TWMS J. Pure Appl. Math., V.13, N.1, 2022, pp.25-37

ON SPACE-LIKE GENERALIZED CONSTANT RATIO HYPERSUFACES IN MINKOWSKI SPACES

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Abstract. In this work, we move the study of generalized constant ratio hypersurfaces started in [6] into the Minkowski space. First, we get some geometrical properties of a non-degenerated GCR hypersurface in an arbitrary dimensional Minkowski space. Then, we obtain the complete classification of space-like GCR hypersurfaces with vanishing Gauss-Kronecker curvature in the Minkowski space $\mathbb{E}^n_1$. We also give some explicit examples.

Keywords: generalized constant ratio hypersurfaces, Minkowski spaces, space-like submanifolds, flat hypersurfaces.

AMS Subject Classification: 53C42, 53D12, 53B25.

1. Introduction

The position vector is one of most elementary geometrical objects of a submanifold of a semi-Euclidean space. In [2] B.-Y. Chen introduced the notion of constant ratio submanifolds. By the definition, a submanifold of a Euclidean space is said to be of *constant ratio* (CR) if the ratio of the length of the tangential and normal components of its position vector is constant (see also [1]).

By using this idea, in [6] Fu and Munteanu gave the definition of generalized constant ratio (GCR) surfaces in the Euclidean space $\mathbb{E}^3$. Later, in [6, 7], classification of GCR surfaces in the Minkowski space $\mathbb{E}^3_1$ was obtained. Recently, in [10] the following definition is given.

Definition 1.1. [10] *Let M be a submanifold in $\mathbb{E}^m$ and x its position vector. M is said to be a GCR submanifold if the tangential part x^T of x is one of principal directions of all shape operators of M.*

GCR surfaces in Euclidean 3-space $\mathbb{E}^3$ are related with *constant slope surfaces* introduced by M. I. Munteanu in [12] because of the following property: Let U and x denote the projection of position vector on the tangent plane of the surface and a generic point in ambient space, respectively. If the projection U makes constant angle with the normal vector of the surface at that point, then U is a canonical principal direction of the surface with the corresponding principal curvature being different from zero. We would like to note that the complete classification of constant slope surfaces in $\mathbb{E}^3_1$ is obtaind by Fu and Wang in [8, 9],

In 2003, B.-Y. Chen established the complete classification of space-like CR submanifolds in pseudo-Euclidean spaces in [4]. Since the position vector of a CR hypersurface is one of principal directions, one can conclude that a CR hypersurface is GCR. However, the converse is not true

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Manuscript received September 2020.

in general, [6]. In the present paper, we get a classifiction of GCR hypersurfaces in Minkowski spaces.

This paper has is divided into 4 sections. In Sect. 2, after we introduce the notation that we are going to use, we give a brief summary of basic definitions in theory of submanifolds of semi-Euclidean spaces. In Sect. 3, we obtain some of geometrical properties of space-like GCR hypersurfaces of an arbitrary dimensional Minkowski space. In Sect. 4, we obtain the complete classification of space-like GCR hypersurfaces of the Minkowski 4-space.

2. Preliminaries

In this section, we present some of basic facts and definitions in the theory of submanifolds of pseudo-Euclidean spaces.

2.1. Isometric immersions into pseudo-Riemannian space forms.

Let $\mathbb{E}^m_s$ denote the pseudo-Euclidean m-space with the canonical pseudo-Euclidean metric tensor g of index s given by

$$\widetilde{g} = \langle\ ,\ \rangle = -\sum_{i=1}^{s} dx_i^2 + \sum_{j=s+1}^{m} dx_j^2.$$

A non-zero vector v in $\mathbb{E}^m_s$ is said to be space-like, time-like and light-like (null) regarding to $\langle v, v\rangle > 0$, $\langle v, v\rangle < 0$ and $\langle v, v\rangle = 0$, respectively. Note that v is said to be causal if it is not space-like.

Let $\widetilde{M}^m_{s,c}$ denote the pseudo-Riemannian space form of dimension m, index s and curvature c. In fact, we have

$$\begin{aligned}\widetilde{M}^m_{s,r^{-2}} = \mathbb{S}^m_s(r^{-2}) &= \{x \in \mathbb{E}^{m+1}_s : \langle x, x\rangle = r^2\},\\ \widetilde{M}^m_{s,-r^{-2}} = \mathbb{H}^m_{s-1}(r^{-2}) &= \{x \in \mathbb{E}^{m+1}_s : \langle x, x\rangle = -r^2\},\end{aligned}$$

when $c > 0$ or $c > 0$, respectively. $\mathbb{S}^{m-1}_s(r^2)$ and $\mathbb{H}^{m-1}_{s-1}(-r^2)$ are the complete pseudo-Riemannian manifolds with constant sectional curvatures r^2 and $-r^2$, respectively. In the Riemannian case, we are going to use the notation $\mathbb{E}^m_0 = \mathbb{E}^m$, $\mathbb{H}^{m-1}_0(-r^2) = \mathbb{H}^{m-1}(-r^2)$ and $\mathbb{S}^{m-1}_0(r^2) = \mathbb{S}^{m-1}(r^2)$.

