\alpha+\nu,\alpha+\mu}\left(-ax^{\alpha+\nu}\right)\right], \tag{4.10}$$

where c is an arbitrary constant.

Example 3. If we put

$$g(x) = x^{\beta+\nu-1} E_{\alpha+\nu,\beta+\nu}\left(-bx^{\alpha+\nu}\right)$$

and apply the following integral formula (see [50]):

$$\int_0^x (x-t)^{\alpha-1} E_{\alpha+\nu,\alpha}\left(-a(x-t)^{\alpha+\nu}\right) t^{\beta+\nu-1} E_{\alpha+\nu,\beta+\nu}\left(-bt^{\alpha+\nu}\right) dt$$
$$= \frac{E_{\alpha+\nu,\beta}\left(-bx^{\alpha+\nu}\right) - E_{\alpha+\nu,\beta}\left(-ax^{\alpha+\nu}\right)}{a-b}\, x^{\beta-1} \qquad (a \neq b), \tag{4.11}$$

we get yet another particular case of the solution (4.4) given by Corollary 4.3.

Corollary 4.3. (see [50]). *The following fractional differintegral equation*:

$$\left(D_{0+}^{\alpha,\mu} f\right)(x) + \frac{a}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)\, dt = x^{\beta+\nu-1} E_{\alpha+\nu,\beta+\nu}\left(-bx^{\alpha+\nu}\right) \tag{4.12}$$

$$\big(0 < \alpha < 1;\ 0 \leq \mu \leq 1;\ \Re(\nu) > 0\big)$$

with the initial condition (4.3) *has its solution in the space* $L(0,\infty)$ *given by*

$$f(x) = cx^{\alpha-\mu(\alpha-1)-1} E_{\alpha+\nu,\alpha-\mu(\alpha-1)}\left(-ax^{\alpha+\nu}\right)$$
$$+ \frac{E_{\alpha+\nu,\beta}\left(-bx^{\alpha+\nu}\right) - E_{\alpha+\nu,\beta}\left(-ax^{\alpha+\nu}\right)}{a-b}\, x^{\beta-1} \qquad (a \neq b), \tag{4.13}$$

where c is an arbitrary constant.

4.2. Kilbas *et al.* [18] established the explicit solution of the Cauchy-type problem for the following fractional differential equation:

$$\left(D_{a+}^{\alpha} f\right)(x) = \lambda \left(\mathbf{E}_{\rho,\alpha,\nu;a+}^{\gamma} f\right)(x) + g(x) \qquad (a \in \mathbb{C};\ g \in L[0,b]) \tag{4.14}$$

in the space of Lebesgue integrable functions $f \in L(0,\infty)$ with the initial conditions:

$$\left(D_{a+}^{\alpha} f\right)(a+) = b_k \qquad \big(b_k \in \mathbb{C} \quad (k = 1, \cdots, n)\big). \tag{4.15}$$

We here consider the following more general Volterra-type fractional differintegral equation:

$$\left(D_{0+}^{\alpha,\mu} f\right)(x) = \lambda \left(\mathbf{E}_{\rho,\alpha,\nu;0+}^{\gamma} f\right)(x) + g(x) \tag{4.16}$$

in the space of Lebesgue integrable functions $f \in L(0,\infty)$ with initial condition:

$$\left(I_{0+}^{(1-\mu)(1-\alpha)} f\right)(0+) = c. \tag{4.17}$$

Theorem 4.2. (see [50]). *The fractional differintegral equation* (4.16) *with the initial condition* (4.17) *has its solution in the space* $L(0,\infty)$ *given by*

$$f(x) = c \sum_{k=0}^{\infty} \lambda^k x^{2\alpha k+\alpha+\mu-\mu\alpha-1} E_{\rho,2\alpha k+\alpha+\mu-\mu\alpha}^{\gamma k}\left(\nu x^{\rho}\right)$$
$$+ \sum_{k=0}^{\infty} \lambda^k \left(\mathbf{E}_{\rho,2\alpha k+\alpha,\nu;0+}^{\gamma k} g\right)(x), \tag{4.18}$$

where c is an arbitrary constant.