For a given isometric immersion $x : (\Omega, g) \hookrightarrow \widetilde{M}^m_{s,c}$, we denote the normal bundle of x by $N^x\Omega$. Then, Gauss and Weingarten formulas are given by

$$\begin{aligned}\widetilde{\nabla}_X Y =& \nabla_X Y + \alpha_x(X, Y),\\ \widetilde{\nabla}_X \xi =& -A^x_\xi(X) + \nabla^\perp_X \xi,\end{aligned} \tag{1}$$

for any vector fields X, Y tangent to $M = x(\Omega)$, where ∇ and $\widetilde{\nabla}$ denote the Levi-Civita connections of M and $\mathbb{E}^{n+1}_1$, respectively, α_x, and $\nabla^\perp$ stand for the second fundamental form and normal connection of x, respectively and A^x_ξ is the shape operator along $\xi \in N^x\Omega$. α_x and A^x are related by

$$\left\langle A^x_\xi X, Y\right\rangle = \langle \alpha_x(X, Y), \xi\rangle. \tag{2}$$

When x is an isometric immersion into $\mathbb{E}^m_1$, M is going to be said to lay in the space-like (resp. time-like) cone if $\langle x, x\rangle > 0$ (resp. $\langle x, x\rangle < 0$).

2.2. **Space-like Hypersurfaces in the Minkowski Space.** Let M be an oriented hypersurface in $\mathbb{E}^{n+1}_1$ with the position vector x and $N \in N^x\Omega$ be unit normal vector associated with the orientation of M. In this case, (1) turns into

$$\begin{aligned} \widetilde{\nabla}_X Y &= \nabla_X Y + h(X,Y), && (3)\\ \widetilde{\nabla}_X N &= -S(X), && (4) \end{aligned}$$

where $h = \alpha_x$ and $S = A^x_N$. The Gauss and Codazzi equations are given, respectively, by

$$\begin{aligned} \langle R(X,Y)Z, W\rangle &= \langle h(Y,Z), h(X,W)\rangle - \langle h(X,Z), h(Y,W)\rangle, && (5)\\ (\widetilde{\nabla}_X h)(Y,Z) &= (\widetilde{\nabla}_Y h)(X,Z), && (6) \end{aligned}$$

where R is the curvature tensor of M and $\widetilde{\nabla} h$ is defined by

$$(\widetilde{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

M is said to be space-like (resp.time-like) if its induced metric $g = \widetilde{g}|_M$ is Riemannian (resp. Lorentzian). This is equivalent to being time-like (resp. space-like) of N at each point of M. We are going to consider the case that M is space-like. Then, the shape operator S is diagonalizable, i.e., there exists a local orthonormal frame field $\{e_1, e_2, \ldots, e_n; N\}$ such that $Se_i = k_i e_i, \quad i = 1, 2, \ldots, n$. In this case, the vector field e_i and smooth function k_i are called a principal direction and a principal curvature of M, respectively. The connection forms ω_{ij} are defined by

$$\omega_{ij}(e_k) = \langle \nabla_{e_k} e_i, e_j\rangle,$$

and satisfies $\omega_{ij} = -\omega_{ji}$. On the other hand, the Codazzi equations (6) for $X = Z = e_j$, $Y = e_i$ and $X = e_i$, $Y = e_j$, $Z = e_k$ imply

$$e_i(k_j) = \omega_{ij}(e_j)(k_i - k_j), \quad i, j = 1, \ldots, n \tag{7}$$

and

$$\omega_{ij}(e_k)(k_i - k_j) = \omega_{ik}(e_j)(k_i - k_k), \quad i, j, k = 1, \ldots, n, \tag{8}$$

respectively.

3. GCR Hypersurfaces of Minkowski Spaces

In this section, we consider GCR hypersurfaces in a Minkowski space $\mathbb{E}^{n+1}_1$.

Let M be a hypersurface in a semi-Euclidean space $\mathbb{E}^{n+1}_s$ and x its position vector. Since x can be considered as a vector field defined on M, it can be expressed as

$$x = x^T + x^{\perp}, \tag{9}$$

where x^T and $x^{\perp}$ denote the tangential and normal parts of x.

Remark 3.1. *If $x^T = 0$ in the decomposition (9), i.e., x is normal to M, then we have $\langle x, x\rangle =$ const which yields that M is an open part of either $\mathbb{S}^n(r^{-2})$ or $\mathbb{H}^n(-r^{-2})$ for some $r > 0$.*

We have the following result for the case of being light-like of x^T.

Theorem 3.1. *Let M be a non-degenerated hypersurface in $\mathbb{E}^{n+1}_1$ with position vector x. If M is GCR, then the tangential part x^T of x can not be light-like.*

Proof. Consider a non-degenerated hypersurface in $\mathbb{E}^{n+1}_1$ such that x^T is light-like. Then, we have M is time-like and

$$x = f_1 + \langle x, x\rangle N \tag{10}$$

for a light-like tangent vector field f_1.