Proof. By taking the Laplace transforms on both sides of (4.16), we get

$$F(s) = c\,\frac{s^{\mu(\alpha-1)}}{s^{\alpha} - \lambda\left[\frac{s^{\rho\gamma-\alpha}}{(s^{\rho}-\nu)^{\gamma}}\right]} + \frac{G(s)}{s^{\alpha} - \lambda\left[\frac{s^{\rho\gamma-\alpha}}{(s^{\rho}-\nu)^{\gamma}}\right]}. \tag{4.19}$$

On the other hand, in light of (1.24), it is not difficult to see that

$$\frac{s^{\mu(\alpha-1)}}{s^{\alpha} - \lambda\left[\frac{s^{\rho\gamma-\alpha}}{(s^{\rho}-\nu)^{\gamma}}\right]} = \mathcal{L}\left(\sum_{k=0}^{\infty} \lambda^{k} x^{2\alpha k+\alpha+\mu-\mu\alpha-1} E^{\gamma k}_{\rho,2\alpha k+\alpha+\mu-\mu\alpha}(\nu x^{\rho})\right)$$

and

$$\frac{G(s)}{s^{\alpha} - \lambda\left[\frac{s^{\rho\gamma-\alpha}}{(s^{\rho}-\nu)^{\gamma}}\right]} = \mathcal{L}\left[\left(\sum_{k=0}^{\infty} \lambda^{k} x^{2\alpha k+\alpha-1} E^{\gamma k}_{\rho,2\alpha k+\alpha}(\nu x^{\rho})\right) * g(x)\right].$$

Upon substituting these last two relations into (4.19), if we apply the inverse Laplace transforms, we arrive at the solution (4.18) asserted by Theorem 4.2. The details involved are being omitted here (see also [50]). □

Each of the following particular cases of Theorem 4.2 are worthy of note here.

Example 4. If we put

$$g(x) = x^{\mu-1}$$

and use the integral formula (4.5), we get the following particular case of the solution (4.18).

Corollary 4.4. (see [50]). *The following fractional differintegral equation*:

$$\left(D^{\alpha,\mu}_{0+} f\right)(x) = \lambda\left(\mathbf{E}^{\gamma}_{\rho,\alpha,\nu;0+} f\right)(x) + x^{\mu-1} \tag{4.20}$$

$$\left(0 < \alpha < 1;\ 0 \leq \mu \leq 1;\ \Re(\nu) > 0\right)$$

with the initial condition (4.17) *has its solution in the space* $L(0,\infty)$ *given by*

$$f(x) = x^{\alpha+\mu-\mu\alpha-1}[c\sum_{k=0}^{\infty}\left(\lambda x^{2\alpha}\right)^{k} E^{\gamma k}_{\rho,2\alpha k+\alpha+\mu-\mu\alpha}(\nu x^{\rho})$$
$$+ \Gamma(\mu)\, x^{\mu\alpha}\sum_{k=0}^{\infty}\left(\lambda x^{2\alpha}\right)^{k} E^{\gamma k}_{\rho,2\alpha k+\alpha+\mu}(\nu x^{\rho})]. \tag{4.21}$$

where c is an arbitrary constant.

Example 5. If we put

$$g(x) = c x^{\mu-\mu\alpha-1} E_{\rho,\mu-\mu\alpha}(\nu x^{\rho})$$

and use the integral formula (4.8), we get the following particular case of the solution (4.18).

Corollary 4.5. (see [50]). *The following fractional differintegral equation*:

$$\left(D^{\alpha,\mu}_{0+} f\right)(x) = \lambda\left(\mathbf{E}^{\gamma}_{\rho,\alpha,\nu;0+} f\right)(x) + c x^{\mu-\mu\alpha-1} E_{\rho,\mu-\mu\alpha}(\nu x^{\rho}) \tag{4.22}$$

$$\left(0 < \alpha < 1;\ 0 \leq \mu \leq 1;\ \Re(\nu) > 0\right)$$

with the initial condition (4.17) *has its solution in the space* $L(0,\infty)$ *given by*

$$f(x) = c\sum_{k=0}^{\infty} \lambda^{k} x^{2\alpha k+\alpha+\mu-\mu\alpha-1}\left[E^{\gamma k}_{\rho,2\alpha k+\alpha+\mu-\mu\alpha}(\nu x^{\rho}) + E^{\gamma k+1}_{\rho,2\alpha k+\alpha+\mu-\mu\alpha}(\nu x^{\rho})\right], \tag{4.23}$$

where c is an arbitrary constant.