Towards contradiction, assume that M is GCR, i.e., f_1 is an eigenvector of S. Then, we have $Sf_1 = k_1 f_1$ for a smooth function k_1. Moreover, there exists a light-like tangent vector field f_2 such that $\langle f_1, f_2\rangle = -1$ and $\langle Sf_1, f_2\rangle = -k_1$ which implies $h(f_1, f_2) = -k_1 N$. By applying f_2 to (10) and considering $\widetilde{\nabla}_{f_2} x = f_2$, we obtain

$$f_2 = \nabla_{f_2} f_1 + h(f_1, f_2) + f_2\left(\langle x, x\rangle\right) N - \langle x, x\rangle S f_2. \tag{11}$$

The normal part of this equation gives

$$k_1 = f_2\left(\langle x, x\rangle\right) = -2.$$

However, by a further computation using (11), we get

$$-1 = \langle x, x\rangle \langle S f_2, f_1\rangle$$

which implies being constant of $\langle x, x\rangle$. Therefore, M is an open part of either $\mathbb{S}^n_1(r^{-2})$ or $\mathbb{H}^n(-r^{-2})$ for some $r > 0$ which yields $x \in N^x\Omega$, i.e., $x^T = f_1 = 0$. Hence, we have a contradiction. □

Remark 3.2. *Because of Remark 3.1 and Theorem 3.1, we, locally, assume $\|x^T\| \neq 0$ in the remaining part of this paper.*

We also need the following lemma given in [3].

Lemma 3.1. *Let $x : M \longrightarrow \mathbb{E}^m_v$ be an isometric immersion of a Riemannian n-manifold into the pseudo-Euclidean space $\mathbb{E}^m_v$. Then, on the open subset $U = \left\{p \in M : x^T \neq 0\right\}$ the distribution D defined by $D_p = \left\{X \in T_pU : \left\langle X, x^T\right\rangle = 0\right\}$ is an integrable distribution,* [3].

In the remaining part of this paper we, put

$$e_1 = \frac{x^T}{|\langle x^T, x^T\rangle|^{1/2}},$$

Then, we have the following result obtained directly from Lemma 3.1.

Corollary 3.1. *Let M be a GCR hypersurface in $\mathbb{E}^{n+1}_1$ Minkowski spaces. Then, $D = \operatorname{span}\{e_2, \ldots, e_n\}$ and $D^\perp = \operatorname{span}\{e_1\}$ are integrable distributions on M.*

Now, we obtain some necessary and sufficient conditions for a hypersurface in $\mathbb{E}^{n+1}_1$ to be GCR.

Proposition 3.1. *Let M be an oriented hypersurface in the Minkowski space $\mathbb{E}^{n+1}_1$ and x its position vector. Then, M is a GCR hypersurface if and only if a curve α is a geodesic of M whenever it is an integral curve of e_1.*

Proof. We are going to consider being space-like or time-like of x^T, separately.

Case 3.1. Let x^T is time-like. In this case, $e_1 = x^T/(-\langle x^T, x^T\rangle)^{1/2}$ is time-like and M is Lorentzian. Thus, we have

$$x = -\langle x, e_1\rangle e_1 + \langle x, N\rangle N.$$

Since $\widetilde{\nabla}_{e_1} x = e_1$, this equation yields

$$e_1 = (1 - \langle x, N\rangle\langle Se_1, e_1\rangle)e_1 - \langle x, e_1\rangle \widetilde{\nabla}_{e_1} e_1 + \langle x, Se_1\rangle N - \langle x, N\rangle Se_1.$$

The tangential part of this equation yields $Se_1 = k_1 e_1$ if and only if $\nabla_{e_1} e_1 = 0$ which is equivalent to being geodesic of all integral curves of e_1.

Case II. Let x^T is space-like. In this case, $e_1 = x^T/(\langle x^T, x^T\rangle)^{1/2}$ is space-like. Thus, we have

$$x = \langle x, e_1\rangle e_1 + \varepsilon\langle x, N\rangle N, \tag{12}$$

where ε is either 1 or -1 regarding to being time-like or space-like of M, respectively.

Similar to Case I, we obtain $Se_1 = k_1 e_1$ if and only if $\nabla_{e_1} e_1 = 0$. Consequently, the proof is completed. □

Now, let M be an oriented space-like GCR hypersurface in $\mathbb{E}^{n+1}_1$ with the position vector, define a function μ by

$$\mu = \sqrt{|\langle x, x\rangle|},$$

and, locally, assume $\mu > 0$. Consider a local orthonormal frame field $\{e_1, e_2, \ldots, e_n; N\}$ consisting of principal directions of M and let $k_1, k_2, \ldots, k_n$ be corresponding principal curvatures.

Case I. M lays on the space-like cone of $\mathbb{E}^{n+1}_1$. In this case, (9) turns into

$$x = \mu\cosh\theta e_1 - \mu\sinh\theta N, \tag{13}$$

for a smooth function θ and we have

$$\begin{aligned} e_1(\mu) &= \cosh\theta, \\ e_j(\mu) &= 0, \quad j = 2, 3, \ldots, n. \end{aligned} \tag{14}$$

By applying e_i to (13) and considering (14), we obtain

$$\begin{aligned} e_i = &\left(\delta_{1i}\cosh^2\theta + \mu\sinh\theta e_i(\theta)\right) e_1 + \mu\cosh\theta\nabla_{e_i} e_1 + \mu\sinh\theta k_i e_i \\ &+ \mu\cosh\theta h(e_i, e_1) - \left(\delta_{1i}\cosh\theta\sinh\theta + \mu\cosh\theta e_i(\theta)\right) N, \end{aligned}$$

from which we get

$$k_1 = -e_1(\theta) - \frac{\sinh\theta}{\mu}, \tag{15a}$$

$$e_j(\theta) = 0, \tag{15b}$$

$$\nabla_{e_j} e_1 = \frac{1 - k_j\mu\sinh\theta}{\mu\cosh\theta} e_j, \quad j = 2, 3, \ldots, n. \tag{15c}$$

Case II. M lays on the time-like cone of $\mathbb{E}^{n+1}_1$. In this case, (9) turns into

$$x = \mu\sinh\theta e_1 - \mu\cosh\theta N. \tag{16}$$

By a similar way to the Case I, we get

$$\begin{aligned} e_1(\mu) &= -\sinh\theta, \\ e_j(\mu) &= 0, \quad j = 2, 3, \ldots, n, \end{aligned} \tag{17}$$

and

$$k_1 = -e_1(\theta) + \frac{\cosh\theta}{\mu}, \tag{18a}$$

$$e_j(\theta) = 0, \tag{18b}$$

$$\nabla_{e_j} e_1 = \frac{1 - k_j\mu\cosh\theta}{\mu\sinh\theta} e_j, \quad j = 2, 3, \ldots, n. \tag{18c}$$

By considering (15b) and (18b), we obtain

Proposition 3.2. *A space-like hypersurface M in the Minkowski space $\mathbb{E}^{n+1}_1$ is GCR if and only if $Y(\theta) = 0$, whenever $Y \in D$, where D is the distribution defined in Lemma 3.1.*

We also would like to state the following result which is a direct result of (15) and (18).

Proposition 3.3. *Let M be a space-like GCR hypersurface in the Minkowski space $\mathbb{E}^{n+1}_1$ and e_1 is a unit normal vector field along x^T. Then there exists a local coordinate function $\hat{s}$ such that $e_1 = \partial_{\hat{s}}$.*

Proof. We consider the case $\langle x, x\rangle < 0$. The other case follows from an analogous computation.

Let $\zeta_1, \zeta_2, \dots, \zeta_n$ be the dual base of $e_1, e_2, \dots, e_n$. By a direct computation using (17) and (18c) , we obtain $d\zeta_1 = 0$, i.e., ζ_1 is closed. Poincare Lemma (see in [5]) yields that it is exact, i.e., there exists a local coordinate function s such that $\zeta_1 = d\hat{s}$. □

4. GCR hypersurfaces in $\mathbb{E}^4_1$

In this section, we consider space-like GCR hypersurfaces with vanishing Gauss-Kronecker curvature in the Minkowski 4-space.

4.1. **Examples of GCR hypersurfaces.** Before we proceed to our main result, we would like to present the following examples of GCR hypersurfaces with vanishing Gauss-Kronecker curvature.

The following two isoparametric hypersurfaces are trivially GCR.

Example 4.1. *Let $y = (x_1, x_2, x_3) : \Omega_1 \hookrightarrow \mathbb{E}^3_1$ be an isometric immersion such that $y(\Omega_1) \subset \mathbb{H}^2(-c^2)$ and $\dim \Omega_1 = 2$. Consider the the hypercylinder M given by $x : \Omega_1 \times I \hookrightarrow \mathbb{E}^4_1$,*

$$x(s,t,u) = \Big(x_1(s,t), x_2(s,t), x_3(s,t), u\Big), \quad (s,t) \in \Omega_1,\ u \in I,$$

where we put either $I = (-1, 1)$ or $I = (1, \infty)$. Note that M is an open part of $\mathbb{H}^2 \times \mathbb{E}^1$ and its unit normal vector field is

$$N(s,t) = c\Big(x_1(s,t), x_2(s,t), x_3(s,t), 0\Big).$$

Therefore, x can be written as $x = u\dfrac{\partial}{\partial u} + \dfrac{1}{c}N$. Since the tangent vector $\dfrac{\partial}{\partial u}$ is the principal direction of M corresponding to the principal curvature $k_1 = 0$, the hypercylinder M is a GCR hypersurface with vanishing Gauss-Kronecker curvature.

Example 4.2. *If M is an open part of $\mathbb{H}(-c) \times \mathbb{E}^2$, then it can be parametrized by $x(s,t,u) = (c\sinh t, c\cosh t, s, u)$. Similar to Example 4.1, M is a GCR hypersurface with vanishing Gauss-Kronecker curvature.*

Example 4.3. *Consider an isometric immersion $\tilde{\alpha} : (I, a^2du^2) \hookrightarrow \mathbb{S}^3_1(1)$ with flat normal bundle, where I is an open interval and a is a non-vanishing function. Let $\tilde{F}_1, \tilde{F}_2 \in N^{\tilde{\alpha}}I$ be parallel orthonormal vector fields such that $\langle \tilde{F}_1, \tilde{F}_1\rangle = -1$. Put $\alpha = \tilde{\alpha} \circ i$ and $F_j = i_*\tilde{F}_j,\ j = 1, 2,$ where $i : S^3_1(1) \subset \mathbb{E}^4_1$ is the inclusion. Consider the hypersurface $M = x(\Omega)$ given by $x : \Omega \hookrightarrow \mathbb{E}^4_1$,*

$$x(s,t,u) = s\alpha(u) - c\Big(\cosh\left(\frac{t}{c}\right) F_1(u) + \sinh\left(\frac{t}{c}\right) F_2(u)\Big), \tag{19}$$

where $\Omega = J \times \mathbb{R} \times I$, $c \in (0, \infty)$ and we put either $J = (-c, c)$ or $I = (c, \infty)$. One can check that the unit normal vector field of M is

$$N(s,t,u) = \cosh\left(\frac{t}{c}\right) F_1(u) + \sinh\left(\frac{t}{c}\right) F_2(u).$$