5. A general family of fractional kinetic differintegral equations

Fractional kinetic equations have gained popularity during the past decade or so due mainly to the discovery of their relation with the CTRW-theory in [16]. These equations are investigated in order to determine and interpret certain physical phenomena which govern such processes as diffusion in porous media, reaction and relaxation in complex systems, anomalous diffusion, and so on (see, for example, [12] and [13]).

In a recent investigation by Saxena and Kalla [35] (see also references to many closely-related works cited in [35]), the following fractional kinetic equation was considered [35, p. 506, Equation (2.1)]:

$$N(t) - N_0\, f(t) = -c^\nu \left(I_{0+}^{\nu} N\right)(t) \qquad \big(\Re(\nu) > 0\big), \tag{5.1}$$

where $N(t)$ denotes the number density of a given species at time t, $N_0 = N(0)$ is the number density of that species at time $t = 0$, c is a constant and (for convenience) $f \in L(0,\infty)$, it being tacitly assumed that $f(0) = 1$ in order to satisfy the initial condition $N(0) = N_0$. By applying the Laplace transform operator $\mathcal{L}$ to each member of (5.1), we readily obtain

$$\begin{aligned}\mathcal{L}[N(t)](s) &= N_0 \left(\frac{F(s)}{1 + c^\nu s^{-\nu}}\right) \\ &= N_0 \left(\sum_{k=0}^{\infty} \left(-c^\nu\right)^k\ s^{-k\nu}\right) F(s) \qquad \left(\left|\frac{c}{s}\right| < 1\right).\end{aligned} \tag{5.2}$$

Remark 5.1. *In view of the fact that*

$$\mathcal{L}\left[t^{\mu-1}\right](s) = \frac{\Gamma(\mu)}{s^\mu} \qquad \big(\Re(s) > 0;\ \Re(\mu) > 0\big), \tag{5.3}$$

it is not possible to compute the inverse Laplace transform of $s^{-k\nu}$ $(k \in \mathbb{N}_0)$ *by setting* $\mu = k\nu$ *in* (5.3)*, simply because the condition* $\Re(\mu) > 0$ *would obviously be violated when* $k = 0$*. Consequently, the claimed solution of the fractional kinetic equation* (5.1) *by Saxena and Kalla* [35, p. 506, Equation (2.2)] *should be corrected to read as follows:*

$$N(t) = N_0 \left(f(t) + \sum_{k=1}^{\infty} \frac{\left(-c^\nu\right)^k}{\Gamma(k\nu)} \left(t^{k\nu-1} * f(t)\right)\right) \tag{5.4}$$

or, equivalently,

$$N(t) = N_0 \left(f(t) + \sum_{k=1}^{\infty} \left(-c^\nu\right)^k \left(I_{0+}^{k\nu} f\right)(t)\right), \tag{5.5}$$

where we have made use of the following relationship between the Laplace convolution and the Riemann-Liouville fractional integral operator $\left(I_{0+}^{\mu} f\right)(x)$ *defined by* (1.1) *with* $a = 0$*:*

$$t^{k\nu-1} * f(t) := \int_0^t (t-\tau)^{k\nu-1}\, f(\tau)d\tau =: \Gamma(k\nu)\left(I_{0+}^{k\nu} f\right)(t) \tag{5.6}$$

$$\big(k \in \mathbb{N};\ \Re(\nu) > 0\big).$$

Remark 5.2. *The solution* (5.5) *would provide the corrected version of the obviously erroneous solution of the fractional kinetic equation* (5.1) *given by Saxena and Kalla* [35, p. 508, Equation (3.2)] *by applying a technique which was employed earlier by Al-Saqabi and Tuan* [3] *for solving fractional differeintegral equations.*

In our conclusion of this section, we choose to consider the following general family of fractional kinetic differintegral equations:

$$a\left(D_{0+}^{\alpha,\beta}N\right)(t) - N_0\, f(t) = b\left(I_{0+}^{\nu}N\right)(t) \tag{5.7}$$

under the initial condition:

$$\left(I_{0+}^{(1-\beta)(1-\alpha)}f\right)(0+) = c, \tag{5.8}$$

where a, b and c are constants and $f \in L(0,\infty)$.