Furthermore, a direct computation yields that $e_1 = \frac{1}{s}x^T = x_*(\partial_s)$ *is the principal direction of* M *corresponding to the principal curvature* $k_1 = 0$. *Consequently, the hypersurface* M *is GCR and its Gauss-Kronecker curvature vanishes identically. We note that one of other principal curvatures of* M *is* $k_2 = \frac{1}{c}$ *with the corresponding principle direction* $e_2 = \partial_t$.

Example 4.4. *Consider an isometric immersion* $\tilde{y} : (\Omega_1, g) \hookrightarrow \mathbb{S}^3_1(1)$ *and let* $\tilde{N} \in N^y\Omega_1$ *be unit, where* g *is a Riemannian metric on* Ω_1 *and* $\dim \Omega_1 = 2$. *Put* $y = \tilde{y} \circ i$ *and* $N = i_*\tilde{N},\ j = 1, 2$, *where* $i : S^3_1(1) \subset \mathbb{E}^4_1$ *is the inclusion and consider the hypersurface* $M = x(\Omega)$ *given by* $x : \Omega \hookrightarrow \mathbb{E}^4_1$

$$x(s, t, u) = sy(t, u) - cN(t, u), \tag{20}$$

for a non-zero constant c, *where* $\Omega = \Omega_1 \times I$ *and we put either* $J = (-c, c)$ *or* $I = (c, \infty)$. *By a direct computation, we see that* N *is the unit normal of the hypersurface of* M *which implies* $x^T = sx_*(\partial_s)$. *Furthermore, from* (20) *we obtain that* $x_{ss} = 0$ *and* $\langle x_{st}, N\rangle = 0$ *and* $\langle x_{tt}, N\rangle = 0$. *So,* $h(x_*(\partial_s), X) = 0$ *is satisfied for all tangent vector* X *on* M *which yields that* $S(x^T) = 0$. *Consequently, the Gauss-Kronecker curvature of the hypersurface* M *vanishes and* $\partial_s = x^T$ *is a principal direction of the hypersurface* M.

4.2. **A Classification of GCR Hypersurfaces in $\mathbb{E}^4_1$.** In this subsection we obtain the complete classification of space-like GCR hypersurfaces with vanishing Gauss-Kronecker curvature in $\mathbb{E}^4_1$.

First, we obtain the following lemma.

Lemma 4.1. *Let* M *be a space-like GCR hypersurface in* $\mathbb{E}^4_1$ *and* $\{e_1, e_2, e_3; N\}$ *be the frame field consisting of principal directions of* M *with corresponding principal curvatures* $k_1, k_2, \ldots, k_n$. *Then, the Levi-Civita connection* $\tilde{\nabla}$ *of* $\mathbb{E}^4_1$ *satisfies*

$$\nabla_{e_1}e_1 = 0, \quad \nabla_{e_1}e_2 = \omega_{23}(e_1)e_3, \qquad \nabla_{e_1}e_3 = -\omega_{23}(e_1)e_2, \tag{21a}$$

$$\nabla_{e_2}e_1 = \omega_{12}(e_2)e_2, \qquad \nabla_{e_2}e_2 = -\omega_{12}(e_2)e_1 + \omega_{23}(e_2)e_3, \tag{21b}$$

$$\nabla_{e_3}e_1 = \omega_{13}(e_3)e_3, \qquad \nabla_{e_3}e_3 = -\omega_{13}(e_3)e_1 - \omega_{23}(e_3)e_2, \tag{21c}$$

$$\nabla_{e_2}e_3 = -\omega_{23}(e_2)e_2, \qquad \nabla_{e_3}e_2 = \omega_{23}(e_3)e_3. \tag{21d}$$

Furthermore the function $\omega_{23}(e_1)$ k_1, k_2 *and* k_3 *satisfy*

$$\omega_{23}(e_1)(k_2 - k_3) = 0, \quad e_2(k_1) = e_3(k_1) = 0, \tag{22}$$

and for $j = 2, 3$ *we have*

$$\omega_{1j}(e_j) = \begin{cases} \dfrac{1 - \mu \sinh\theta k_j}{\mu\cosh\theta} & \text{if } M \text{ lays on the space-like cone,} \\ \dfrac{1 - \mu\cosh\theta k_j}{\mu\sinh\theta} & \text{if } M \text{ lays on the time-like cone.} \end{cases} \tag{23}$$

Proof. Because of Proposition 3.1, we have

$$\omega_{12}(e_1) = 0, \tag{24a}$$

which gives (21a). Furthermore, (15c) and (18c) give

$$\omega_{13}(e_2) = \omega_{12}(e_3) = 0, \tag{24b}$$

which implies (21b), (21c) and (23). On the other hand, we combine (7) and (8) with (24) to get (22). □

Lemma 4.2. *Let M be a space-like hypersurface in the Minkowski space $\mathbb{E}^4_1$. If the Gauss-Kronecker curvature of M vanishes, then the principal curvature k_1 of M vanishes identically and the equation*

$$e_1(\theta) = \begin{cases} -\dfrac{\sinh\theta}{\mu} & \text{if } M \text{ lays on the space-like cone,} \\ \dfrac{\cosh\theta}{\mu} & \text{if } M \text{ lays on the time-like cone} \end{cases} \tag{25}$$

is satisfied.

Proof. By combining (13), (16) and (23) with the Codazzi equation (7) for $i = 1, j = 2$, we get

$$e_1(k_2) = \frac{(1 - \langle x, N\rangle k_2)(k_1 - k_2)}{\langle x, e_1\rangle}. \tag{26}$$

We are going to show that the open subset $\mathcal{O} = \{p \in M | k_1(p) \neq 0\}$ of M is empty. Suppose that M is a space-like hypersurface with vanishing Gauss-Kronecker curvature, i.e., $k_1k_2k_3 = 0$. Then, without loss of generality, we may assume the existance of a connected open subset $\mathcal{O}_2$ of $\mathcal{O}$ on which $k_2 = 0$ is satisfied. However, in this case (26) gives $k_1 = 0$ on $\mathcal{O}_2$ which yields a contradiction unless $\mathcal{O}$ is empty. Hence, we have $k_1 = 0$ and (15a) and (18a) implies (25). □

Next, by considering Lemma 4.1 and Lemma 4.2, we construct a local coordinate system on M.

Proposition 4.1. *Let M be a space-like GCR hypersurface with vanishing Gauss-Kronecker curvature in the Minkowski space $\mathbb{E}^4_1$. Then, there exists a local coordinate system s, t, u on M such that $e_1 = \partial_s$ and*

$$\operatorname{span}\{e_2, e_3\} = \operatorname{span}\{\partial_t, \partial_u\}.$$

Moreover, the position vector x of M is decomposed as

$$x(s, t, u) = se_1(t, u) - cN(t, u), \tag{27}$$

for a non-zero constant c.

Proof. Let M be a space-like hypersurface with vanishing Gauss-Kronecker curvature in the Minkowski space $\mathbb{E}^4_1$. The the equation (25) is satisfied because of Lemma 4.2. On the other hand, because of Corollary 3.1, the distributions $D = \operatorname{span}\{e_2, e_3\}$ and $D^\perp = \operatorname{span}\{e_1\}$ are integrable distributions on M. Therefore, there exist $(\hat{s}, t, u)$ local coordinate system such that $D^\perp = \operatorname{span}\{\partial_{\partial_{\hat{s}}}\}$ and $D = \operatorname{span}\{\partial_t, \partial_u\}$ (See [11, Lemma on p. 182]). Consequently, we have

$$e_1 = a\partial_{\hat{s}} \tag{28}$$

for a smooth functions a. Note that Lemma 4.1 implies

$$[e_1, Y] \in D \qquad \text{whenever } Y \in D. \tag{29}$$

By combining (28) with (29), we get $\partial_t(a) = \partial_u(a) = 0$, i.e., $a(\hat{s}, t, u) = a(\hat{s})$. Therefore, by defining s by

$$s = \int_{s_0}^{\hat{s}} \frac{d\xi}{a(\xi)},$$

we get $e_1 = \partial_s$. Hence, the local coordinate system (s, t, u) satisfies the desired properties.

In order to obtain (27) we are going to consider the cases $\langle x, x\rangle > 0$ and $\langle x, x\rangle < 0$ separately.

Case I. M lays on the space-like cone. In this case, x can be decomposes as given in (13). By considering (25) and equations given in (14), we get

$$e_1(\mu \cosh \theta) = 1, \qquad e_1(\mu \sinh \theta) = 0. \tag{30}$$

Therefore, $\mu \cosh \theta = f$ for a function f satisfying $e_1(f) = 1$, $e_j(f) = 0$ for $j = 2, 3$ and $\mu \sinh \theta = c$ for a non-zero constant c. By considering, the local coordinate system that we have obtained, we can assume $f = s$. Hence, (13) gives

$$x(s, t, u) = se_1(s, t, u) - cN(t, u). \tag{31}$$

On the other hand, the first equation in (21a) gives $\dfrac{\partial e_1}{\partial_s} = 0$. Hence, (31) turns into (27).
Case II. M lays on the time-like cone. In this case, x can be decomposes as given in (16). By an analogous computation we get

$$e_1(\mu \sinh \theta) = 1, \qquad e_1(\mu \cosh \theta) = 0. \tag{32}$$

Similar to the Case I, (13) and (32) imply (31) which gives (27). □

Let M be a GCR hypersurface with vanishing Gauss-Kronecker curvature in $\mathbb{E}^4_1$. Then, in terms of local coordinate system (s, t, u) that we have constructed in Proposition 4.1, (23) turns into

$$\omega_{1j}(e_j) = \frac{1 - ck_j}{s}, j = 2, 3. \tag{33}$$

Therefore, the first equations in (21b), (21c) become

$$\nabla_{e_j} e_1 = \frac{1 - ck_j}{s} e_2. \tag{34}$$

Therefore $(\nabla_{e_j} e_1)_p = 0$ if and only if $S_p(e_j) = \frac{1}{c} e_j$ for any $p \in M$. Hence, we have