By suitably making use of the Laplace transform method as in our demonstrations of the results proven in the preceding sections, we can obtain the following explicit solution of (5.7) under the initial condition (5.8).

Theorem 5.1. (see [50]). *The fractional kinetic differintegral equation* (5.7) *with the initial condition* (5.8) *has its explicit solution given by*

$$N(t) = \frac{N_0}{a}\sum_{k=0}^{\infty}\left(\frac{b}{a}\right)^k \frac{\left(t^{\alpha+k(\nu+\alpha)-1} * f(t)\right)}{\Gamma\big(\alpha+k(\nu+\alpha)\big)} + c\sum_{k=0}^{\infty}\left(\frac{b}{a}\right)^k \frac{t^{\alpha-\beta(1-\alpha)+k(\nu+\alpha)-1}}{\Gamma\big(\alpha-\beta(1-\alpha)+k(\nu+\alpha)\big)} \qquad (a \neq 0) \tag{5.9}$$

or, equivalently, by

$$N(t) = \frac{N_0}{a}\sum_{k=0}^{\infty}\left(\frac{b}{a}\right)^k \left(I_{0+}^{\alpha+k(\nu+\alpha)}f\right)(t) + c\sum_{k=0}^{\infty}\left(\frac{b}{a}\right)^k \frac{t^{\alpha-\beta(1-\alpha)+k(\nu+\alpha)-1}}{\Gamma\big(\alpha-\beta(1-\alpha)+k(\nu+\alpha)\big)} \qquad (a \neq 0), \tag{5.10}$$

where a, b and c are constants and $f \in L(0,\infty)$.

The case of the explicit solution of the fractional kinetic differintegral equation (5.7) with the initial condition (5.8) when $b \neq 0$ can be considered similarly. Several illustrative examples of Theorem 5.1 involving some appropriately-chosen special values of the function $f(t)$ can also be derived fairly easily. We choose to leave the details involved in these derivations as an exercise for the interested reader.

6. Further observations and concluding remarks

First of all, we observe that an interesting and potentially useful family of λ-generalized Hurwitz-Lerch zeta functions, which *further* extend the multi-parameter Hurwitz-Lerch zeta function

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z,s,a)$$

defined by (1.23), was introduced and investigated systematically in a recent paper by Srivastava [39]. Among various properties of this and related novel families of the λ-generalized Hurwitz-Lerch zeta functions, Srivastava [39] presented many potentially useful results involving some of these λ-generalized Hurwitz-Lerch zeta functions including (for example) their partial differential equations, new series and Mellin-Barnes type contour integral representations (which are associated with Fox's H-function [45]) and several other summation formulas involving them. Furthermore, Srivastava [39] discussed their potential application in Number Theory by appropriately constructing a presumably new continuous analogue of Lippert's Hurwitz measure and also considered some other statistical applications of these families of the λ-generalized Hurwitz-Lerch zeta functions in probability distribution theory (see also the references to several related earlier works cited by Srivastava [39]).

The so-called pathway integral transform, that is, the $\mathcal{P}_\delta$-transform $\mathcal{P}_\delta[f(t);s]$, of a function $f(t)$ $(t \in \mathbb{R})$ is a function $F_{\mathcal{P}}(s)$ of a complex variable s, which is defined by (see, for example, [23])

$$\mathcal{P}_\delta[f(t);s] = F_{\mathcal{P}}(s) := \int_0^\infty [1+(\delta-1)s]^{-\frac{t}{\delta-1}} f(t)\,dt \qquad (\delta > 1), \tag{6.1}$$

provided that the sufficient existence conditions are satisfied.

Remark 6.1. *By closely comparing the definitions in* (1.7) *and* (6.1), *it is easily observed that the $\mathcal{P}_\delta$-transform is essentially the same as the classical Laplace transform with the following rather trivial parameter change in* (1.7) :

$$s \mapsto \frac{\ln[1+(\delta-1)s]}{\delta-1} \qquad (\delta > 1). \tag{6.2}$$

Nevertheless, the current literature on various families of extended Mittag-Leffler type functions vis-à-vis operators of fractional integrals and fractional derivatives is flooded by investigations claiming at least implicitly that the $\mathcal{P}_\delta$-transform $\mathcal{P}_\delta[f(t);s]$ defined by (6.1) *is a generalization of the classical Laplace transform defined by* (1.7) (see also Remark 6.2 below).