Corollary 4.1. *Consider a GCR hypersurface M with vanishing Gauss-Kronecker curvature in $\mathbb{E}^4_1$ with the position vector x given in (27) for a non-zero constant c. For a $p \in M$ define V_p as a kernel of the endomorphism*

$$\Phi : D_p \to D_p, \quad \Phi(Y_p) = \nabla_{Y_p} e_1.$$

Then, $V_p = W_p$, where W_p is the eigenspace of S_p corresponding to $\frac{1}{c}$, i.e., $W_p = \{Y_p \in T_pM | S_p(Y_p) = \frac{1}{c} Y_p\}$.

Now, we are ready to prove the classification theorem.

Theorem 4.1. *Let M be a space-like hypersurface with vanishing Gauss-Kronecker curvature in the Minkowski space $\mathbb{E}^4_1$. If M is a GCR hypersurface, then it is locally congruent to one of the following four types of hypersurfaces.*

(i) *An open part of the hypercylinder $\mathbb{H}^2(-c^2) \times \mathbb{E}^1$ given in Example 4.1,*
(ii) *An open part of the hypercylinder $\mathbb{H}(-c) \times \mathbb{E}^2$ given in Example 4.2,*
(iii) *A hypersurface parametrized with (19) in Example 4.3,*
(iv) *A hypersurface parametrized with (20) in Example 4.4.*

Proof. Let $p \in M$ and put $\dim V_p = r$. Consider a local coordinate system (s, t, u) constructed in Proposition 4.1. Then, we have

$$e_1 = \frac{\partial}{\partial s}, \tag{35a}$$
$$e_2 = a_{22}\frac{\partial}{\partial t} + a_{23}\frac{\partial}{\partial u}, \tag{35b}$$
$$e_3 = a_{32}\frac{\partial}{\partial t} + a_{33}\frac{\partial}{\partial u} \tag{35c}$$

for some smooth functions $a_{22}, a_{23}, a_{32}, a_{33}$. Note that there exists a neighborhood $\mathcal{N}_p$ on p such that $\dim V_q = r$ whenever $q \in \mathcal{N}_p$ because of Corollary 4.1. We consider the cases $r = 0$, $r = 1$ and $r = 2$, separately and find a parametrization of $\mathcal{N}_p$ for each cases.

Case 1. $r = 2$. In this case, Corollary 4.1 implies that $k_2 = k_3 = \dfrac{1}{c}$ on $\mathcal{N}_p$. Therefore, $\mathcal{N}_p$ is a space-like isoparametric hypersurface with principal curvatures $0, 1/c, 1/c$. Consequently, $\mathcal{N}_p$ is a part of the hypercylinder $\mathbb{H}^2(-c^2) \times \mathbb{E}^1$ given in Example 4.1 (see [13]). Hence, we have the case (i) of the theorem.

Case 2. $r = 1$. Without loss of generality, we assume $\nabla_{e_2}e_1 = 0$ and $\nabla_{e_3}e_1 \neq 0$. In this case, Corollary 4.1 implies $k_2 = \frac{1}{c} \neq k_3$ on $\mathcal{N}_p$. We have two subcases:

Case 2a. k_3 vanishes identically on $\mathcal{N}_p$. In this case, similar to the Case 1, $\mathcal{N}_p$ is an open part of $\mathbb{H}(-c) \times \mathbb{E}^2$. Hence, we have the case (ii) of the theorem.

Case 2b. There exists a point $q \in \mathcal{N}_p$ such that $k_3(q) \neq 0$. In this case, by shrinking $\mathcal{N}_p$ if necessary, we assume that k_3 doesn't vanish on $\mathcal{N}_p$. Then, k_3 is not constant. Therefore, (22) and the Codazzi equation (7) give

$$\omega_{23}(e_1) = \omega_{23}(e_2) = \omega_{12}(e_2) = 0, \qquad \omega_{13}(e_3) \neq 0. \tag{36}$$

By combining (36) with (21), we obtain

$$\widetilde{\nabla}_{e_1}e_1 = \widetilde{\nabla}_{e_1}e_2 = 0, \tag{37a}$$
$$\widetilde{\nabla}_{e_2}e_1 = \widetilde{\nabla}_{e_2}e_3 = 0, \qquad \widetilde{\nabla}_{e_2}e_2 = -\frac{1}{c}N, \tag{37b}$$
$$\widetilde{\nabla}_{e_3}e_1 = \frac{1 - ck_3}{s}e_3 \neq 0, \qquad \widetilde{\nabla}_{e_2}N = -\frac{1}{c}e_2 \tag{37c}$$

and $[e_1, e_2] = 0$. This equation and (35) give $\partial_s(a_{22}) = \partial_s(a_{23}) = 0$ which gives $a_{2j} = a_{2j}(t, u)$ for $j = 2, 3$.

Now, we define a new local coordinate system (S, T, U) by $S = s$, $T = \Phi_1(t, u)$ and $T = \Phi_2(t, u)$, where Φ_1 and Φ_2 satisfy

$$a_{22}(t, u)(\Phi_1)_t + a_{23}(t, u)(\Phi_1)_u = 1,$$
$$a_{22}(t, u)(\Phi_2)_t + a_{23}(t, u)(\Phi_2)_u = 0,$$

respectively. Then, by a direct computation using (35), we obtain

$$e_1 = \frac{\partial}{\partial S}, \quad e_2 = \frac{\partial}{\partial T}, \quad e_3 = \tilde{a}_{32}\frac{\partial}{\partial T} + \tilde{a_{33}}\frac{\partial}{\partial U}$$

for some smooth functions $\tilde{a}_{32}$ and $\tilde{a}_{33}$. By abusing the notation, in the rest of the proof of the Case 2b, we put $S = s$, $T = t$, $U = u$, $\tilde{a}_{32} = a_{32}$ and $\tilde{a}_{33} = a_{33}$. Consequently, the first equations in (37) give $e_1 = e_1(u)$ and $e_1'(u) \neq 0$. Therefore, one can define an immersion $\tilde{\alpha} : I \hookrightarrow \mathbb{S}^3_1(1)$, by

$$\alpha(u) = (\tilde{\alpha} \circ i)(u) = e_1(u), \tag{38}$$

where $i : S^3_1(1) \subset \mathbb{E}^4_1$ is the inclusion.

On the other hand, the second equations in (37b) and (37c) turn into $\widetilde{\nabla}_{e_2} e_2 = x_{tt} = -\frac{1}{c} N$ and $\widetilde{\nabla}_{e_2} N = N_t = -\frac{1}{c} x_t$, respectively. By combining these two equations, we get $c^2 N_{tt} - N = 0$ whose solution is given by

$$N(t,u) = \cosh\left(\frac{t}{c}\right) F_1(u) + \sinh\left(\frac{t}{c}\right) F_2(u). \tag{39}$$

for some vector valued functions F_1, F_2 satisfying

$$\langle F_1, F_1\rangle = -1, \quad \langle F_2, F_2\rangle = 1, \quad \langle F_1, F_2\rangle = 0. \tag{40}$$

because $\langle N, N\rangle = -1$. From (27), (38) and (39) we get (19). Furthermore, by the assumptions we have $\langle \partial_s, N\rangle = \langle \widetilde{\nabla}_{\partial_s} \partial_s, N\rangle = \langle \partial_u, x\rangle = 0$. By combining these equations with (19) and (40) we get

$$\langle \alpha, F_j\rangle = \left\langle \alpha', F_j\right\rangle = \left\langle F_1', F_2\right\rangle = \left\langle F_2', F_1\right\rangle = 0, \ j = 1, 2. \tag{41}$$

Therefore $\{\tilde{F}_1, \tilde{F}_2\}$ is a parallel orthonormal base of $N^{\tilde{\alpha}} I$, where $\tilde{F}_j$ is defined by $F_j = i_* \tilde{F}_j$. Consequently, α has parallel normal bundle and the induced metric of α is Riemannian. Hence, $\mathcal{N}_p$ is congruent to the hypersurface given in Example 4.3 and we have the case (iii) of the theorem.

Case 3. $r = 0$. In this case, the endomorphism Φ defined in Corolary 4.1 is one-to-one. Therefore, one can define an immersion $\tilde{y} : \Omega_1 \hookrightarrow \mathbb{S}^3_1(1)$ by

$$y(t,u) = (\tilde{y} \circ i)(t,u) = e_1(t,u), \tag{42}$$

where $i : S^3_1(1) \subset \mathbb{E}^4_1$ is the inclusion. Consequently, (27) turns into (20).

On the other hand, by combining $\langle \partial_s, N\rangle = \langle \partial_t, N\rangle = \langle \partial_u, N\rangle = 0$ and (20), we obtain $\langle y, N\rangle = \langle y_t, N\rangle = \langle y_u, N\rangle = 0$. So, the vector field $\tilde{N}$ defined by $N = i_* \tilde{N}$ belongs to $\in N^{\tilde{y}} \Omega_1$ and it satisfies $\langle \tilde{N}, \tilde{N}\rangle = -1$. Consequently, the induced metric of $\tilde{y}$ is Riemannian. Hence, $\mathcal{N}_p$ is congruent to the hypersurface given in Example 4.4 and we have the case (iv) of the theorem. □

Acknowledgements

This paper is a part of PhD thesis of the first named author who was supported by The Scientific and Technological Research Council of Turkey (TUBITAK) as a PhD scholar.

Conclusion

In this paper we study generalized constant ratio hypersurfaces. We first move the study initiated in [6] to the Minkowski space $\mathbb{E}^{n+1}_1$ with the arbitrary dimension. In Theorem 3.1 and Proposition 3.2 we obtain several geometric properties of these class of hypersurfaces. Next, by considering these results we focus on the hypersurfaces of the Minkowski 4-space. We prove Theorem 4.1 which presents the complete classification of space-like generalized constant ratio hypersurfaces in $\mathbb{E}^4_1$. In the future the results appearing on this paper may extend into time-like hypersurfaces of Minkowski spaces as well as pseudo-Euclidean spaces of arbitrary index.

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