We now turn to another widely-claimed generalization of the familiar Riemann-Liouville fractional integral operator $\left(I_{0+}^{\mu} f\right)(x)$ of order μ, which is defined by (1.1) *with* $a = 0$. Indeed, in all of these many publications which are much too numerous to cite here, the so-called pathway fractional integral operator $\left(\mathfrak{P}_{+0}^{(\eta,\alpha,\beta)} f\right)(x)$ is defined by (see, for example, [28] *and* the references to several earlier works on the subject, which are cited therein)

$$\begin{aligned}\left(\mathfrak{P}_{+0}^{(\eta,\alpha,\beta)} f\right)(x) &:= x^\eta \int_0^{\frac{x}{(1-\alpha)\beta}} \left(1-\frac{(1-\alpha)\beta t}{x}\right)^{\frac{\eta}{1-\alpha}} f(t)\,dt \\ &= x^{-\frac{\eta\alpha}{1-\alpha}} \int_0^{\frac{x}{(1-\alpha)\beta}} [x-(1-\alpha)\beta t]^{\frac{\eta}{1-\alpha}} f(t)\,dt,\end{aligned} \tag{6.3}$$

where $f(t)$ is suitably constrained Lebesgue integrable function, $\alpha < 1$, $\beta > 0$ and $\Re(\eta) > 0$. For an obvious change of the variable of integration in (6.3), we set

$$t = \frac{\tau}{(1-\alpha)\beta} \qquad \text{and} \qquad dt = \frac{d\tau}{(1-\alpha)\beta}.$$

We thus find from the definition (6.3) that

$$\begin{aligned}\left(\mathfrak{P}_{+0}^{(\eta,\alpha,\beta)} f\right)(x) = &\frac{x^{-\frac{\eta\alpha}{1-\alpha}}\,\Gamma\left(\frac{\eta}{1-\alpha}+1\right)}{(1-\alpha)\beta} \\ &\cdot \frac{1}{\Gamma\left(\frac{\eta}{1-\alpha}+1\right)} \int_0^x (x-\tau)^{\left(\frac{\eta}{1-\alpha}+1\right)-1} f\left(\frac{\tau}{(1-\alpha)\beta}\right) d\tau,\end{aligned}$$

which would lead us immediately to Remark 6.2 below.

Remark 6.2. *The so-called pathway fractional integral in* (6.3) *is, in fact, essentially the same as the extensively- and widely-investigated Riemann-Liouville fractional integral in* (1.1) *with, of course, some obvious straightforward parameter, variable and notational changes* (see also

Remark 6.1 above). *Furthermore, two of the three parameters* η, α and β, *which are involved in the definition* (6.3), *are obviously redundant.*

We choose to conclude this presentation by reiterating the fact that the extensively-investigated *and* celebrated special function named after the famous Swedish mathematician, Magnus Gustaf (Gösta) Mittag-Leffter (16 March 1846–07 July 1927), as well as its various extensions and generalizations including (among others) those that are considered here, have found remarkable applications in the solutions of a significantly wide variety of problems in the physical, biological, chemical, earth and engineering sciences (see, for example, [51]). However, in a presentation of this *modest* size, it is naturally hard to justify and elaborate upon the tremendous potential for applications of all those Mittag-Leffler type functions in one and more variables which have appeared in the existing literature on the subject. In our presentation here, we have focussed mainly on the problems and prospects involving some of the Mittag-Leffler type functions in the areas of variois families of fractional differintegral equations. In many recent investigations (see, for example, [2], [4], [6], [10], [43], and [52] to [66]), one form or the other of the Mittag-Leffler type functions (which we have considered in this survey-cum-expository article) have found interesting applications in the solutions of a wide variety of well-known (rather classical) ordinary as well as partial differential equations of Mathematical Physics and Applied Mathematics when such differential equations are studied in the context of *local fractional calculus* (that is, local fractional integrals and local fractional derivatives).

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